Math 411: Honors Algebra I
Problem Set 6
due: October 23, 2019

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Exercise 1. Let $P = \{2, 3, 5, 7, 11, 13, \ldots\}$ be the (countably infinite) set of prime integers. Use the unique factorization into primes to define an isomorphism between the group $\mathbb{Q}^{\times}_{>0} = (\mathbb{Q}_{>0}, \times, 1)$ of positive rationals under multiplication and the **free abelian group** $\oplus_P \mathbb{Z}$ on the set P.[1]

Exercise 2.

- List all of the subgroups of D_8, the group of symmetries of the square.[2]
- Indicate which of these subgroups are normal.[3]

Exercise 3. Prove[4] that the set of "upper triangular" matrices — $n \times n$ matrices $A = (a_{ij})_{1 \leq i,j \leq n}$ with $a_{ij} = 0$ if $i > j$ and with $a_{ii} \neq 0$ — defines a subgroup of $GL_n(\mathbb{R})$.

Exercise 4. Find an example that shows that the union of two subgroups $H, K \subset G$ of a common group G is not necessarily a subgroup of G.

Exercise 5. A group G is **finitely generated** if there exists finitely many elements $g_1, \ldots, g_n \in G$ so that the subgroup generated by these elements is all of G. Prove that the group $\mathbb{Q} = (\mathbb{Q}, +, 0)$ is *not* finitely generated by showing that any subgroup generated by only finitely many rational numbers $q_1, \ldots, q_n$ does not contain some rational number $q \in \mathbb{Q}$.

Exercise 6. For any subset $N \subset G$ of a group G define

$$gN = \{gn \mid n \in N\} \qquad \text{and} \qquad Ng = \{ng \mid n \in N\}.$$

Let N be a subgroup of G. Prove that the following are equivalent.

(i) N is a *normal* subgroup of G.
(ii) For all $g \in G$, $gNg^{-1} \subset N$.
(iii) For all $g \in G$, $gNg^{-1} = N$.
(iv) For all $g \in G$, $gN \subset Ng$.
(v) For all $g \in G$, $Ng \subset gN$.
(vi) For all $g \in G$, $gN = Ng$.

In terminology we will introduce the equivalence (i) $\Leftrightarrow$ (vi) says that N is normal in G if and only if each **left coset** gN equals the **right coset** Ng.

Exercise 7. Find an example of a group G, a subgroup $H \subset G$, and an element $g \in G$ so that $gH \neq Hg$.

Exercise 8. The **index** $[G, H]$ of a subgroup $H \subset G$ is the number of left cosets gH for that subgroup.[5] Suppose $H \subset G$ is a subgroup of index 2. Prove that H is normal in G. (Hint: use Exercise 6(vi) and the fact that the cosets partition G.)

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[1] Note that free *abelian* groups are different from (smaller than) free groups. You can read more about them in your book.
[2] Hint: we'll prove soon that the order of a subgroup must divide the order of the group.
[3] Hint: Exercise 8 will help identify one normal subgroup.
[4] You don't need to write up a bunch of messy algebra. Just list the things you would have to check to prove this and then wave your hands.
[5] Two left cosets are the same, in symbols $gH = g'H$, if these sets have the same elements, which is the case iff $g^{-1}g' \in H$.