Math 411: Honors Algebra I
Problem Set 5
due: October 9, 2019

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Exercise 1. Recall the dihedral group D_{2n} was defined as the subgroup of S_n comprised of those permutations that define symmetries of a regular n-gon whose vertices are labeled $1, 2, \ldots, n$ in cyclic order. Prove that $|D_{2n}| = 2n$, justifying Aluffi's notation.[1]

Exercise 2. The five platonic solids are the tetrahedron, cube, octahedron, dodecahedron, and icosahedron.

(i) Draw a picture of each of these figures.
(ii) Referring to your picture as appropriate determine the orders of each group of symmetries.[2]

Exercise 3. Prove that if $m, n \in \mathbb{N}$ are positive integers with $\mathrm{gcd}(m, n) = 1$, then $\mathbb{Z}/mn \cong \mathbb{Z}/m \times \mathbb{Z}/n$. (Hint: to prove that these groups are isomorphic, it suffices to define *one* bijective homomorphism. Why?)

Exercise 4. Prove that if A and B are abelian groups, then $A \times B$ satisfies the universal property of the *coproduct* in the category Ab of abelian groups. Explain why the commutativity hypothesis is necessary.

Exercise 5.

(i) Fix an element g in a group G. Prove that the conjugation function $x \mapsto gxg^{-1}$ defines a homomorphism $\gamma_g \colon G \to G$.
(ii) Prove that the function $g \mapsto \gamma_g$ defines a homomorphism $\gamma \colon G \to \mathrm{Aut}(G)$. The image of this function is the subgroup of *inner automorphisms* of G.
(iii) Prove that γ is the zero homomorphism if and only if G is abelian.

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[1] Hint: recall that elements of S^n are bijective functions from the set $\{1, \ldots, n\}$ to itself. This problem asks you to count the number of such bijections that define symmetries of the regular n-gon.

[2] Hint: the cube has eight vertices so its group of symmetries can be understood as a subgroup of S_8. The question asks how many elements are in this subgroup.