Metric Spaces Worksheet 1

Metric spaces

A metric space is a formalisation of the notion of a set of 'points' with a well-defined means of measuring a 'distance' between each pair.

Definition 1 (metric space)**.** A *metric* on a set X is a function $d : X \times X \to [0, \infty) \subseteq \mathbb{R}$ which satisfies the following properties for all $x, y, z \in X$:

M1 $d(x,x) = 0$

M2 $d(x,y) = 0$ implies $x = y$

M3 $d(x,y) = d(y,x)$

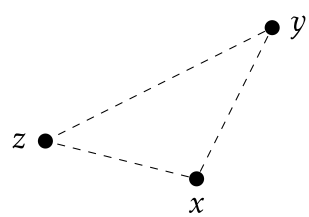
M4 $d(x,z) \leq d(x,y) + d(y,z)$

A pair of a set and metric defined on it, (X,d), is termed a *metric space*. ⌟

In the course we will refer to M2 as the separation axiom, M3 as the symmetry axiom, and M4 as the triangle inequality.

These axioms capture several different intuitions about the properties that a distance measurement device, the function d, should satisfy. M1 says that the distance from any point to itself is 0, as it should be. M2 says that the only way that two points are zero distance apart is when they are the same. M3 says that it doesn't matter whether we choose to measure the distance *from x to y* or *from y to x*, the answer will be the same.

In most cases M4 is the most difficult to check, and at first reading, the most difficult to understand. To develop an intuition for it, and shed some light on the *triangle* part of the triangle inequality, consider the following picture.



In this picture, the distance covered by traversing from x to y and from y to z along the dashed lines is greater than the distance covered by going directly from x to z along the third dashed line. This is the statement of the triangle inequality: *taking detours cannot shorten your trip*.

Now we are ready to look at some familiar-ish examples of metric spaces. To begin we'll need the following definition.

Definition 2 (absolute value function)**.** The *absolute value* of a real number $x \in \mathbb{R}$ is defined to be

$$|x| :\equiv \begin{cases} x, & x \geq 0 \\ -x, & x < 0 \end{cases} .$$

⌟

By arguing on cases (is 'a' non-negative?), you should try proving the following:

Lemma 3 (absolute value facts)**.** *For all real numbers $a \in \mathbb{R}$,*

1. $|a| = 0 \leftrightarrow a = 0$,
2. $|a| = |-a|$,
3. $-|a| \leq a$ *and* $a \leq |a|$,
4. *if* $b \in \mathbb{R}$ *has* $b \geq 0$ *then* $|a| \leq b \leftrightarrow -b \leq a \leq b$.

Complete the proof here

$\blacksquare$

With this lemma in hand we're ready to give our first example of a metric space.

Example 4 (*Euclidean metric space* $\mathbb{R}$)

Consider the real line, $\mathbb{R}$. Let us define the function $d : \mathbb{R} \times \mathbb{R} \to [0, \infty)$ as

$$d(x, y) = |x - y| \quad .$$

We will show that $(\mathbb{R}, d)$ is a metric space, and to do this we must show that d is a metric. To do this, we'll tackle each requirement M1-M4 in turn.

To see that M1 holds, consider that $d(x, x) = |x - x| = |0| = 0$. For M2 consider that if $d(x, y) = 0$ then $|x - y| = 0$ which, by our lemma 3 (1), means $x = y$. M3 is a consequence of lemma 3 (2); do you see how? Now for M4.

Unlike the previous requirements, M4 requires a 'trick' of sorts. Let's take $x, y, z \in \mathbb{R}$ and apply lemma 3 (3) to the numbers $x - z$ and $z - y$. That is, we derive the inequalities

$$\begin{aligned} -|x - z| &\leq x - z \leq |x - z| \\ -|z - y| &\leq z - y \leq |z - y| \quad . \end{aligned}$$

By adding these inequalities we see that $-(|x - z| + |z - y|) \leq x - y \leq (|x - z| + |z - y|)$. In this inequality note that $|x - z| + |z - y| \geq 0$ and so lemma 3 (4) is applicable with $b = |x - z| + |z - y|$. Thus our previous inequality is logically equivalent, by lemma 3 (4), to the inequality

$$|x - y| \leq |x - z| + |z - y| \quad ,$$

but this is precisely the statement of M4!

Now that we have established that d is a metric we know that $(\mathbb{R}, d)$ is a metric space. This particular metric is called the *Euclidean metric* on $\mathbb{R}$, and when we refer to the Euclidean metric space $\mathbb{R}$, we mean $\mathbb{R}$ with the Euclidean metric. We will sometimes write $|\cdot|$ for d, so as to help in reminding ourselves which metric we're talking about. In this notation, the Euclidean metric space $\mathbb{R}$ is the pair $(\mathbb{R}, |\cdot|)$.

This example is the 'lowest-dimensional' version of a sequence of metric spaces called the Euclidean metric spaces. We will explore the next step up the ladder, but not the general case. Our proof will make use of a result which you may have seen in a calculus or a linear algebra course before. We will not be interested in establishing the truth of the following theorem, so we take it for granted.

