Math 411: Honors Algebra I
Problem Set 2
due: September 18, 2019

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Exercise 1. Fix a set A. The aim of this exercise is establish a bijection between

- the set of equivalence relations on A,
- the set of partitions on A, and
- the set of surjective functions with domain A up to an isomorphism between the codomains of two such surjective functions.[1]

To that end:

(i) Consider an equivalence relation $\sim$ on A and prove that the set of equivalence classes define a partition of A. For $x, y \in A$, we write $[x] = [y]$ and say that x and y belong to the same **equivalence class** iff $x \sim y$.
(ii) Consider a partition $A = \coprod_{i \in I} A_i$ of A into a disjoint union of non-empty subsets and define a surjective function $\pi \colon A \to I$ in such a way that A_i is recovered as the **fiber** of π over $i \in I$.
(iii) Consider a surjective function $f \colon A \to B$ and show that the relation $\sim$ on A defined by the rule $x \sim y$ iff $f(x) = f(y)$ is an equivalence relation.
(iv) Briefly observe that if you start with an equivalence relation $\sim$ and go through the constructions of (i), (ii), and (iii) in sequence you recover the equivalence relation you started with.

Exercise 2. Describe as explicitly as you can all of the terms in the canonical decomposition of the function $\mathbb{R} \to \mathbb{C}$ defined by $x \mapsto e^{2\pi i x}$.

Exercise 3. Let C be a category. Define a category C^{op}, called the *opposite category* of C as follows:

- the objects of C^{op} are the same as the objects of C
- For each morphism $f \colon x \to y$ in C there is a corresponding morphism $f^{\mathrm{op}} \colon y \to x$ in C^{op}.

Complete this definition by solving the following:

(i) Define identity morphisms and the composition of morphisms in C^{op}.
(ii) Prove that composition is associative and unital.

Exercise 4. A morphism $i \colon A \to B$ in a category C admits a *left inverse* or a *retraction* if there exists a morphism $r \colon B \to A$ so that $r \circ i = 1_A$. In this case, i is called a *split monomorphism*.

(i) Prove that split monomorphisms are in fact monomorphisms.
(ii) State the dual definition of a *split epimorphism* in any category.[2]
(iii) Prove that any morphism that is both a split monomorphism and an epimorphism is an isomorphism. Conclude by duality that any morphisms that is both a split epimorphism and a monomorphism is an isomorphism.

Exercise 5. Consider a commutative triangle of morphisms in any category C

$$\begin{tikzcd} A \arrow[rr, "h"] \arrow[dr, "f"'] & & C \\ & B \arrow[ur, "g"'] & \end{tikzcd}$$

(i) Prove that if f and g are monomorphisms so is their composite h.
(ii) Prove that if h is a monomorphism then so is f.
(iii) Find an example to show that it is possible for h to be a monomorphism while g is not.

Exercise 6. Prove that the collection of isomorphisms in any category C define a *subcategory* of C, with the same objects and with composition and identities defined by restricting these operations from C. This category is called the *maximal subgroupoid* or sometimes the *groupoid core* of C.

[1] The intention of the phrase "up to an isomorphism between the codomains of two such surjective functions" is that the names of the elements in the codomain set should not matter. If you're not sure what this means, feel free to ignore it, and you'll probably be fine.

[2] Here's how to "state a dual definition." Write out the definition of split monomorphism in C^{op}. Then reinterpret that definition in the category C. This will give the definition of a split epimorphism in C.

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