Math 301: Introduction to Proofs
Problem Set 8
due: never

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Read. `coq_lecture.html` (lecture notes) on the course website.

In this problem set we'll depart from the usual routine of solving problems based on Newstead's *An Infinite Descent into Pure Mathematics* to look at a certain way of doing mathematics, as introduced by the Coq section of the course.

This homework is all about *constructive* mathematics, and we'll see how it naturally develops from the assumptions on propositions we have made. Constructive mathematics is, loosely, the idea that in order to prove that something is true one must be able to *produce* a proof of that fact. In the context of tasks (say, tying shoe laces or baking), the idea of constructive mathematics is tantamount to saying that we don't care to hear that a task is not impossible, we want to know *how* to do it.

For the purposes of this problem set, we adopt the following definition.

defn. A *proposition* is a set. If P is a proposition then elements $p \in P$ are thought of as *proofs* that P is true.

Thus to prove a proposition P we must *give* an element of the set P.

defn. Given two sets X and Y, we denote the *set of functions* from X to Y by $X \to Y$. Thus if $f \in X \to Y$ then f is a function $f\colon X \to Y$.

defn. If P and Q are propositions, then:

(i) The proposition "P implies Q" is defined to be set $P \to Q$.
(ii) The proposition "$\neg P$" is defined to be the set of functions $P \to \varnothing$, where $\varnothing$ is the empty set.
(iii) The proposition "$P \wedge Q$" is defined to be set $P \times Q$, the set whose elements are pairs $(p \in P, q \in Q)$.
(iv) The proposition "$P \vee Q$" is defined to be the set $P \sqcup Q$, the set whose elements $x \in P \sqcup Q$ are such that either $x \in P$ or $x \in Q$, exclusively.

From this we make the following observations:

(i) To prove $P \to Q$ we must construct a function from P to Q, that is, we may assume that we have an element $p \in P$ and then build an element $q \in Q$.
(ii) To prove $\neg P$ we must construct a function $P \to \varnothing$. Of course there is no way to give elements of the empty set, and so when we assume an element $p \in P$ the only way we can give an element of the empty set is to derive a contradiction (logical falsehood) and then infer the existence of $x \in \varnothing$ from our contradiction.[1]
(iii) To prove $P \wedge Q$ we must give a pair (p, q), that is, an element $p \in P$ as well as an element $q \in Q$.
(iv) To prove $P \vee Q$ we may either give an element of P or an element of Q.

[1] Note that for any proposition Q, there is a unique function of the form $\varnothing \to Q$. One way to interpret this is to say "from falsehood we may infer anything."

With the above definitions, complete the following exercises. For exercises 1-7, you may choose to either write your constructive proof on paper or write your proof in Coq.[2]

Exercise 1 (modus ponens)**.** For any propositions P and Q, define a function which is an element of

$$(P \wedge (P \to Q)) \to Q.$$

By our definitions above, this is the same as as proving that the proposition $(P \wedge (P \to Q)) \to Q$ is true.

Exercise 2 (law of non-contradiction)**.** For any proposition P, define an element of

$$\neg(P \wedge \neg P).$$

By our definitions above, this is the same as proving that P and $\neg P$ cannot simultaneously be true.

Exercise 3 (transitivity of implication)**.** For any propositions P, Q, and R, define an element of

$$(P \to Q) \wedge (Q \to R) \to (P \to R).$$

Exercise 4 (modus tollens)**.** For any propositions P, Q define an element of

$$((P \to Q) \wedge \neg Q) \to \neg P.$$

In the Coq reading exercises there is a proof that, for propositions P, Q, and R,

$$(P \wedge Q \to R) \leftrightarrow (P \to (Q \to R)). \tag{†}$$

When we expand our definitions, this gives us two, equivalent ways to think about functions. In particular, the left-to-right direction might be read as the following:

> If $f \in P \wedge Q \to R$ this means that f is a function $f\colon P \times Q \to R$. That is, f is a *function of two variables* and a pair $(p, q) \in P \times Q$ is sent to $f(p, q) \in R$. A proof of (†) sends this function of two variables to an element $\tilde{f}\colon P \to (Q \to R)$. The way to read this is that $\tilde{f}$ is a function which takes in a $p \in P$ and gives another *function*, namely $\tilde{f}(p) \in Q \to R$. The proof given in the Coq file constructs the function $\tilde{f}$ as the function defined by $\tilde{f}(p) :\equiv f(p, -)$; that is, when $\tilde{f}(p)$ is applied to $q \in Q$ its value is $(\tilde{f}(p))(q) :\equiv f(p, q)$.

Using this technique of defining functions of the form $P \to (Q \to R)$, which when given an input $p \in P$ give as output a function $Q \to R$, perform the following exercises.

Exercise 5 (modus ponens)**.** For any propositions P and Q, define a function which is an element of

$$P \to ((P \to Q) \to Q).$$

Exercise 6 (double negation introduction)**.** For any proposition P, prove that

$$P \to \neg\neg P.$$

That is, come up with an element of the set represented by this proposition.[3]

[2] A Coq source file `coq_exercises.v` is available on the course website with statements of these exercises.

[3] This can be tricky to think about. Think carefully about what $\neg\neg P$ is supposed to be. Hint, it is a set of functions of some form.

Exercise 7 (DeMorgan)**.** Let P and Q be propositions, prove

$$(\neg P) \wedge (\neg Q) \to \neg(P \vee Q)$$

and also prove

$$\neg(P \vee Q) \to (\neg P) \wedge (\neg Q).$$

Together, these prove the tautology

$$(\neg P) \wedge (\neg Q) \leftrightarrow \neg(P \vee Q).$$[4]

Exercise 8. It turns out that we cannot, in general, construct an element of the set

$$\neg\neg P \to P.$$

Speculate why this is the case. You needn't have a *proof* of this fact (that's a very subtle and difficult matter). Instead, tell us about your intuitions: Why can't we construct this function? How would its output be defined?

Observe that this final exercise means that P and $\neg\neg P$ are not necessarily logically equivalent! This is the very portal to constructive mathematics, a more general type of mathematics that the classical mathematics we have been learning.

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[4] Again, it's helpful to break down the propositions that are presented here into the sets that they represent, and to determine what their elements should look like.