Math 401: Introduction to Abstract Algebra
Problem Set 8
due: April 8, 2019

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Read. §8, §9, §11

Exercise 1. Let G be the 12-element symmetry group of the tetrahedron. There is an injective homomorphism $\phi\colon G \to S_6$ defined by labeling the six edges of the tetrahedron. This homomorphism sends a symmetry of the tetrahedron to the induced permutation of these six edges. Describe the subgroup of S_6 that arises as the image of ϕ.

Exercise 2. Carry out the procedure described in the proof of Cayley's theorem to obtain a subgroup of S_6 which is isomorphic to D_3.

Exercise 3. Prove that the matrices

$$\left\{ \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \right\}$$

form a subgroup of SO_3 and describe the corresponding rotations of $\mathbb{R}^3$.

Exercise 4.

(i) Show that the rotation of $\mathbb{R}^3$ of angle θ around the positive z-axis defines an element of SO_3 by writing down the corresponding matrix and verify that it is a matrix in SO_3.
(ii) If $A \in SO_3$ and $B \in O_3$ verify that the matrix $B^{-1}AB \in SO_3$.
(iii) Use (i) and (ii) to argue that any rotation of $\mathbb{R}^3$ which fixes the origin is represented by a matrix in SO_3.

Exercise 5. Suppose that H and K are finite subgroups of a group G and that the orders of H and K are relatively prime. Prove that $H \cap K = \{e\}$.

Exercise 6. Lagrange's theorem says that for any finite group G and any subgroup H, the order of H divides the order of G. Determine whether the following "converse" statement holds — "if n divides the order of G, then G has a subgroup of order n" — by either supplying a proof or finding a counterexample.

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