Theorem 5 (Cauchy-Schwartz inequality)**.** *For all points* $(a_1, a_2), (b_1, b_2) \in \mathbb{R}^2$,

$$(a_1 b_1 + a_2 b_2)^2 \leq ({a_1}^2 + {a_2}^2)({b_1}^2 + {b_2}^2) \quad .$$

Example 6 (*Euclidean metric space* $\mathbb{R}^2$)

Consider the set of points in the real plane, $\mathbb{R}^2 :\equiv \mathbb{R}\times\mathbb{R}$. Let us define the function $d : \mathbb{R}^2 \times \mathbb{R}^2 \to [0,\infty)$ as

$$d((u_1,u_2),(v_1,v_2)) :\equiv \sqrt{(u_1-v_1)^2+(u_2-v_2)^2} \quad .$$

From this definition we see that M1 follows. For M2 we note that if

$$d((u_1,u_2),(v_1,v_2)) = 0$$

then in particular $(u_1-v_1)^2+(u_2-v_2)^2 = 0$. As both summands are positive, this is only possible if both $(u_1-v_1)^2$ and $(u_2-v_2)^2$ are zero, and in turn this implies that $u_1 = v_1$ and $u_2 = v_2$ and thus that M2 holds. For M3 we observe that we have $(u_1-v_1)^2 = (v_1-u_1)^2$ and similarly for the second components so that $d(\vec{u},\vec{v}) = d(\vec{v},\vec{u})$ as desired.

Finally we arrive at the frightening requirement of M4, which is where we make use of Cauchy-Schwartz. To begin then fix $\vec{u},\vec{v},\vec{w}\in\mathbb{R}^2$. Our first step is to observe that it suffices to show

$$d(\vec{u},\vec{w})^2 \leq (d(\vec{u},\vec{v})+d(\vec{v},\vec{w}))^2 \quad .$$

Can you see why this might be enough? Mashing out some calculations we find that we have the equalities

$$\begin{aligned} d(\vec{u},\vec{w})^2 &= (u_1-w_1)^2+(u_2-w_2)^2 \\ &= ((u_1-v_1)+(v_1-w_1))^2+((u_2-v_2)+(v_2-w_2))^2 \\ &= (u_1-v_1)^2+(u_2-v_2)^2+(v_1-w_1)^2+(v_2-w_2)^2 \\ &\quad +2(u_1-v_1)(v_1-w_1)+2(u_2-v_2)(v_2-w_2) \quad , \end{aligned}$$

and similarly for the other side,

$$\begin{aligned} (d(\vec{u},\vec{v})+d(\vec{v},\vec{w}))^2 &= d(\vec{u},\vec{v})^2+d(\vec{v},\vec{w})^2+2d(\vec{u},\vec{v})d(\vec{v},\vec{w}) \\ &= (u_1-v_1)^2+(u_2-v_2)^2+(v_1-w_1)^2+(v_2-w_2)^2 \\ &\quad +2\sqrt{((u_1-v_1)^2+(u_2-v_2)^2)((v_1-w_1)^2+(v_2-w_2)^2)} \quad . \end{aligned}$$

Now if we apply Cauchy-Schwartz with $\vec{a} = \vec{u}-\vec{v}$ and $\vec{b} = \vec{v}-\vec{w}$ we see that we have the inequality

$$((u_1-v_1)(v_1-w_1)+(u_2-v_2)(v_2-w_2))^2 \leq ((u_1-v_1)^2+(u_2-v_2)^2)((v_1-w_1)^2+(v_2-w_2)^2)$$

so that, by our second calculation,

$$\begin{aligned} (d(\vec{u},\vec{v})+d(\vec{v},\vec{w}))^2 &\geq (u_1-v_1)^2+(u_2-v_2)^2+(v_1-w_1)^2+(v_2-w_2)^2 \\ &\quad +2\sqrt{((u_1-v_1)(v_1-w_1)+(u_2-v_2)(v_2-w_2))^2} \\ &= d(\vec{u},\vec{w})^2 \quad , \end{aligned}$$

as desired. Thus $\mathbb{R}^2$ with this function is a metric space. Much like in the previous case we call this the Euclidean metric too.

We will content ourselves now with having established that the usual notions of distance in $\mathbb{R}$ and $\mathbb{R}^2$ are in fact metrics. There is more general theory, beyond the scope of this course, which stands to make efficient the proof that $\mathbb{R}^n$ is a metric space when equipped with the usual notion of distance, for each $n \in \mathbb{N}$ – efficient specifically in a way that does not involve the awful calculations of example 6. These spaces are the Euclidean spaces, of which we have seen two examples, and with which we are very familiar[1].

For the remainder of this worksheet we will acquaint ourselves with further examples of metric spaces. Well, where this worksheet says "we will acquaint ourselves" it really means to state "you will acquaint yourself" because all of what follows is for you to fill in – for each example you meet, try to show that M1-M4 are satisfied.

