Math 411: Honors Algebra I

Problem Set 4[1]

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Exercise 1. Let $\phi\colon G \to H$ be a group homomorphism whose underlying function is a bijection. Prove that ϕ defines an isomorphism between the groups G and H by defining its inverse homomorphism.

Exercise 2. Given groups G and H their **product** $G \times H$ is the group whose:

- elements are ordered pairs (g, h) with $g \in G$ and $h \in H$.
- multiplication law is defined componentwise:

$$(g_1, h_1) \cdot (g_2, h_2) := (g_1 g_2, h_1 h_2).$$

(i) What is the identity element for $G \times H$?
(ii) What is the inverse of $(g, h) \in G \times H$?
(iii) Define a pair of canonical "projection" homomorphisms $\pi_G\colon G \times H \to G$ and $\pi_H\colon G \times H \to H$.[2]
(iv) Let $\phi\colon K \to G$ and $\psi\colon K \to H$. Define a unique homomorphism $\zeta\colon K \to G \times H$ so that $\phi = \pi_G \circ \zeta$ and $\psi = \pi_H \circ \zeta$. (You should check that the function ζ that you define is a homomorphism but do not need to verify that it is unique with the commutativity property.)

Exercise 3. The **Klein four group**[3] is the group with four elements defined by the multiplication table which forms an "iterated battenberg cake":

	e	i	j	k
e	e	i	j	k
i	i	e	k	j
j	j	k	e	i
k	k	j	i	e

Define an isomorphism between the Klein four group and the product $\mathbb{Z}/2 \times \mathbb{Z}/2$ of the cyclic group with two elements[4] with itself.

Exercise 4.

(i) Are there any non-zero homomorphisms $\mathbb{Z}/n \to \mathbb{Z}$ for $n \in \mathbb{N}$? If so, define one. If not, explain why not.
(ii) How many homomorphisms are there from $\mathbb{Z}$ to $\mathbb{Z}/n$?

Exercise 5. Let $p, n \in \mathbb{N}$ with p prime. For which n does there exist a non-zero group homomorphism $\mathbb{Z}/p \to \mathbb{Z}/n$?

Exercise 6.

(i) Prove that there are no non-zero group homomorphisms between the Klein four group and $\mathbb{Z}/7$.
(ii) Define a non-zero homomorphism from the Klein four group to $\mathbb{Z}/4$.

Exercise 7. Let $S_n = \mathrm{Aut}_{\mathsf{Set}}(\{1, 2, \ldots, n\})$ denote the group of permutations of an n-element set.

- Define an element of order d in S_n for any $d < n$.
- For which n is S_n abelian? Give a proof or supply a counterexample for each $n \geq 1$.

Exercise 8. Is the product of cyclic groups cyclic? If so, give a proof. If not, find a counterexample.

Exercise 9*. Classify the homomorphisms from $\mathbb{Z}/n$ to $\mathbb{Z}/m$ for any $n, m \in \mathbb{N}$. In particular, how many homomorphisms are there in the set $\mathrm{Hom}_{\mathsf{Group}}(\mathbb{Z}/n, \mathbb{Z}/m)$? How many automorphisms are there of the group $\mathbb{Z}/n$?

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[1] Problems labelled n^* are optional (fun!) challenge exercises that will not be graded.
[2] Note that the multiplication on $G \times H$ is *defined* so that the projection functions π_G and π_H become group homomorphisms.
[3] It's also the name of an early aughts a cappella group; google "finite simple group of order 2."
[4] Aka the "finite simple group of order 2."