Math 411: Honors Algebra I
Problem Set 5
due: October 9, 2019

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Exercise 1. Recall the dihedral group D_{2n} was defined as the subgroup of S_n comprised of those permutations that define symmetries of a regular n-gon whose vertices are labeled $1, 2, \ldots, n$ in cyclic order. Prove that $|D_{2n}| = 2n$, justifying Aluffi's notation.[1]

Proof. Label the vertices of the regular n-gone by $\{0, \ldots, n-1\}$ in a counter-clockwise fashion. This allows us to work "mod n." Elements $\sigma \in D_{2n} \subset S_n$ are automorphisms of the set $\{0, \ldots, n\}$. Any symmetry of the regular n-gon sends adjacent vertices (sharing an edge between them) to adjacent vertices. So if $\sigma(i) = j$, then there exactly two possible values for σ on the vertices that are adjacent to i: $\phi(i+1) = \phi(i) \pm 1$. Then necessarily $\phi(i-1)$ must be the other adjacent vertex to $\phi(i)$, and in this way the value of ϕ is determined on all other vertices. There are n choices of $\phi(i)$ and two choices of $\phi(i+1)$ so this tells us there are $2n$ elements of D_{2n} in total. □

Exercise 2. The five platonic solids are the tetrahedron, cube, octahedron, dodecahedron, and icosahedron.

(i) Draw a picture of each of these figures.
(ii) Referring to your picture as appropriate determine the orders of each group of symmetries.[2]

Proof. I'll leave the pictures to your imagination. To count the automorphism we're going to use the "rigidity" of each motion and argue that once you've positioned a single corner and edge in an allowable position the entire automorphism is determined.

Color your favorite vertex of the tetrahedron and one of the edges that meets that vertex. An automorphism of the tetrahedron is determined uniquely by the image of the colored vertex (four choices) together with the image of the colored edge (three choices, dependent on the first choice). So there are 12 automorphisms.

Similarly, an automorphism of the cube is determined by the image of a single vertex (8 choices) and an edge adjacent to it (3 choices). So there are 24 automorphisms.

An automorphism of the octahedron is determined by the image of a single vertex (6 choices) and an edge adjacent to it (4 choices), so there are 24 automorphisms.

An automorphism of the dodecahedron is determined by the image of a single vertex (20 choices) and an edge adjacent to it (3 choices), so there are 60 automorphisms.

An automorphism of the icosahedron is determined by the image of a single vertex (12 choices) and an edge adjacent to it (5 choices), so there are 60 automorphisms.

It's not a coincidence that the symmetry groups of the octahedron and dodecahedron have the same order: these are the same. To see this, note that an octahedron can be embedded in a cube (or visa-versa) where the vertices of the octahedron are the center points of the faces of the cube. So a rotation of one figure also defines a rotation of the embedded figure. There is a similar "duality" between the dodecahedron and the icosahedron, which is why these groups are also the same. The tetrahedron is self dual: the embedded platonic solid is another tetrahedron. □

Exercise 3. Prove that if $m, n \in \mathbb{N}$ are positive integers with $\gcd(m, n) = 1$, then $\mathbb{Z}/mn \cong \mathbb{Z}/m \times \mathbb{Z}/n$. (Hint: to prove that these groups are isomorphic, it suffices to define *one* bijective homomorphism. Why?)

Proof. It suffices to define a bijective homomorphism $\mathbb{Z}/mn \to \mathbb{Z}/m \times \mathbb{Z}/n$ because we proved on the last problem set that a bijective homomorphism defines an isomorphism in the category of groups (its inverse is automatically a homomorphism). Homomorphisms ϕ whose domain is the cyclic group of order mn correspond bijectively to choices of image for the generating element $[1]_{mn}$ and this generating element must have order dividing mn. In this case we want to find an element $\phi([1]_{mn}) \in \mathbb{Z}/m \times \mathbb{Z}/n$ whose order is exactly mn because this implies that our function will be injective (and hence also surjective, since the two groups have the same order). The element $([1]_m, [1]_n)$ has this property. □

[1] Hint: recall that elements of S^n are bijective functions from the set $\{1, \ldots, n\}$ to itself. This problem asks you to count the number of such bijections that define symmetries of the regular n-gon.

[2] Hint: the cube has eight vertices so its group of symmetries can be understood as a subgroup of S_8. The question asks how many elements are in this subgroup.

Exercise 4. Prove that if A and B are abelian groups, then $A \times B$ satisfies the universal property of the *coproduct* in the category Ab of abelian groups. Explain why the commutativity hypothesis is necessary.

Proof. The first step in showing $A \times B$ is the coproduct is to define group homomorphisms $\iota_A \colon A \to A \times B$ and $\iota_B \colon B \to A \times B$. These are defined by

$$\iota_A(a) = (a, e_B) \qquad \text{and} \qquad \iota_B(b) = (e_A, b).$$

The homomorphism property is easy to check.

Now given homomorphisms $\phi \colon A \to Z$ and $\psi \colon B \to Z$ where Z is an abelian group we may define $\zeta \colon A \times B \to Z$ by $\zeta(a, b) = \phi(a) + \psi(b)$. Note this definition has the property that $\zeta \circ \iota_A = \phi$ and $\zeta \circ \iota_B = \psi$. Normally ζ would not be a homomorphism but in an abelian group it is:

$$\zeta(a, b) + \zeta(a', b') = \phi(a) + \psi(b) + \phi(a') + \phi(b')$$
$$= \phi(a) + \phi(a') + \psi(b) + \psi(b') = \phi(a + a') + \psi(b + b') = \zeta(a + a', b + b').$$

□

Exercise 5.

(i) Fix an element g in a group G. Prove that the conjugation function $x \mapsto gxg^{-1}$ defines a homomorphism $\gamma_g \colon G \to G$.

(ii) Prove that the function $g \mapsto \gamma_g$ defines a homomorphism $\gamma \colon G \to \mathrm{Aut}(G)$. The image of this function is the subgroup of *inner automorphisms* of G.

(iii) Prove that γ is the zero homomorphism if and only if G is abelian.

Proof. For (i) we have $\gamma_g(x) \cdot \gamma_g(y) = gxg^{-1}gyg^{-1} = gxyg^{-1} = \gamma_g(xy)$.

For (ii) we must show that for all $g, h \in G$ $\gamma_{gh} = \gamma_g \circ \gamma_h$ since $\circ$ is the multiplication in the group $\mathrm{Aut}(G)$. These automorphisms agree if and only if they act the same way on elements $x \in G$. We can check $\gamma_{gh}(x) = ghx(gh)^{-1} = ghxh^{-1}g^{-1} = \gamma_g(\gamma_h(x)) = (\gamma_g \circ \gamma_h)(x)$.

For (iii), γ is the zero homomorphism iff for all $g \in G$ $\gamma_g = \mathrm{id}_G$, the identity element of $\mathrm{Aut}(G)$. This is the case iff $gxg^{-1} = x$ for all g and all x and this is the case iff $gx = xg$, i.e., iff G is abelian. □

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