Math 411: Honors Algebra I
Problem Set 3
due: September 25, 2019

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Exercise 1. Prove that $(gh)^{-1} = h^{-1}g^{-1}$ for all elements g, h in a group G.

Exercise 2. The multiplication operation $- \cdot - \colon G \times G \to G$ for a group G can be specified by writing down its **multiplication table**: the columns and the rows are each labelled by the elements of G, and then the entry in row g and column h is the product $g \cdot h$.

G	e	g	h	$\cdots$	k
e	e	g	h	$\cdots$	k
g	g	g^2	$g \cdot h$	$\cdots$	$g \cdot k$
h	h	$h \cdot g$	h^2	$\cdots$	$h \cdot k$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$
k	k	$k \cdot g$	$k \cdot h$	$\cdots$	k^2

(i) Explain the pattern that you see in the first row and first column of the table (indexed by the identity element e).
(ii) Prove that every row and every column of the multiplication table of a group contains all elements of that group exactly once (like a Sudoko diagram).

Exercise 3. Let g be an element of finite order in a group G and let $n \in \mathbb{Z}$. Prove that $g^n = e$ if and only if the order of g divides n.

Exercise 4. Prove that every element in a finite group has finite order.

Exercise 5. The hour-hand group has twelve elements $\{1, 2, 3, \ldots, 12\}$ with addition defined by "addition of hours": e.g. $8 + 6 = 2$ because six hours after 8 o'clock is 2 o'clock. Prove that this defines a group by specifying an identity element, justifying associativity (it's okay to wave your hands on this point), and calculating the inverse of each elements. Have we encountered this group by another name?

Exercise 6.
(i) Sketch a proof that the unit circle centered at the origin in $\mathbb{R} \times \mathbb{R}$ defines a group with identity element $(1, 0)$ and with addition defined by "adding angles," where the angle of a unit vector is measured counterclockwise starting from the positive x axis.
(ii) Is this group abelian?
(iii) How does this group relate to the group $(\mathbb{C}^\times, \times, 1)$?[1]

Exercise 7. Let C be any category and fix an object $A \in \mathsf{C}$. Let $\mathrm{Aut}_{\mathsf{C}}(A)$ be the set of **automorphisms** of A in C:

$$\mathrm{Aut}_{\mathsf{C}}(A) := \{f \colon A \to A \in \mathsf{C} \mid f \text{ is an isomorphism}\}.$$

(i) Prove that $\mathrm{Aut}_{\mathsf{C}}(A)$ is a group with composition as its multiplication operation. What is the identity element?
(ii) Explain why the set $\mathrm{End}_{\mathsf{C}}(A)$ of **endomorphisms** of A in C defined by

$$\mathrm{End}_{\mathsf{C}}(A) := \{f \colon A \to A \in \mathsf{C}\}$$

is not a group under composition.

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[1] We don't have the language to describe this relationship precisely yet, but use your own words to describe your intuition.