Math 411: Honors Algebra I
Problem Set 6
due: October 23, 2019

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Exercise 1. Let $P = \{2, 3, 5, 7, 11, 13, \ldots\}$ be the (countably infinite) set of prime integers. Use the unique factorization into primes to define an isomorphism between the group $\mathbb{Q}^{\times}_{>0} = (\mathbb{Q}_{>0}, \times, 1)$ of positive rationals under multiplication and the **free abelian group** $\oplus_P \mathbb{Z}$ on the set P.[1]

Proof. Note the group of positive rationals is abelian. By the universal property of the free abelian group $\oplus_P\mathbb{Z}$ to define a homomorphism $\phi\colon \oplus_P \mathbb{Z} \to \mathbb{Q}^{\times}_{>0}$ it suffices to chose arbitrary elements $\phi(p) \in \mathbb{Q}^{\times}_{>0}$ for every prime $p \in P$. Define $\phi(p) := p$. This uniquely determines the homomorphism ϕ.

Arbitrary elements of $\oplus_P\mathbb{Z}$ are sequences of integers $(a_1, a_2, \ldots)$ with all but finitely many of the a_i equal to zero. The homomorphism ϕ determined by the universal property above is then defined by

$$\phi(a_1, a_2, a_3 \ldots) = 2^{a_1}3^{a_2}5^{a_3}\cdots,$$

where the product on the right is finite because only finitely many of the a_i are non-zero. (This is the unique homomorphism that extends the definition $\phi(p) = p$ for each prime p.)

Now we just need to check that ϕ is injective and surjective. For surjectivity, note that any positive rational number has a unique expression as a fraction $\frac{a}{b}$ with $a, b \in \mathbb{N}$ and $\gcd(a, b) = 1$. Any natural number has a unique factorization into a product of primes: $a = p_1^{e_1} \cdots p_n^{e_n}$ and $b = q_1^{b_1} \cdots q_m^{b_m}$. By the hypothesis that our fractions are reduced the prime factors of a and b are distinct. Now you can define an element of $\oplus_P\mathbb{Z}$ which has the integers e_i in the slots corresponding to the primes p_i and the integers $-b_j$ in the slots corresponding to the primes q_j and ϕ sends this element to $\frac{a}{b}$. Injectivity follows because factorizations into primes are unique. (If you like, you can check that the kernel of ϕ is zero.) □

Exercise 2.

- List all of the subgroups of D_8, the group of symmetries of the square.[2]
- Indicate which of these subgroups are normal.[3]

Proof. For (i), there is the cyclic subgroup $C_4 \cong \mathbb{Z}/4$ of all rotations, and this has a further subgroup $C_2 \cong \mathbb{Z}/2$ of 180 degree rotations. Then D_8 has four additional subgroups of order 2, each generated by one of the four reflections (two through diagonal axes and the other two through the axes that bisect opposite sides of the square).

For (ii), none of the subgroups generated by the reflections are normal but both of the subgroups generated by rotations are. □

Exercise 3. Prove[4] that the set of "upper triangular" matrices — $n \times n$ matrices $A = (a_{ij})_{1 \le i,j \le n}$ with $a_{ij} = 0$ if $i < j$ and with $a_{ii} \neq 0$ — defines a subgroup of $GL_n(\mathbb{R})$.

Proof. The point is to note that the product of two upper triangular matrices is again an upper triangular matrix. □

Exercise 4. Find an example that shows that the union of two subgroups $H, K \subset G$ of a common group G is not necessarily a subgroup of G.

Proof. Basically any example will do. Eg, $2\mathbb{Z} \cup 3\mathbb{Z}$ contains both 2 and 3 but not 5 so is not a subgroup of $\mathbb{Z}$. □

Exercise 5. A group G is **finitely generated** if there exists finitely many elements $g_1, \ldots, g_n \in G$ so that the subgroup generated by these elements is all of G. Prove that the group $\mathbb{Q} = (\mathbb{Q}, +, 0)$ is *not* finitely generated by showing that any subgroup generated by only finitely many rational numbers $q_1, \ldots, q_n$ does not contain some rational number $q \in \mathbb{Q}$.

