Math 411: Honors Algebra I
Problem Set 9
due: November 15, 2017

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Exercise 1. Prove that if $0 = 1$ in a ring then the ring is the zero ring.

Exercise 2. A ring R is **Boolean** if $a^2 = a$ for every $a \in R$. For any set X prove that the set of subsets of X becomes a Boolean ring with

$$A + B := A \cup B - A \cap B \qquad \text{the symmetric difference}$$
$$A \cdot B := A \cap B \qquad \text{the intersection}$$

by verifying the ring axioms.

Exercise 3. Prove or find a counter-example. If R is a ring and $a, b \in R$ are zero divisors then $a + b$ is a zero divisor.

Exercise 4. Construct a field with 4 elements. The underlying abelian group is $\mathbb{Z}/2 \times \mathbb{Z}/2$ with $(0, 0)$ as the zero element and $(1, 0)$ as the multiplicative identity. The question is to define the multiplication table so that you get a *field* and not just a ring.

Exercise 5. Let R be a commutative integral domain and consider the polynomial ring $R[x]$. Prove that the only units in $R[x]$ are the constant polynomials $f(x) = a_0$ where a_0 is a unit in R and explain what goes wrong in your proof if R is *not* an integral domain.

Exercise 6. Let R be a commutative ring and consider the ring of power series $R[\![x]\!]$. Prove that $1 - x$ is a unit in $R[\![x]\!]$ by computing its inverse.[1]

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[1] More generally, a power series $a_0 + a_1 x + a_2 x^2 + \cdots$ is a unit in $R[\![x]\!]$ if and only if a_0 is a unit in R.