Math 411: Honors Algebra I

Problem Set 1

due: September 13, 2017

Emily Riehl

Exercise 1. For each of the following functions determine whether they are injective, surjective, and bijective and construct a left, right, or two-sided inverse whenever these exist.

(i) The function $f \colon \mathbb{N} \to \mathbb{N}$ defined by $f(x) = x^2$.
(ii) The function $f \colon \mathbb{R} \to \mathbb{R}$ defined by $f(x) = x^2$.
(iii) The function $f \colon \mathbb{Z} \times \mathbb{Z}_{>0} \to \mathbb{Q}$ defined by $(a, b) \mapsto \frac{a}{b}$; here $\mathbb{Z}_{>0}$ denotes the set of positive integers.
(iv) The function $\pi_B \colon A \times B \to B$ defined by $\pi_B(a, b) = b$.
(v) The function $\pi \colon A \to A_{/\sim}$ associated to an equivalence relation $\sim$ on A defined by $\pi(a) = [a]_\sim$.
(vi) The function from the set of subsets of $\mathbb{N}$ to the set of countably-infinite binary sequences $(0, 1, 0, 0, 0, 1, \ldots)$ that sends a subset $S \in \mathbb{N}$ to the sequence that has a 1 in the nth coordinate if and only if $n \in S$.
(vii) Writing $10 = \{0, \ldots, 9\}$ and $[0, 1] = \{x \in \mathbb{R} \mid 0 \leq x \leq 1\}$, the function $f \colon 10^{\mathbb{N}} \to [0, 1]$ that sends a sequence of decimal digits $(x_n)_{n \in \mathbb{N}}$ to the real number $0.x_1 x_2 x_3 \ldots$.

Exercise 2. Write B^A for the set of functions from A to B.

(i) Express the cardinality of B^A in terms of the cardinalities of A and B, assuming these are finite sets.
(ii) Express the cardinality of the powerset 2^A of A in terms of the cardinality of A, assuming that A is a finite set.
(iii) Explain why (ii) is a special case of (i).

Exercise 3. How many functions are there from a set of n elements to itself? How many bijections are there between a set with n elements and itself?

Exercise 4.

(i) Let $f \colon A \to B$ be a function that has a left inverse $g \colon B \to A$ and also a right inverse $h \colon B \to A$. Prove that $h = g$.
(ii) Prove that $f \colon A \to B$ is a bijection if and only if f is an isomorphism without using (i).

Exercise 5.

(i) For any function $f \colon A \to B$ define an explicit isomorphism between A and the graph $\Gamma_f \subset A \times B$.
(ii) Define a natural function $\Gamma_f \to B$. Is it necessarily injective? Is it necessarily surjective?

Exercise 6.

(i) Define an isomorphism between any set A and the set A^1 of functions from a singleton set to A.
(ii) Argue that a function $f \colon A \to B$ is uniquely determined by its collection of composites with functions $1 \to A$.

Exercise 7. Explain in your own words why all sets with three elements are isomorphic and speculate why I don't care what we call the elements of a 3-element set.

Exercise 8. Suppose $p\colon A \to B$ is a surjective function. Explain how the fibers of p define an equivalence relation on A and prove that B is isomorphic to the set of equivalence classes for this equivalence relation.

Dept. of Mathematics, Johns Hopkins Univ., 3400 N Charles St, Baltimore, MD 21218
E-mail address: `eriehl@math.jhu.edu`