Math 411: Honors Algebra I
Problem Set 2
due: September 20, 2017

Emily Riehl

Exercise 1. Describe as explicitly as you can all of the terms in the canonical decomposition of the function $\mathbb{R} \to \mathbb{C}$ defined by $x \mapsto e^{2\pi i x}$.

Exercise 2. Let C be a category. Define a category C^{op}, called the *opposite category* of C as follows:

- the objects of C^{op} are the same as the objects of C
- For each morphism $f\colon x \to y$ in C there is a corresponding morphism $f^{\mathrm{op}}\colon y \to x$ in C^{op}.

Complete this definition by solving the following:

(i) Define identity morphisms and the composition of morphisms in C^{op}.
(ii) Prove that composition is associative and unital.

Exercise 3. Let c be an object in the category C. How does $(c/\mathsf{C})^{\mathrm{op}}$ relate to $(\mathsf{C}^{\mathrm{op}})/c$?

Exercise 4. A morphism $i\colon A \to B$ in a category C admits a *left inverse* or a *retraction* if there exists a morphism $r\colon B \to A$ so that $r \circ i = 1_A$. In this case, i is called a *split monomorphism*.

(i) Prove that split monomorphisms are in fact monomorphisms.
(ii) State the dual definition of a *split epimorphism* in any category.
(iii) Prove that any morphism that is both a split monomorphism and an epimorphism is an isomorphism. Conclude by duality that any morphisms that is both a split epimorphism and a monomorphism is an isomorphism.

Exercise 5. Consider a commutative triangle of morphisms in any category C

$$\begin{tikzcd} A \arrow[rr, "h"] \arrow[dr, "f"'] & & C \\ & B \arrow[ur, "g"'] & \end{tikzcd}$$

(i) Prove that if f and g are monomorphisms so is their composite h.
(ii) Prove that if h is a monomorphism then so is f.
(iii) Find an example to show that it is possible for h to be a monomorphism while g is not.

Exercise 6. Prove that the collection of isomorphisms in any category C define a *subcategory* of C, with the same objects and with composition and identities defined by restricting these operations from C. This category is called the *maximal subgroupoid* or sometimes the *groupoid core* of C.

Dept. of Mathematics, Johns Hopkins Univ., 3400 N Charles St, Baltimore, MD 21218
E-mail address: `eriehl@math.jhu.edu`