Math 401: Introduction to Abstract Algebra
Problem Set 7
due: March 25, 2019

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Read. §6, §7

Exercise 1. Show that group isomorphisms preserve the order of every element: if $\phi\colon G \to H$ is an isomorphism then for all $g \in G$ the order of g is equal to the order of $\phi(g)$.

Exercise 2. Show that the composite of two group homomorphisms is a group homomorphism.

Exercise 3. Show that the correspondence $x \leftrightarrow x^{-1}$ defines an isomorphism from a group G to itself if and only if G is abelian.

Exercise 4.

(i) Are there any non-zero homomorphisms $\mathbb{Z}/n \to \mathbb{Z}$ for $n \in \mathbb{N}$? If so, define one. If not, explain why not.
(ii) How many homomorphisms are there from $\mathbb{Z}$ to $\mathbb{Z}/n$?

Exercise 5. Let $p, n \in \mathbb{N}$ with p prime. For which n does there exist a non-zero group homomorphism $\mathbb{Z}/p \to \mathbb{Z}/n$?

Exercise 6.

(i) Show that the subset of S_7 comprised of those permutations that fix the elements $2, 5, 7$ form a subgroup. What is the order of this subgroup?
(ii) Show that the subset of S_7 comprised of those permutations that permute the elements $2, 5, 7$ among themselves form a subgroup. What is the order of this subgroup?

Exercise 7. Let $S_n = \mathrm{Aut}_{\mathsf{Set}}(\{1, 2, \ldots, n\})$ denote the group of permutations of an n-element set.

- Find an element of order d in S_n for any $d < n$.
- For which n is S_n abelian? Give a proof or supply a counterexample for each $n \geq 1$.

Exercise 8. Prove that the order of an element $\alpha \in S_n$ is the least common multiple of the cycle lengths obtained when α is written as a product of disjoint cycles.

Exercise 9*. Classify the homomorphisms from $\mathbb{Z}/n$ to $\mathbb{Z}/m$ for any $n, m \in \mathbb{N}$. In particular, how many homomorphisms are there from $\mathbb{Z}/n$ to $\mathbb{Z}/m$?

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