Math 411: Honors Algebra I
Practice Midterm
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True or False

(1 point) Indicate whether each of the following statements is true or false (circle one).
(2 points) For each true statement, give a short (one to two sentence) justification, explaining the essential reason why the assertion is correct; for each false statement, provide either a counter-example or, in the case where a counter-example would not make sense, a short disproof.

1. (T or F) A bijective function $f\colon A \to B$ has a unique inverse function.

True. To say that f is bijective means for each $b \in B$ there exists a unique $a \in A$ so that $f(a) = b$. Define the inverse function $f^{-1}\colon B \to A$ by $f^{-1}(b) = a$. This is the only function that is the inverse to f in the sense that $f^{-1} \circ f = 1_A$ and $f \circ f^{-1} = 1_B$ so it is unique.

2. (T or F) A finite group can have elements of infinite order.

False. If $g \in G$ has infinite order then that means that the elements $g^n \in G$ are all distinct for each $n \in \mathbb{Z}$. This clearly cannot happen in a finite group.

3. (T or F) The symmetric group S_4 has 256 elements.

False. The elements of the symmetric group are *automorphisms* of the set $\{1, 2, 3, 4\}$. There are 24=4! ways to define a bijection from this set to itself, so $|S_4| = 24$.

4. (T or F) The Klein four group is abelian.

True. The Klein four group is the product $\mathbb{Z}/2 \times \mathbb{Z}/2$ of two abelian groups and thus abelian. Another way to see this is to draw the multiplication table and observe that it is symmetric across the diagonal axes.

5. (T or F) The function $-+10\colon \mathbb{Z} \to \mathbb{Z}$, adding 10 to any integer, defines a group homomorphism.

False. A group homomorphism must preserve the identity and since $0 + 10 = 10 \neq 0$ this is not the case for this function.

6. (T or F) There exists a non-zero homomorphism $\mathbb{Z} \to S_6$.

True. In fact there are $719 = 6! - 1$ non-zero homomorphisms $\mathbb{Z} \to S_6$. The group $\mathbb{Z}$ is the free group on one generator which means that to define a group homomorphism $\mathbb{Z} \to G$, where G is any group, it suffices to choose *any* element $g \in G$. The homomorphism is then defined by $n \mapsto g^n$. So in the case $G = S_6$ we may choose any non-identity permutation $\sigma \in S_6$ and get a non-zero homomorphism that sends $1 \in \mathbb{Z}$ to $\sigma \in S_6$.

7. (T or F) Every subgroup of $\mathbb{Z}$ is cyclic.

True. A subgroup of $\mathbb{Z}$ is a subset $H \subset \mathbb{Z}$ that is closed under addition, contains zero, and contains the inverse of any element in that subgroup. Let $d \in H$ be the smallest non-zero element. (If this does not exist then $H = e$ is the trivial subgroup which is cyclic of order 1.) Then clearly $d\mathbb{Z} \subset H$. In fact, we claim that $H \subset d\mathbb{Z}$ because otherwise you get a contradiction by division (we proved this in class). Since $d\mathbb{Z} \cong \mathbb{Z}$ is cyclic, this is true.

8. (T or F) Let $r_1 \in D_{10}$ denote the reflection through the axis of symmetry that bisects the 1st vertex of the pentagon. The subgroup generated by r_1 is a normal subgroup of D_{10}.

False. To give names to the elements of D_{10} we identify D_{10} with a subgroup of S_5 by labeling the vertices of the pentagon in clockwise order. The reflection r_1 in question then corresponds to the element $(25)(34)$. It generates a subgroup of order 2 in D_{10}. This is NOT a normal subgroup because we can conjugate r_1 by other elements of D_{10} and get elements that are not in this subgroup. For example, we can conjugate with the reflection $r_3 = (24)(15)$ and get

$$r_3 r_1 r_3^{-1} = r_3 r_1 r_3 = (24)(15)(25)(34)(24)(15) = (23)(14) = r_5,$$

which is not in the subgroup generated by r_1.

9. (T or F) For any non-zero group homomorphism $\phi\colon G \to H$ if $g_1, g_2 \in G$ are so that $g_1 \cdot g_2 = g_2 \cdot g_1$ in G, then $\phi(g_1) \cdot \phi(g_2) = \phi(g_2) \cdot \phi(g_1)$ in H.

True. By the homomorphism property,

$$\phi(g_1) \cdot \phi(g_2) = \phi(g_1 \cdot g_2) = \phi(g_2 \cdot g_1) = \phi(g_2) \cdot \phi(g_1).$$

10. (T or F) For any non-zero group homomorphism $\phi\colon G \to H$ if $g_1, g_2 \in G$ are so that $\phi(g_1) \cdot \phi(g_2) = \phi(g_2) \cdot \phi(g_1)$ in H, then $g_1 \cdot g_2 = g_2 \cdot g_1$ in G.

False. Consider the sign homomorphism $\sigma\colon S_n \to \mathbb{Z}/2$ where n is at least 3. Since $\mathbb{Z}/2$ is abelian, any two elements in the image will commute. But S_n is not abelian, so this statement is false. (Another counter-example we have briefly discussed in class is the determinant homomorphism $GL_n(\mathbb{R}) \to \mathbb{R}^{\times}$.)