Math 401: Introduction to Abstract Algebra
Problem Set 1
due: February 4, 2019

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defn. A **function** $f\colon A \to B$ is given by specifying for each $a \in A$ a unique element $f(a) \in B$.

Exercise 1. For any functions $f\colon A \to B$ and $g\colon B \to C$ define the **composite function** $g \circ f\colon A \to C$ by specifying its action on elements.

defn. A function $f\colon A \to B$ is
- **injective** iff $f(a) = f(a')$ implies that $a = a'$;
- **surjective** iff for every $b \in B$ there exists $a \in A$ so that $f(a) = b$;
- **bijective** iff it is both injective and surjective.

Exercise 2. How many functions are there from a set of n elements to itself, where $n \in \mathbb{N}$? How many bijections are there between a set with n elements and itself?

Exercise 3. Assume $A, B \neq \emptyset$. Prove that $f\colon A \to B$ is an injective if and only if it has a **left inverse**: a function $g\colon B \to A$ so that $g \circ f = \mathrm{id}_A$.[1].

Exercise 4. Assume $A, B \neq \emptyset$. Prove that $f\colon A \to B$ is an surjective if and only if it has a **right inverse**: a function $g\colon B \to A$ so that $f \circ g = \mathrm{id}_B$.

Exercise 5. Let $f\colon A \to B$ be a function that has a left inverse $g\colon B \to A$ and also a right inverse $h\colon B \to A$. Prove that $h = g$.

Exercise 6. Prove that function $f\colon A \to B$ is a bijection iff it has a **two-sided inverse**: a function $g\colon B \to A$ so that $g \circ f = \mathrm{id}_A$ and $f \circ g = \mathrm{id}_B$. The data of A, B, f, and g define a **isomorphism**.

Exercise 7. Two sets are **isomorphic** if there exists a bijection between them. Explain in your own words why all sets with three elements are isomorphic.

Exercise 8. Find a formula for the sum $1 + 3 + 5 + \cdots + (2n - 1)$ of the first $n + 1$ odd numbers (for n a positive integer) and prove (by induction) that your formula is correct.[2]

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[1] For any set A, the **identity function** $\mathrm{id}_A\colon A \to A$ is defined by $\mathrm{id}_A(a) = a$.

[2] Hint: To discover the formula, first compute this sum for some particular small values of n and look for a pattern.