Math 411: Honors Algebra I
Problem Set 2
due: September 18, 2019

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Exercise 1. Fix a set A. The aim of this exercise is establish a bijection between

- the set of equivalence relations on A,
- the set of partitions on A, and
- the set of surjective functions with domain A up to an isomorphism between the codomains of two such surjective functions.[1]

To that end:

(i) Consider an equivalence relation $\sim$ on A and prove that the set of equivalence classes define a partition of A. For $x, y \in A$, we write $[x] = [y]$ and say that x and y belong to the same **equivalence class** iff $x \sim y$.

(ii) Consider a partition $A = \coprod_{i \in I} A_i$ of A into a disjoint union of non-empty subsets and define a surjective function $\pi \colon A \to I$ in such a way that A_i is recovered as the **fiber** of π over $i \in I$.

(iii) Consider a surjective function $f \colon A \to B$ and show that the relation $\sim$ on A defined by the rule $x \sim y$ iff $f(x) = f(y)$ is an equivalence relation.

(iv) Briefly observe that if you start with an equivalence relation $\sim$ and go through the constructions of (i), (ii), and (iii) in sequence you recover the equivalence relation you started with.

Proof. (i) Every x is a member of its own equivalence class $[x]$ so $\cup_{x \in A}[x] = A$. To see that the equivalence classes partition A we must show that if $z \in [x] \cap [y]$ then $[x] = [y]$. But if $z \in [x] \cap [y]$ then by symmetry $x \sim z$ and $z \sim y$ so by transitivity $x \sim y$. Thus $x \in [y]$ and $y \in [x]$. By transitivity any $w \in [x]$ has $w \sim x$ and $x \sim y$ so $w \in [y]$. Thus $[x] \subset [y]$ and exchanging x and y we conclude that $[x] = [y]$.

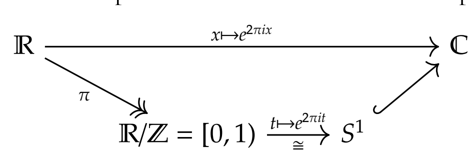
(ii) Define $\pi \colon \coprod_{i \in I} A_i \to A$ by the rule that $\pi(x) = i$ for all $x \in A_i$. Note the fiber $\pi^{-1}(i) = A_i$. Since each $A_i \neq \emptyset$, this proves that π is surjective.

(iii) We must show that the relation defined by $x \sim y$ iff $f(x) = f(y)$ is reflexive, surjective, and transitive. This follows from the fact that equality has these properties and f is a well-defined function.

(iv) Starting with an equivalence relation $\sim$ on A, the equivalence classes partition A. Thus, the function constructed in (ii) has the equivalence classes as fibers. Thus y and z are related by the relation encoded by this function iff they are in the same fiber which is the case iff $y \sim z$ in the original equivalence relation. □

Exercise 2. Describe as explicitly as you can all of the terms in the canonical decomposition of the function $\mathbb{R} \to \mathbb{C}$ defined by $x \mapsto e^{2\pi i x}$.

Proof. The image is the unit circle S^1 in $\mathbb{C}$: the set of complex numbers of norm 1. The quotient is $\mathbb{R}_{/\sim}$ where $x \sim y$ if and only if $x - y$ is an integer. These equivalence classes can be represented by elements in the half-open interval $[0, 1)$. The bijection $[0, 1) \cong S^1$ sends $0 \leq t < 1$ to the complex number $e^{2\pi i t}$. This defines a parametrization of the unit circle.



□

Exercise 3. Let C be a category. Define a category C^{op}, called the *opposite category* of C as follows:

- the objects of C^{op} are the same as the objects of C
- For each morphism $f \colon x \to y$ in C there is a corresponding morphism $f^{\mathrm{op}} \colon y \to x$ in C^{op}.

Complete this definition by solving the following:

(i) Define identity morphisms and the composition of morphisms in C^{op}.

[1] The intention of the phrase "up to an isomorphism between the codomains of two such surjective functions" is that the names of the elements in the codomain set should not matter. If you're not sure what this means, feel free to ignore it, and you'll probably be fine.

(ii) Prove that composition is associative and unital.

Proof. The identity morphism at x in C^{op} is 1_x^{op}, the formal opposite of the specified identity morphism at x in C. Composition of $f^{\mathrm{op}} : y \to x$ with $g^{\mathrm{op}} \colon z \to y$ in C^{op} is defined by the rule

$$f^{\mathrm{op}} \circ g^{\mathrm{op}} := (g \circ f)^{\mathrm{op}},$$

the opposite of the composite $g \circ f$ of the morphisms $f \colon x \to y$ and $g \colon y \to z$ in C.

