Math 401: Introduction to Abstract Algebra
Problem Set 2
due: February 11, 2019

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Exercise 1. For any two sets A and B their **cartesian product** is the set

$$A \times B = \{(a, b) \mid a \in A \quad \text{and} \quad b \in B\}$$

whose elements are ordered pairs, where the first coordinate is an element of A and the second coordinate is an element of B.

(i) If A is a set with n elements and B is a set with m elements, what is the cardinality of $A \times B$?
(ii) If A and B are both countably infinite sets, what is the cardinality of $A \times B$?

Exercise 2.

(i) For any function $f \colon A \to B$ define an explicit isomorphism between A and the graph $\Gamma_f \subset A \times B$, the subset defined by the property that for each $a \in A$ there is exactly one pair $(a, b) \in \Gamma_f$ whose first coordinate is a.[1]
(ii) Define a natural function $\Gamma_f \to B$. Is it necessarily injective? Is it necessarily surjective?

Exercise 3. Write B^A for the set of functions from A to B.

(i) Express the cardinality of B^A in terms of the cardinalities of A and B, assuming these are finite sets.
(ii) Express the cardinality of the powerset of A in terms of the cardinality of A, assuming that A is a finite set.
(iii) Explain why (ii) is a special case of (i) by defining an explicit isomorphism between the powerset of A and a set of functions. This isomorphism explains the notation "2^A" which is commonly used for the powerset.

Exercise 4.

(i) Define an isomorphism between any set A and the set A^1 of functions from a singleton set to A.
(ii) Argue that a function $f \colon A \to B$ is uniquely determined by its collection of composites with functions $1 \to A$.

Exercise 5. For each of the following functions determine whether they are injective, surjective, and bijective.

(i) The function $f \colon \mathbb{Z} \times \mathbb{Z}_{>0} \to \mathbb{Q}$ defined by $(a, b) \mapsto \frac{a}{b}$; here $\mathbb{Z}_{>0}$ denotes the set of positive integers.
(ii) The function $\pi_B \colon A \times B \to B$ defined by $\pi_B(a, b) = b$.
(iii) The function from the set of subsets of $\mathbb{N}$ to the set of countably-infinite binary sequences $(0, 1, 0, 0, 0, 1, \ldots)$ that sends a subset $S \in \mathbb{N}$ to the sequence that has a 1 in the nth coordinate if and only if $n \in S$.
(iv) Writing $10 = \{0, \ldots, 9\}$ and $[0, 1] = \{x \in \mathbb{R} \mid 0 \leq x \leq 1\}$, the function $f \colon 10^{\mathbb{N}} \to [0, 1]$ that sends a sequence of decimal digits $(x_n)_{n \in \mathbb{N}}$ to the real number $0.x_1x_2x_3\ldots$.

[1] Think about the relationship between this definition and the "vertical line test."

Exercise 6. Show that the set $\mathbb{Q}$ of rational numbers is countable.

Exercise 7. Show that the sets $2^{\mathbb{N}}$ and $\mathbb{R}$ have the same cardinality.

Exercise 8. Let p be a prime number. Prove that $\sqrt{p}$ is irrational.

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