Math 401: Introduction to Abstract Algebra
Practice Midterm
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Conventions: In what follows $\mathbb{Z}$ denotes the group of integers under addition, $\mathbb{Z}/n$ denotes the cyclic group under addition modulo n, D_n denotes the dihedral group of symmetries of a regular n-gon, and S_n denotes the symmetric group of permutations of n elements.

True or False

(1 point) Indicate whether each of the following statements is true or false (circle one).

(2 points) For each true statement, give a short (one to two sentence) justification, explaining the essential reason for its correctness; for each false statement, provide either a counter-example or, if a counter-example would not make sense, a short disproof.

1. (T or F) The set of all 2×2 matrices with real coefficients forms a group under matrix multiplication.

False. The element

$$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$

does not have a multiplicative inverse.

2. (T or F) The Klein four group is abelian.

True. The multiplication table for the Klein four group is

	e	i	j	k
e	e	i	j	k
i	i	e	k	j
j	j	k	e	i
k	k	i	j	e

which is symmetric about the upper left, lower right diagonal. This is a sign that the group is abelian.

3. (T or F) The dihedral group D_5 is generated by the 72° rotation through the center of mass of the pentagon.

False. The 72° rotation is an element of order 5 so it generates a subgroup of order 5. Since D_5 has 10 elements, this subgroup is not all of D_5.

4. (T or F) Every non-zero[1] subgroup of $\mathbb{Z}$ is cyclic.

True. Every non-zero subgroup of $\mathbb{Z}$ contains a minimum positive element d. Necessarily then all multiples dn of d, for any $n \in \mathbb{N}$, are in the subgroup (closure under addition and inverses). If there exists some element g in the subgroup that is not a multiple of d then by the division theorem $g - qd = r$ for some $q \in \mathbb{N}$ and $0 < r < d$, contradicting the claim that d is the smallest element of the subgroup. So the subgroup is the cyclic subgroup generated by d.

[1] Technically, the trivial group with a single element is a cyclic group, but we're excluding it to avoid confusion.

5. (T or F) Let $\phi\colon G \to H$ define a group homomorphism and consider two elements $g_1, g_2 \in G$. If $g_1 g_2 = g_2 g_1$ in G, then $\phi(g_1)\phi(g_2) = \phi(g_2)\phi(g_1)$ in H.

True. A homomorphism has the property that $\phi(xy) = \phi(x) \cdot \phi(y)$ for all $x, y \in G$. So

$$\phi(g_1)\phi(g_2) = \phi(g_1 g_2) = \phi(g_2 g_1) = \phi(g_2)\phi(g_1).$$

6. (T or F) Let $\phi\colon G \to H$ define a group homomorphism and consider two elements $g_1, g_2 \in G$. If $\phi(g_1)\phi(g_2) = \phi(g_2)\phi(g_1)$ in H, then $g_1 g_2 = g_2 g_1$ in G.

False. The sign of a permutation defines a homomorphism $\phi\colon S_n \to \{1, -1\}$ from the symmetric group S_n to the group with two elements under multiplication. The two-element group is abelian so $\phi(g_1)\phi(g_2) = \phi(g_2)\phi(g_1)$ for all permutations $g_1, g_2 \in S_n$. But the permutations $g_1 = (12)$ and $g_2 = (23)$ do not commute.

7. (T or F) There exists a non-zero homomorphism $\mathbb{Z}/17 \to \mathbb{Z}$.

False. If there were such a $\phi\colon \mathbb{Z}/17 \to \mathbb{Z}$ then since 1 is an element of order 17 in $\mathbb{Z}/17$, $\phi(1)$ would have to be an element of order at most 17 in $\mathbb{Z}$. But all non-zero elements of $\mathbb{Z}$ have infinite order. So this homomorphism cannot exist unless $\phi(1) = 0$ in which case ϕ is the zero homomorphism.

8. (T or F) For $2 \leq k \leq n$ a k-cycle $(a_1 \cdots a_k) \in S_n$ is an even permutation if and only if k is odd.

True. The cycle

$$(a_1 a_2 \cdots a_k) = (a_1 a_k)(a_1 a_{k-1}) \cdots (a_1 a_4)(a_1 a_3)(a_1 a_2).$$

is a product of $k-1$ transpositions. The sign of each transposition is -1. Since sign is a homomorphism, the sign of $k-1$ transpositions is $(-1)^{k-1}$ which is 1 iff k is odd and -1 iff k is even.