To prove composition is associative consider morphisms in C^{op}:

$$w \xrightarrow{h^{\mathrm{op}}} z \xrightarrow{g^{\mathrm{op}}} y \xrightarrow{f^{\mathrm{op}}} x$$

These correspond to morphisms

$$w \xleftarrow{h} z \xleftarrow{g} y \xleftarrow{f} x$$

in C. Now by the definition of composition in C^{op} we have

$$f^{\mathrm{op}} \circ (g^{\mathrm{op}} \circ h^{\mathrm{op}}) = f^{\mathrm{op}} \circ (h \circ g)^{\mathrm{op}} = ((h \circ g) \circ f)^{\mathrm{op}} = (h \circ (g \circ f))^{\mathrm{op}} = (g \circ f)^{\mathrm{op}} \circ h^{\mathrm{op}} = (f^{\mathrm{op}} \circ g^{\mathrm{op}}) \circ h^{\mathrm{op}}$$

where the middle equality holds by associativity of composition in C.

Similarly by the definition of composition and the identity in C^{op} for any $f^{\mathrm{op}} \colon y \to x$ we have

$$f^{\mathrm{op}} \circ 1_y^{\mathrm{op}} = (1_y \circ f)^{\mathrm{op}} = f^{\mathrm{op}} = (f \circ 1_x)^{\mathrm{op}} = 1_x^{\mathrm{op}} \circ f^{\mathrm{op}},$$

where the middle two equalities use the unital property of composition with identities 1_x and 1_y in C. $\square$

Exercise 4. A morphism $i \colon A \to B$ in a category C admits a *left inverse* or a *retraction* if there exists a morphism $r \colon B \to A$ so that $r \circ i = 1_A$. In this case, i is called a *split monomorphism*.

(i) Prove that split monomorphisms are in fact monomorphisms.
(ii) State the dual definition of a *split epimorphism* in any category.
(iii) Prove that any morphism that is both a split monomorphism and an epimorphism is an isomorphism. Conclude by duality that any morphisms that is both a split epimorphism and a monomorphism is an isomorphism.

Proof. For (i) consider a pair of morphisms $f, g \colon X \to A$ and suppose i has a left inverse r. If $if = ig$ then $rif = rig$ so $f = g$ since $ri = 1_A$.

For (ii) a **split epimorphism** is a morphism $r \colon B \to A$ that admits a right inverse s, meaning so that there exists a morphism $s \colon A \to B$ so that $rs = 1_A$.

For (iii) suppose i is a split monomorphism and also an epimorphism. We know that $ri = 1_A$. We want to show that $ir = 1_B$ so that i is an isomorphism. Since $ri = 1_A$ we know that $iri = i$. By the epimorphism property we may cancel i from the right and conclude that $ir = 1_B$. $\square$

Exercise 5. Consider a commutative triangle of morphisms in any category C

$$\begin{array}{ccc} A & \xrightarrow{\quad h \quad} & C \\ & {\scriptstyle f}\searrow \quad \nearrow{\scriptstyle g} & \\ & B & \end{array}$$

(i) Prove that if f and g are monomorphisms so is their composite h.
(ii) Prove that if h is a monomorphism then so is f.
(iii) Find an example to show that it is possible for h to be a monomorphism while g is not.

Proof. For (i), consider a pair of morphisms $x, y \colon X \to A$ and suppose $hx = hy$. Then $gfx = gfy$. Since g is mono, then $fx = fy$, but since f is mono, then $x = y$.

For (ii), consider a pair of morphisms $xy \colon X \to A$ and suppose $fx = fy$. Then $gfx = gfy$ so $hx = hy$. Since h is mono then $x = y$.

For (iii) Let $A = 1$ a singleton set, $B = 2 = \{\top, \bot\}$, and $C = \mathbb{Z}$. Define h to be the function whose image is $0 \in \mathbb{Z}$. Define f to be the function whose image is $\top \in 2$. Note that both of these are monomorphisms. Now define g to be the function that is constant at 0, i.e., so that $g(\top) = g(\bot) = 0$. Then the triangle commutes but g is not a monomorphism since it's not injective. $\square$

Exercise 6. Prove that the collection of isomorphisms in any category C define a *subcategory* of C, with the same objects and with composition and identities defined by restricting these operations from C. This category is called the *maximal subgroupoid* or sometimes the *groupoid core* of C.

Proof. Write G for the subcategory of C with the same objects but whose morphisms are only the isomorphisms in C. To show that G is indeed a subcategory first observe that the identities in C are all isomorphisms: the inverse of $1_A \colon A \to A$ is just $1_A \colon A \to A$. We must also verify that if $f \colon A \to B$ and $g \colon B \to C$ are isomorphisms in C and so lie in G then their composite $g \circ f$ in C is again an isomorphism, and so lies in G. But the inverse to $g \circ f$ is $f^{-1} \circ g^{-1}$, the composite of the inverses of f and g. Now associativity and unitality of composition in G are inherited from the analogous properties of G so there is nothing more to check. □

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