Math 401: Introduction to Abstract Algebra
Problem Set 10[1]
due: April 22, 2019

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Read. §15, §16

Exercise 1. How many elements of S_8 are conjugate to $(12)(345)(678)$?

Exercise 2. The **center** of a group G is the subgroup

$$Z(G) = \{g \in G \mid \forall x \in G, gx = xg\}$$

made up of all elements that commute with all other elements of g.

(i) Prove that $Z(G)$ is a subgroup.
(ii) Prove that $g \in Z(G)$ if and only if the conjugacy class of g contains a single element.
(iii) Prove that $Z(G)$ is normal.
(iv) Use the calculation of the conjugacy classes in O_2 to calculate the center of O_2.

Exercise 3. For any subset $N \subset G$ of a group G define

$$gN = \{gn \mid n \in N\} \qquad \text{and} \qquad Ng = \{ng \mid n \in N\}.$$

Let N be a subgroup of G. Prove that the following are equivalent.

(i) N is a *normal* subgroup of G.
(ii) For all $g \in G$, $gNg^{-1} \subset N$.
(iii) For all $g \in G$, $gNg^{-1} = N$.
(iv) For all $g \in G$, $gN \subset Ng$.
(v) For all $g \in G$, $Ng \subset gN$.
(vi) For all $g \in G$, $gN = Ng$.

In terminology we will introduce the equivalence (i) $\Leftrightarrow$ (vi) says that N is normal in G if and only if each **left coset** gN equals the **right coset** Ng.

Exercise 4. The **index** $[G, H]$ of a subgroup $H \subset G$ is the number of left cosets gH for that subgroup.[2] Suppose $H \subset G$ is a subgroup of index 2. Prove that H is normal in G. (Hint: use Exercise 3(vi) and the fact that the cosets partition G.)

Exercise 5. Prove that a homomorphism $\phi\colon G \to H$ is injective if and only if its kernel is the subgroup $\{e\}$.

Exercise 6. A group G is **finitely generated** if there exists finitely many elements $g_1, \ldots, g_n \in G$ so that the subgroup generated by these elements is all of G. Prove that the group $\mathbb{Q} = (\mathbb{Q}, +, 0)$ is *not* finitely generated by showing that any subgroup generated by only finitely many rational numbers $q_1, \ldots, q_n$ does not contain some rational number $q \in \mathbb{Q}$.

[1] Problems labelled n^* are optional (fun!) challenge exercises that will not be graded.

[2] Two left cosets are the same, in symbols $gH = g'H$, if these sets have the same elements, which is the case iff $g^{-1}g' \in H$.

Exercise 7. Define a presentation for the dihedral group D_{2n} with two generators r and s and justify the relations you enumerate by arguing that every element of the dihedral group has a unique representation as $r^m s^n$ where $m, n \geq 0$ and are each less than the orders of r and s respectively.

Exercise 8*. Prove that the free group on 26 generators $a, b, c, \ldots, z$ modulo pronunciation in English is trivial.[3]

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[3]Alternatively, google "homophonic quotients of free groups."