Math 411: Honors Algebra I
Problem Set 4[1]
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Exercise 1. Let $\phi\colon G \to H$ be a group homomorphism whose underlying function is a bijection. Prove that ϕ defines an isomorphism between the groups G and H by defining its inverse homomorphism.

Proof. Let $\phi^{-1}\colon H \to G$ denote the inverse function. We must show that $\forall h, k \in H$, $\phi^{-1}(h \cdot k) = \phi^{-1}(h) \cdot \phi^{-1}(k)$. Since ϕ^{-1} and ϕ are inverse functions $h = \phi(\phi^{-1}(h))$ and similarly for k. So

$$\begin{aligned}\phi^{-1}(h \cdot k) &= \phi^{-1}(\phi(\phi^{-1}(h)) \cdot \phi(\phi^{-1}(k))) = \phi^{-1}(\phi(\phi^{-1}(h) \cdot \phi^{-1}(k))) \\ &= (\phi^{-1} \circ \phi) \circ (\phi^{-1}(h) \cdot \phi^{-1}(k)) = \phi^{-1}(h) \cdot \phi^{-1}(k),\end{aligned}$$

the second equality by the homomorphism property for ϕ, the third by the definition of composition of functions, and the fourth because $\phi^{-1} \circ \phi$ is the identity. □

Exercise 2. Given groups G and H their **product** $G \times H$ is the group whose:

- elements are ordered pairs (g, h) with $g \in G$ and $h \in H$.
- multiplication law is defined componentwise:

$$(g_1, h_1) \cdot (g_2, h_2) := (g_1 g_2, h_1 h_2).$$

(i) What is the identity element for $G \times H$?
(ii) What is the inverse of $(g, h) \in G \times H$?
(iii) Define a pair of canonical "projection" homomorphisms $\pi_G\colon G \times H \to G$ and $\pi_H\colon G \times H \to H$.[2]
(iv) Let $\phi\colon K \to G$ and $\psi\colon K \to H$. Define a unique homomorphism $\zeta\colon K \to G \times H$ so that $\phi = \pi_G \circ \zeta$ and $\psi = \pi_H \circ \zeta$. (You should check that the function ζ that you define is a homomorphism but do not need to verify that it is unique with the commutativity property.)

Proof. (e, e) is the identity and (g^{-1}, h^{-1}) is the inverse of (g, h). The projection homomorphisms are defined by $(g, h) \mapsto g$ and $(g, h) \mapsto h$, respectively. The homomorphism ζ is defined by $\zeta(k) = (\phi(k), \psi(k))$. Note that

$$\zeta(k \cdot k') = (\phi(k \cdot k'), \psi(k \cdot k')) = (\phi(k) \cdot \phi(k'), \psi(k) \cdot \psi(k')) = (\phi(k), \psi(k)) \cdot (\phi(k'), \psi(k')) = \zeta(k) \cdot \zeta(k'),$$

since ϕ and ψ are homomorphisms. □

Exercise 3. The **Klein four group**[3] is the group with four elements defined by the multiplication table:

	e	i	j	k
e	e	i	j	k
i	i	e	k	j
j	j	k	e	i
k	k	j	i	e

Define an isomorphism between the Klein four group and the product $\mathbb{Z}/2 \times \mathbb{Z}/2$ of the cyclic group with two elements[4] with itself.

Proof. Let K_4 denote the Klein four group. Define $\phi\colon K_4 \to \mathbb{Z}/2 \times \mathbb{Z}/2$ by $e \mapsto (0, 0)$, $i \mapsto (1, 0)$, $j \mapsto (0, 1)$, and $k \mapsto (1, 1)$. You can check that this preserves the multiplication table. Since ϕ is a bijective homomorphism, Exercise 1 implies that it is an isomorphism. □

Exercise 4.

(i) Are there any non-zero homomorphisms $\mathbb{Z}/n \to \mathbb{Z}$ for $n \in \mathbb{N}$? If so, define one. If not, explain why not.
(ii) How many homomorphisms are there from $\mathbb{Z}$ to $\mathbb{Z}/n$?

[1] Problems labelled n^* are optional (fun!) challenge exercises that will not be graded.
[2] Note that the multiplication on $G \times H$ is *defined* so that the projection functions π_G and π_H become group homomorphisms.
[3] It's also the name of an a cappella group; google "finite simple group of order 2."
[4] Aka the "finite simple group of order 2."

Proof. (i) No because such homomorphisms correspond to elements in $\mathbb{Z}$ of order dividing n. $0 \in \mathbb{Z}$ is the only such element.

(ii) There are n such homomorphisms, since these correspond to elements of $\mathbb{Z}/n$, chosen to be the image of $1 \in \mathbb{Z}$, and any of the n choices is possible. □

Exercise 5. Let $p, n \in \mathbb{N}$ with p prime. For which n does there exist a non-zero group homomorphism $\mathbb{Z}/p \to \mathbb{Z}/n$?

Proof. There exists a non-zero homomorphism $\mathbb{Z}/p \to \mathbb{Z}/n$ iff $\mathbb{Z}/n$ has an element of order p (the image of $[1] \in \mathbb{Z}/p$), which is the case just when p divides n. □

Exercise 6.

(i) Prove that there are no non-zero group homomorphisms between the Klein four group and $\mathbb{Z}/7$.
(ii) Define a non-zero homomorphism from the Klein four group to $\mathbb{Z}/4$.

Proof. To define a non-zero homomorphism from K_4 to $\mathbb{Z}/7$ we need a non-zero image for one of the elements i, j, k of order 2. But $\mathbb{Z}/7$ contains no elements of order 2 so this can't be done.

A non-zero homomorphism from $K_4 \cong \mathbb{Z}/2 \times \mathbb{Z}/2$ to $\mathbb{Z}/4$ (using the isomorphism of Exercise 3) is given by the composite of either of the projections $\mathbb{Z}/2 \times \mathbb{Z}/2 \to \mathbb{Z}/2$ of Exercise 2 with any of the non-zero homomorphisms $\mathbb{Z}/2 \to \mathbb{Z}/4$ of Exercise 5. □

Exercise 7. Let $S_n = \mathrm{Aut}_{\mathsf{Set}}(\{1, 2, \ldots, n\})$ denote the group of permutations of an n-element set.

- Define an element of order d in S_n for any $d < n$.
- For which n is S_n abelian? Give a proof or supply a counterexample for each $n \geq 1$.

Proof. The d-cycle $(123 \cdots d)$ has order d.

$S_1 = \{e\}$ and $S_2 = \mathbb{Z}/2$ are both abelian. For $n > 2$, S_n contains (12) and (13) and these transpositions do not commute. □

Exercise 8. Is the product of cyclic groups cyclic? If so, give a proof. If not, find a counterexample.

Proof. No, not necessarily, since $\mathbb{Z}/2 \times \mathbb{Z}/2$ is the Klein four group which is not cyclic (all elements having order 2 and none having order 4). □

Exercise 9*. Classify the homomorphisms from $\mathbb{Z}/n$ to $\mathbb{Z}/m$ for any $n, m \in \mathbb{N}$. In particular, how many homomorphisms are there in the set $\mathrm{Hom}_{\mathsf{Group}}(\mathbb{Z}/n, \mathbb{Z}/m)$?

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