Metric Spaces Worksheet 5

Topology I

With our understanding of metric spaces and sequences cemented, we'll turn to examine a notion which is supported by every metric space, and in some ways subsumes the concepts we have seen so far.

Definition 1 (open ball)**.** Let (X, d) be a metric space, $x \in X$ a point and $r \in (0, \infty)$ a non-negative real number. The *open ball of radius r centred on x*, written $B_r(x)$, is the subset $B_r(x) :\equiv \{y \in X \mid d(x, y) < r\} \subseteq X$. ⌟

We now calculate open balls in Euclidean metric spaces. To describe open balls in the Euclidean line, we need the notion of an *open interval* in $\mathbb{R}$. For any $a, b \in \mathbb{R}$, with $a < b$, let

$$(a, b) :\equiv \{z \in \mathbb{R} \mid a < z < b\}.$$

Example 2 (*open balls in Euclidean spaces*)

1. In the Euclidean metric space $\mathbb{R}$, the open ball $B_r(x) = \{y \in \mathbb{R} \mid |x - y| < r\}$ is the open interval $(x - r, x + r)$. Conversely, every open interval (a, b) for $a < b \in \mathbb{R}$, is an open ball of some radius $r = \frac{b-a}{2}$ centred about the midpoint $\frac{a+b}{2}$.

2. In the Euclidean metric space $\mathbb{R}^2$, the open ball

$$B_r((u_1, u_2)) = \{(v_1, v_2) \in \mathbb{R}^2 \mid (u_1 - v_1)^2 + (u_2 - v_2)^2 < r^2\}$$

is comprised of all points inside the circle of radius r centred at the point (u_1, u_2). This explains the name "open ball" given to the sets $B_r(x)$ in general metric spaces.

Question 3. What are the possible open balls in a discrete metric space (X, d)?

Complete the proof here

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Definition 4 (open set)**.** A subset $U \subseteq X$ in a metric space (X, d) is *open* if for every $u \in U$ there exists an $\varepsilon \in (0, \infty)$ such that $B_\varepsilon(u) \subseteq U$. ⌟

Example 5 (*an open set in the Euclidean space* $\mathbb{R}$)

For any $a \in \mathbb{R}$, the open ray
$$(a, \infty) :\equiv \{x \in \mathbb{R} \mid a < x\}$$
is an open set.

To see this we must prove, for every point $u \in (a, \infty)$, that there exists some $\varepsilon \in (0, \infty)$ such that $B_\varepsilon(u) \subseteq (a, \infty)$. To that end, consider a point $u \in (a, \infty)$. We know that $u - a > 0$, so we may choose ε to be any real number so that $0 < \varepsilon < u - a$. (For sake of concreteness, we might pick $\varepsilon = \frac{u-a}{2}$, but it's also not necessary to specify a concrete value of ε.)

Now if $x \in B_\varepsilon(u)$, then by example 2 item 1, $u - \varepsilon < x < u + \varepsilon$. Since $u - \varepsilon > a$ we conclude that $x > a$ so $x \in (a, \infty)$. Since we've shown that $\forall x \in B_\varepsilon(u)$, $x \in (a, \infty)$ this demonstrates that $B_\varepsilon(u) \subseteq (a, \infty)$ as required. Thus (a, ∞) is an open set.

Non-example 6 (*sets which are not open in the Euclidean space* $\mathbb{R}$)

1. The set $\{0\} \subseteq \mathbb{R}$ is not open because there is no ε small enough so that $B_\varepsilon(0) \subset \{0\}$.
2. For any $a \in \mathbb{R}$, the closed ray
$$[a, \infty) :\equiv \{x \in \mathbb{R} \mid a \leq x\}$$
is not an open set. The argument given in example 5 proves that for every $u \in [a, \infty)$ if $a \neq u$ then there exists $\varepsilon \in (0, \infty)$ so that $B_\varepsilon(u) \subset [a, \infty)$. However, there is no open ball that contains the point a and is contained within $[a, \infty)$.

   To see this, take $\varepsilon \in (0, \infty)$. Then by example 2 item 1 the point $a - \frac{\varepsilon}{2} \in B_\varepsilon(a)$. But since $a - \frac{\varepsilon}{2} < a$, $a - \frac{\varepsilon}{2} \notin [a, \infty)$. Thus $B_\varepsilon(a) \nsubseteq [a, \infty)$.

Question 7. What are the open sets in a discrete metric space (X, d)?

Complete the proof here

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As we might hope, the subsets we were calling *open* balls are indeed open.

Proposition 8 (open balls are open sets)**.** *Let (X,d) be a metric space, $x \in X$ be a point, $r \in (0,\infty)$ be a non-negative real number. The subset $B_r(x) \subseteq X$ is open.*

Complete the proof here

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Corollary 9 (open intervals are open sets)**.** *In the Euclidean metric space $\mathbb{R}$, all open intervals (a,b) are open.*

Complete the proof here

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It turns out that open sets can be combined in certain ways and the result is always again an open set.

Theorem 10 (open set laws)**.** *In a metric space (X, d),*

1. *X and $\emptyset$ are open sets.*
2. *If $\mathcal{F}$ is a family of open sets in X then $\cup_{U \in \mathcal{F}} U$ is open.*
3. *If $U, V \subseteq X$ are open sets then $U \cap V$ is open.*

Complete the proof here

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Surprise 11 (*intersection of opens is not generally open*)

In the Euclidean metric space $\mathbb{R}$, the subset $I :\equiv \bigcap_{n \in \mathbb{N}} \left(0, \frac{n+2}{n+1}\right) \subseteq \mathbb{R}$ is not open.

Compute I and prove this fact.

Complete the proof here

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