Math 401: Introduction to Abstract Algebra
Problem Set 11[1]
due: April 29, 2019

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Read. §17, §13, §18, §19

Exercise 1. The group $\mathbb{Z}/2$ acts on $\mathbb{C}$ by complex conjugation.

(i) Any group action on a set defines a partition of that set into orbits. Describe the resulting partition of the complex plane into orbits.
(ii) An element $z \in \mathbb{C}$ is **fixed** by the complex conjugation action if its orbit is a singleton. What are the fixed points of this action?

Exercise 2. Here are six group actions on $\mathbb{R}^2$.

(i) The usual action of $GL_2(\mathbb{R})$ on $\mathbb{R}^2$ by linear transformations.
(ii) The action of the subgroup O_2 on $\mathbb{R}^2$ by orthogonal transformations.
(iii) The action of the subgroup SO_2 on $\mathbb{R}^2$ by orientation preserving orthogonal transformations.
(iv) Identify $\mathbb{R}^2$ with $\mathbb{C}$ and let $GL_1(\mathbb{C}) = \mathbb{C}^\times$ act on $\mathbb{R}^2$ by complex multiplication.
(v) Let the group $\mathbb{Z}^2 \cong \mathbb{Z} \times \mathbb{Z}$ act on $\mathbb{R}^2$ by addition: the element $(n, m) \in \mathbb{Z}^2$ sends $(x, y) \in \mathbb{R}^2$ to $(x + n, y + m) \in \mathbb{R}^2$.
(vi) The action of $\mathbb{Z}/2$ on $\mathbb{C}$ by complex conjugation discussed in the previous exercise.

Describe the orbits and stabilizers for each of these actions.

Exercise 3. Let $H \subset G$ be a subgroup. Then G acts on the set of left cosets G/H by left multiplication as discussed in class.

(i) What is the orbit of the left coset H?
(ii) What is the stabilizer of the left coset H?
(iii) What is the orbit of a generic left coset gH?
(iv) What is the stabilizer of a generic left coset gH?

Exercise 4. Prove that the center of S_n is trivial for $n \geq 3$.

Exercise 5. If $H \subset G$ is a subgroup its conjugate subgroups are the subgroups of the form

$$gHg^{-1} = \{ghg^{-1} \mid h \in H\}$$

for some $g \in G$.

(i) Prove that gHg^{-1} is a subgroup of G.
(ii) Define a bijective group homomorphism $H \to gHg^{-1}$.
(iii) The group G acts on the set of subgroups of G by conjugation: the action of a group element $g \in G$ on a subgroup $H \subset G$ is defined by $H \mapsto gHg^{-1} \subset H$. Rephrase the condition of H being a normal subgroup in terms of the orbits of this action.

[1]Problems labelled $n*$ are optional (fun!) challenge exercises that will not be graded.

Exercise 6. Let p be a prime number. Show that the matrices

$$\begin{pmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{pmatrix}$$

with coefficients $a, b, c \in \mathbb{Z}/p$ form a non-abelian group of order p^3.

Exercise 7*. A Rubik's cube is built from 26 little cubes called *cubies*; the expected 27th cubie at the very center of the cube is missing.[2] The *Rubik's cube group* is generated by six elements of order four R, L, F, B, U, D which act on the Rubik's cube by performing one counterclockwise rotation of the right, left, front, bottom, upwards, and downards faces, respectively. The Rubik's cube action identifies the Rubik's cube group with a subgroup of S_{26}.

(i) Any group action on a set defines a partition of that set into orbits. Describe the resulting partition of the set of 26 cubies into orbits.

(ii) A cubie is **fixed** by the Rubik's cube action if its orbit is a singleton. What are the fixed points of the Rubik's cube action?

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[2]For the purposes of this problem we will consider the cubies to be unoriented.