Example 7 (*discrete metric spaces*)

For any inhabited set X, the function $d : X \times X \to [0, \infty)$ defined by

$$d(x,y) :\equiv \begin{cases} 0, & x = y \\ 1, & \text{otherwise} \end{cases}$$

equips X with the structure of a metric space.

Example 7 reveals that every inhabited set is naturally endowed with the structure of a metric space. This naturally occurring metric is called the *discrete metric*, and we'll see why it has that name later on.

Complete the proof here

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[1] After all, we essentially live in one — just don't tell this to a physicist.

You may have noticed that we have not yet shown that *every* set is naturally equipped with a metric for we assumed above that X was inhabited. Fear not,

Example 8 (*empty metric space*)

The empty set supports the structure of a metric space. There is, in some sense, *nothing* to verify. In fact, there is a unique metric on the empty set.

Complete the proof here

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It can be useful to isolate recurring pattern in our proofs that functions are metrics. This lemma will help us reduce the number of cases we might have to check.

Lemma 9. *Let $d : X \times X \to [0, \infty)$ be a function which satisfies M1 and M3 above. Then for all $x, y, z \in X$ M4 holds iff for all distinct points $x, y, z \in X$ M4 holds.*

Complete the proof here

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For our next example we turn to the sky. Many airlines operate on a "hub and spokes" model, in which every airport is connected via a direct flight to a central hub airport, say in Minneapolis. Then to fly between two other cities in the network, say Baltimore and Boston, one would have to fly from Baltimore to Minneapolis *and then* from Minneapolis to Boston.

This "hub and spokes" notion of distance inspires the following examples.

Example 10 (*British Rail metric*)

Given a set X, a chosen point $x_0 \in X$, and a function $f : X \to [0, \infty)$ for which $f(x) = 0 \leftrightarrow x = x_0$ holds, we may define a function $d_f : X \times X \to [0, \infty)$ as

$$d_f(x, y) :\equiv \begin{cases} 0, & x = y \\ f(x) + f(y), & \text{otherwise} \end{cases} .$$

This function is a metric and this construction is known variously as the "British Rail Metric", the "hub and spoke metric", or the "Memphis metric."

Hint 11. Make sure you use $f(x) = 0 \to x = x_0$.

Complete the proof here

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You might be wondering what role the function $f : X \to [0, \infty)$ plays in the definition of the British Rail metric.

Lemma 12. *In the British Rail metric space (X, d_f), prove that for every point $x \in X$,*

$$f(x) = d_f(x, x_0) = d_f(x_0, x).$$

Complete the proof here

$\blacksquare$

Exercise 13. Using Lemma 12, explain why the definition of d_f given in Example 10 encodes the idea of a "hub and spokes" notion of distance.

Complete the proof here

$\blacksquare$

Items marked with an asterisk (*) are more challenging.

Example* 14 ($L^\infty(\mathbb{R}^2)$)

Consider the set of real numbers $\mathbb{R}^2$, and define the function $d : \mathbb{R}^2 \times \mathbb{R}^2 \to [0, \infty)$ as

$$d((u_1, u_2), (v_1, v_2)) :\equiv \max\{|u_1 - v_1|, |u_2 - v_2|\} \quad .$$

This metric is known as the L^∞ metric (pronounced "ell infinity") on $\mathbb{R}^2$.

We don't yet know how to compare metrics, but there is a sense in which this is the same as the Euclidean metric on $\mathbb{R}^2$, but not the same as the discrete metric.

Complete the proof here

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When we first defined metrics we ensured that $d(x, y) < \infty$ for all $x, y \in X$. One might wonder whether there are interesting metrics which are bounded by some finite value instead.

Definition 15 (bounded metrics)**.** A metric d on X is *bounded* if there exists $k \in \mathbb{R}$ so that for all $x, y \in X$, $d(x, y) < k$. ⌟

Question 16. Is the Euclidean metric on $\mathbb{R}$ bounded? Is the discrete metric on $\mathbb{R}$ bounded?

Complete the proof here

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Lemma* 17 (any metric may be replaced by a bounded metric)**.** *Given a metric space* (X, d) *we may define a function* $\hat{d} : X \times X \to [0, \infty)$ *via the formula*

$$\hat{d}(x, y) :\equiv \min\{d(x, y), 1\} \quad .$$

Then $\hat{d}$ *is a bounded metric.*

Complete the proof here

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Definition 18 (sub-metric-spaces)**.** If (X, d) is a metric space and $Y \subseteq X$, then the restriction of the function d to the subset $Y \times Y \subseteq X \times X$, written $d_{\restriction_{Y\times Y}}$, defines a metric on Y and so endows Y with the structure of a metric space. The metric $d_{\restriction_{Y\times Y}}$ on Y is called the *subspace metric* of (X, d) and $(Y, d_{\restriction_{Y\times Y}})$ is called a *sub-metric-space* or *subspace* of (X, d). ⌟

Can you see how $(Y, d_{\restriction_{Y\times Y}})$ is a metric space?