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[1] Note that free *abelian* groups are different from (smaller than) free groups. You can read more about them in your book.
[2] Hint: we'll prove soon that the order of a subgroup must divide the order of the group.
[3] Hint: Exercise **??** will help identify one normal subgroup.
[4] You don't need to write up a bunch of messy algebra. Just list the things you would have to check to prove this and then wave your hands.

Proof. Let $\frac{a_1}{b_1}, \ldots, \frac{a_n}{b_n}$ be any finite set of generators, expressed in lowest terms. The subgroup they generate is comprised of all sums

$$k_1 \frac{a_1}{b_1} + \cdots + k_n \frac{a_n}{b_n},$$

where the k_i are arbitrary integers. Remembering how you add fractions these elements can all be written as fractions with denominator $b_1 \cdots b_n$ (possibly not in lowest terms) and no larger denominators are possible. So in particular $\frac{1}{b_1 \cdots b_n + 1}$ is not in this subgroup. □

Exercise 6. For any subset $N \subset G$ of a group G define

$$gN = \{gn \mid n \in N\} \qquad \text{and} \qquad Ng = \{ng \mid n \in N\}.$$

Let N be a subgroup of G. Prove that the following are equivalent.

(i) N is a *normal* subgroup of G.
(ii) For all $g \in G$, $gNg^{-1} \subset N$.
(iii) For all $g \in G$, $gNg^{-1} = N$.
(iv) For all $g \in G$, $gN \subset Ng$.
(v) For all $g \in G$, $Ng \subset gN$.
(vi) For all $g \in G$, $gN = Ng$.

In terminology we will introduce the equivalence (i) ⇔ (vi) says that N is normal in G if and only if each **left coset** gN equals the **right coset** Ng.

Proof. This is in your book. (i)⇔(ii) by the definition of normality.

Now assume (ii). We know that $gNg^{-1} \subset N$ and also $g^{-1}Ng \subset N$ (since g^{-1} is another element of G). So $N = gg^{-1}Ngg^{-1} \subset gNg^{-1}$, which tells us that $N = gNg^{-1}$, proving (iii).

Assuming (iii), that is $gNg^{-1} = N$ for all g, we can multiply on the right by g to prove (vi) $gN = Ng$. This trivially implies (iv) and (v).

To finish we just need to show that (iv) or (v) implies (ii) and the proofs in both cases are the same. Assuming (iv) we have $gN \subset Ng$ so multiplying on the right by g^{-1} we have $gNg^{-1} \subset N$ which is the definition of normality. □

Exercise 7. Find an example of a group G, a subgroup $H \subset G$, and an element $g \in G$ so that $gH \neq Hg$.

Proof. Any non-normal subgroup will do. For instance, take $G = S_3$ and $H = \{e, (12)\}$. For certain choices of g it may be the case that $gH = Hg$ but by Exercise **??** this won't be true for all of them. Here $g = (13)$ gives $gH = \{(13), (123)\}$ but $Hg = \{(13), (132)\}$. □

Exercise 8. The **index** $[G, H]$ of a subgroup $H \subset G$ is the number of left cosets gH for that subgroup.[5] Suppose $H \subset G$ is a subgroup of index 2. Prove that H is normal in G. (Hint: use Exercise **??**(vi) and the fact that the cosets partition G.)

Proof. $H \subset G$ is normal iff for all $g \in G$ $gH = Hg$. If $g \in H$ then $gH = H = Hg$, so we assume $g \notin H$. But then G is a disjoint union of the left cosets H and gH. But G is also a disjoint union of the right cosets H and Hg so taking complements of H we see that $gH = Hg$. □

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[5] Two left cosets are the same, in symbols $gH = g'H$, if these sets have the same elements, which is the case iff $g^{-1}g' \in H$.