Math 411: Honors Algebra I
Problem Set 11
due: December 4, 2019

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Exercise 1. Let R be a commutative ring.

(i) Prove that if $I \subset R$ is an ideal and $1 \in I$ then $I = R$.[1]
(ii) Prove that the only ideals in a field are the zero ideal $I = \{0\}$ or the ideal containing every element.
(iii) Prove that every ring homomorphism $\Bbbk \to K$ between fields is an injection by arguing that the kernel must be zero.[2]

Exercise 2.

(i) The subset $\mathbb{Z} \subset \mathbb{Z}[x]$ of constant polynomials is a subgroup under addition but is not an ideal. Nonetheless we can form the quotient group $\mathbb{Z}[x]/\mathbb{Z}$. The cosets $f(x) + \mathbb{Z}$ are represented by polynomials $f(x) = a_1 x + \cdots + a_n x^n$ with no constant term. Prove that the formula
$$(f(x) + \mathbb{Z}) \cdot (g(x) + \mathbb{Z}) := (f(x) \cdot g(x)) + \mathbb{Z}$$
is not well-defined by finding explicit polynomials in the same cosets that have products that do not belong to the same cosets.
(ii) Now let I be an ideal in a commutative ring R. Prove that the formula
$$(a + I) \cdot (b + I) := (a \cdot b) + I$$
defines a well-defined operation on the abelian group R/I of cosets of I. This proves that R/I inherits the structure of a ring in a unique way making the canonical projection $\pi \colon R \twoheadrightarrow R/I$ into a ring homomorphism.

Exercise 3.

(i) Let $a \in R$ be any element in a commutative ring. Prove that $(a) = \{a \cdot r \mid r \in R\}$ is an ideal. This is called the **principal ideal** generated by a.
(ii) If u is a unit prove that $(u) = (1) = R$.
(iii) If $a = ub$ and u is a unit prove that $(a) = (b)$.
(iv) If $(a) = (b)$ and R is an integral domain prove that there exists a unit u so that $a = ub$.

Exercise 4. Consider the principal ideal $(x) \subset \mathbb{Z}[x]$. Compute the quotient ring $\mathbb{Z}[x]/(x)$.

Exercise 5. Consider the principal ideal $(x^2 + 1) \subset \mathbb{R}[x]$.

(i) Define a ring homomorphism $\phi \colon \mathbb{R}[x] \to \mathbb{C}$ that sends $\mathbb{R} \subset \mathbb{R}[x]$ to $\mathbb{R} \subset \mathbb{C}$ and sends x to $i = \sqrt{-1}$.[3]
(ii) Prove that the homomorphism ϕ is surjective.
(iii) Prove that $(x^2 + 1)$ is contained in the kernel of ϕ. In fact $(x^2 + 1)$ is a **maximal ideal** so this implies that $\ker \phi = (x^2 + 1)$.
(iv) Apply a theorem from class to prove that the quotient ring $\mathbb{R}[x]/(x^2 + 1)$ is isomorphic to $\mathbb{C}$.

Exercise 6. Consider the principal ideal $(x^2 + x + 1) \in (\mathbb{Z}/2)[x]$.

(i) Prove that any polynomial $f(x) \in (\mathbb{Z}/2)[x]$ with coefficients in $\mathbb{Z}/2$ is equivalent modulo $(x^2 + x + 1)$ to a polynomial of the form $a + bx$ with $a, b \in \mathbb{Z}/2$.
(ii) How many elements are in the quotient ring $(\mathbb{Z}/2)[x]/(x^2 + x + 1)$?
(iii) Prove that the quotient ring $(\mathbb{Z}/2)[x]/(x^2 + x + 1)$ is a field by writing down the multiplication table and verifying that every non-zero element is a unit.
(iv) Explain what this has to do with question 5 on Problem Set 10.

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[1] This is why ideals $I \subset R$ do not define subrings. Most ideals do not contain 1.
[2] An injective ring homomorphism $\Bbbk \hookrightarrow K$ defines a **field extension** of $\Bbbk$.
[3] Pick one square root of -1.