

Math 411: Honors Algebra I
Problem Set 10
due: November 20, 2019

Emily Riehl

Exercise 1. Let R be a ring. Prove that

- (i) $u \in R$ is a left unit iff left multiplication by u defines a surjective function $u \cdot -: R \rightarrow R$.
- (ii) If u is a left unit then right multiplication $- \cdot u: R \rightarrow R$ is also injective (units are not zero divisors).
- (iii) The (two-sided) units form a group R^\times under multiplication.

Exercise 2. Prove that if $0 = 1$ in a ring then the ring is the zero ring.

Exercise 3. A ring R is **Boolean** if $a^2 = a$ for every $a \in R$.

- (i) For any set X prove that the set of subsets of X becomes a Boolean ring with

$$\begin{aligned} A + B &:= A \cup B - A \cap B && \text{the symmetric difference} \\ A \cdot B &:= A \cap B && \text{the intersection} \end{aligned}$$

by verifying the ring axioms.

- (ii) A ring has **characteristic two** if $a + a = 0$ for every $a \in R$. Prove that every Boolean ring has characteristic two.¹
- (iii) Using (ii) prove that every Boolean ring is commutative.²

Exercise 4. Prove or find a counter-example. If R is a ring and $a, b \in R$ are zero divisors then $a + b$ is a zero divisor.

Exercise 5. Construct a field with 4 elements. The underlying abelian group is $\mathbb{Z}/2 \times \mathbb{Z}/2$ with $(0, 0)$ as the zero element and $(1, 0)$ as the multiplicative identity. The question is to define the multiplication table so that you get a *field* and not just a ring.

Exercise 6. Let R be a commutative integral domain and consider the polynomial ring $R[x]$. Prove that the only units in $R[x]$ are the constant polynomials $f(x) = a_0$ where a_0 is a unit in R and explain what goes wrong in your proof if R is *not* an integral domain.

Exercise 7. Let R be a commutative ring and consider the ring of power series $R[[x]]$. Prove that $1 - x$ is a unit in $R[[x]]$ by computing its inverse.³

DEPT. OF MATHEMATICS, JOHNS HOPKINS UNIV., 3400 N CHARLES ST, BALTIMORE, MD 21218
E-mail address: eriehl@math.jhu.edu

¹Hint: consider $(a + a)^2$.

²Hint: consider $(a + b)^2$.

³More generally, a power series $a_0 + a_1x + a_2x^2 + \dots$ is a unit in $R[[x]]$ if and only if a_0 is a unit in R .