301 PS3 Answers

Exercise 1

1. $\forall a \in A, \exists b \in B, a \mapsto b$
2. $\forall a \in A, \forall b, b' \in B, (a \mapsto b \wedge a \mapsto b') \to b = b'$

Exercise 2

(a) Here the would be function is the assignment of $x \in \mathbb{Z}$ to the number $-2x \in \mathbb{Z}$. As we can see, it is certainly the case that for every $x \in \mathbb{Z}$ the number $-2x \in \mathbb{Z}$ is assigned, and moreover a given x is sent to precisely one element of $\mathbb{Z}$. Thus this is a function.

(b) As there are infinitely many primes, it is the case that every $n \in \mathbb{N}$ is assigned to another number in $\mathbb{N}$. Moreover, the property "$(\text{n+1})^{\text{th}}$ prime number" uniquely specifies the output so this is a function.

(c) While it is the case that for those $y \in \mathbb{Q}$ assigned to an value of $1/y \in \mathbb{Q}$, the assignment is uniquely determined, it is not the case that every $y \in \mathbb{Q}$ is assigned such a number of $\mathbb{Q}$. Precisely one $y \in \mathbb{Q}$ cannot be assigned to a number $1/y \in \mathbb{Q}$, viz., $y = 0 \in \mathbb{Q}$. Thus this is not a function.

(d) While it is the case that every $m \in \mathbb{N}$ has some $d \in \mathbb{N}$ assigned, it is not the case that this assignment satisfies our uniqueness criterion. For example, $m = 42 \in \mathbb{N}$ is assigned to all of 1,2, 3,6,7,14,21, and 42 in $\mathbb{N}$ and so this is not a function.

Exercise 3

(a) We count the elements as follows: a given element $a \in A$ may be paired with every $b \in B$ and so there are β many partners for $a \in A$. There are α many different $a \in A$ so that the number of elements of $A \times B$ is $\alpha\beta$.

(b) An element $x \in A \sqcup B$ is either an element $x \in A$ xor an element $x \in B$ – there is no overlap. As such there are $\alpha + \beta$ many elements in $A \sqcup B$.

(c) To define a function, as we have seen, we must specify for every input a unique output. Thus to give a function $f : A \to B$ me must specify a single $b \in B$ for each $a \in A$. For a fixed $a \in A$ there are β many choices of the assignment $a \mapsto b$, and we must make one such α many times. That is, we have $\underbrace{\beta \cdot \beta \cdots \beta}_{\alpha \text{ many times}}$ and so there are β^α many functions $A \to B$.

As we can see, the operations of $\times$, $\sqcup$, and exponentiation perform the operations of $\times$, $+$, and exponentiation on the sizes of finite sets and so we may more suggestively write $A + B$ for $A \sqcup B$, in line with the rest of our notation.

Exercise 4

(a) We define the assignment $\chi_S : A \to 2$ as follows: $a \mapsto \begin{cases} \top, & a \in S \\ \bot, & a \notin S \end{cases}$

As we can see this does determine a function for every $a \in A$ is assigned an element of 2 and moreover this assignment is unique.

(b) Now we must define a function assigning $S \in P(A)$ to a function $A \to 2$. That means for $S \in P(A)$, or equivalently $S \subseteq A$, we may assigned the element $\chi_S \in 2^A$ defined in part (a) – we proved already that χ_S is a function and so it is an element of the set 2^A. We can see that the way in which the assignment is done guarantees that every element $S \in P(A)$ is assigned to some element of 2^A and that this assignment is unique. Thus we have defined a function $\chi : P(A) \to 2^A$ via $\chi(S) = \chi_S$.

(c) To prove that this determines a bijection we will define a function $k : 2^A \to P(A)$ and demonstrate that $k \circ \chi = \mathrm{id}_{P(A)}$ and $\chi \circ k = \mathrm{id}_{2^A}$.

Our function k is the assignment $f \in 2^A \mapsto \{a \in A \mid f(a) = \top\}$. We note that $\{a \in A \mid f(a) = \top\} \subseteq A$ by construction and so $\{a \in A \mid f(a) = \top\} \in P(A)$ as desired, and moreover this assignment works for every f and is uniquely determined. Thus $k : 2^A \to P(A)$ is a function.

