Math 411: Honors Algebra I
Problem Set 1
due: September 11, 2019

Emily Riehl

Exercise 1. For each of the following functions determine whether they are injective, surjective, and bijective and construct a left, right, or two-sided inverse whenever these exist.

(i) The function $f\colon \mathbb{N} \to \mathbb{N}$ defined by $f(x) = x^2$.
(ii) The function $f\colon \mathbb{R} \to \mathbb{R}$ defined by $f(x) = x^2$.
(iii) The function $f\colon \mathbb{Z} \times \mathbb{Z}_{>0} \to \mathbb{Q}$ defined by $(a, b) \mapsto \frac{a}{b}$; here $\mathbb{Z}_{>0}$ denotes the set of positive integers.
(iv) The function $\pi_B\colon A \times B \to B$ defined by $\pi_B(a, b) = b$.
(v) The function $\pi\colon A \to A_{/\sim}$ associated to a non-identity equivalence relation $\sim$ on A defined by $\pi(a) = [a]_\sim$.[1]
(vi) The function from the set of subsets of $\mathbb{N}$ to the set of countably-infinite binary sequences $(0, 1, 0, 0, 0, 1, \ldots)$ that sends a subset $S \subset \mathbb{N}$ to the sequence that has a 1 in the nth coordinate if and only if $n \in S$.
(vii) Writing $10 = \{0, \ldots, 9\}$ and $[0, 1] = \{x \in \mathbb{R} \mid 0 \leq x \leq 1\}$, the function $f\colon 10^{\mathbb{N}} \to [0, 1]$ that sends a sequence of decimal digits $(x_n)_{n\in\mathbb{N}}$ to the real number $0.x_1x_2x_3 \ldots$.

Exercise 2. Write $2 = \{\bot, \top\}$ or $2 = \{0, 1\}$ for the set with two elements. (Your choice which notation you want to use for its elements.) Let $P(A)$ denote the set of all subsets of a set A and let 2^A denote the set of functions from A to 2.

(i) Let $S \subset A$. Define a function $\chi_S\colon A \to 2$ that is related to S in some natural way.
(ii) Use part (i) to define a natural function $\chi\colon P(A) \to 2^A$.
(iii) Show that the function $\chi\colon P(A) \to 2^A$ that you've defined in part (ii) is a bijection.[2]

Exercise 3. Write B^A for the set of functions from A to B.

(i) Express the cardinality of B^A in terms of the cardinalities of A and B, assuming these are finite sets.
(ii) Express the cardinality of the powerset 2^A of A in terms of the cardinality of A, assuming that A is a finite set.
(iii) Explain why (ii) is a special case of (i).

Exercise 4. How many functions are there from a set of n elements to itself? How many bijections are there between a set with n elements and itself?

Exercise 5.

(i) Let $f\colon A \to B$ be a function that has a left inverse $g\colon B \to A$ and also a right inverse $h\colon B \to A$. Prove that $h = g$.
(ii) Prove that $f\colon A \to B$ is a bijection if and only if f is an isomorphism without using (i).

Exercise 6.

(i) For any function $f\colon A \to B$ define an explicit isomorphism between A and the graph $\Gamma_f \subset A \times B$.
(ii) Define a natural function $\Gamma_f \to B$. Is it necessarily injective? Is it necessarily surjective?

Exercise 7.

(i) For any non-empty set A, define an isomorphism between the set $A \times A$ and the set A^2 of functions from the set with two elements to the set A.
(ii) For any non-empty set A and positive natural number n, define an isomorphism between the n-fold cartesian product $\prod_n A \coloneqq A \times \cdots \times A$ and the set A^n of functions from the set with n elements to A.[3]

Exercise 8. Explain in your own words why all sets with three elements are isomorphic and speculate why I don't care what we call the elements of a 3-element set.

Exercise 9. Suppose $p\colon A \to B$ is a surjective function. Explain how the fibers of p define an equivalence relation on A and prove that B is isomorphic to the set of equivalence classes for this equivalence relation.

[1] If the equivalence relation is defined by $x \sim y$ iff $x = y$, then π is the identity function. Please exclude this case.

[2] If the function you've defined in part (ii) is not a bijection, you might need to redefine the function χ.

[3] In fact, it i for any set A and any set I (possible empty and possibly infinite) it is also the case that the product $\prod_I A$ is isomorphic to the set of functions A^I. The proof is by the same argument, but requires somewhat more complicated notation. In fact there is a sense in which the product $\prod_I A$ of the indexed family of sets $(A)_{i\in I}$ is *defined* to be the set of functions A^I.

Dept. of Mathematics, Johns Hopkins Univ., 3400 N Charles St, Baltimore, MD 21218
Email address: `eriehl@math.jhu.edu`