Now we check $k \circ \chi = \mathrm{id}_{P(A)}$. Given $S \in P(A)$ we see that $(k \circ \chi)(S) = k(\chi(S)) = k(\chi_S) = \{a \in A \mid \chi_S(a) = \top\}$. If we unwind the definition of χ_S we see that $\chi_S(a) = \top \leftrightarrow a \in S$ and so we may simplify $\{a \in A \mid \chi_S(a) = \top\} = \{a \in A \mid a \in S\} = S$ – that is, $(k \circ \chi)(S) = S$ as desired.

Finally we check that $\chi \circ k = \mathrm{id}_{2^A}$. Given $f \in 2^A$ we have $(\chi \circ k)(f) = \chi(k(f)) = \chi_{\{a \in A \mid f(a) = \top\}}$. If we want to prove that $\chi(k(f)) = \mathrm{id}_{2^A}(f) = f$ we must now compare the two functions $\chi(k(f))$ and f for equality. This means ensuring that they have the same values on every element $a \in A$. Thus we expand

$$\chi_{\{a \in A \mid f(a) = \top\}}(x) = \begin{cases} \top, & x \in \{a \in A \mid f(a) = \top\} \\ \bot, & x \notin \{a \in A \mid f(a) = \top\} \end{cases}$$

Looking at this we notice that $x \in \{a \in A \mid f(a) = \top\} \leftrightarrow f(x) = \top$ and similarly $x \notin \{a \in A \mid f(a) = \top\} \leftrightarrow \neg(f(x) = \top)$. Now we appeal to a decision procedure on 2 to transform $\neg(f(x) = \top)$ into the logically equivalent $f(x) = \bot$. With these simplifications we may rewrite our expanded χ above as

$$\chi_{\{a \in A \mid f(a) = \top\}}(x) = \left\{ \begin{matrix} \top, & f(x) = \top \\ \bot, & f(x) = \bot \end{matrix} \right\} = f(x)$$

Whence $\chi(k(f)) = \mathrm{id}_{2^A}(f)$ and so k, χ witness a bijection between $P(A)$ and 2^A.

Exercise 5

Let us define $p : A \times A \to A^2$ as the assignment taking (a, b) to the function $p_{(a,b)} : 2 \to A$ defined via $l \mapsto a$ and $r \mapsto b$. We may verify that $p_{(a,b)}$ is a function and that so too is p – every element $(a, b) \in A \times A$ is assigned to a unique element $p_{(a,b)} \in A^2$.

Now let us define $q : A^2 \to A$ as the assignment $f \mapsto (f(l), f(r))$ – again seen to be a function. With these two functions in hand we will prove that $p \circ q = \mathrm{id}_{A^2}$ and $q \circ p = \mathrm{id}_{A \times A}$ thereby completing the proof.

On $f \in A^2$ we see that $(p \circ q)(f) = p(q(f)) = p(f(l), f(r)) = p_{(f(l), f(r))}$. As in the previous exercise to show that $p_{(f(l), f(r))} = f$ we must compare these two functions for equality on every element $x \in 2$.

$$p_{(f(l), f(r))}(x) = \left\{ \begin{matrix} f(l), & x = l \\ f(r), & x = r \end{matrix} \right\} = f(x)$$

Similarly, on (a,b) we have $(q\circ p)(a,b) = q(p(a,b)) = q(p_{(a,b)}) = (p_{(a,b)}(l), p_{(a,b)}(r)) = (a,b)$ so that we have proven the existence of a bijection between $A\times A$ and A^2.

§2.3

9. Let $I = \{0,1\}$, $A_0 = \{a\} = B_1$, $A_1 = \{b\} = B_0$, with $a \neq b$. Then $A_0 \cap B_0 = A_1 \cap B_1 = \emptyset$ so that $\emptyset = \cup_I (A_i \cap B_i) \neq (\cup_I A_i) \cap (\cup_I B_i) = \{a,b\} \cap \{a,b\} = \{a,b\}$.

10. $x \in P(A\cap B) \leftrightarrow x \subseteq A \cap B \leftrightarrow (x \subseteq A) \wedge (x \subseteq B) \leftrightarrow (x \in P(A)) \wedge (x \in P(B)) \leftrightarrow x \in P(A) \cap P(B)$, and thus $P(A\cap B) = P(A) \cap P(B)$ for they have the same elements.

12 (a) $x \in \cup_I (A_i \cup B_i) \leftrightarrow \exists i \in I, x \in (A_i \cup B_i) \leftrightarrow \exists i \in I, (x \in A_i \vee x \in B_i) \leftrightarrow (\exists i \in I, x \in A_i) \vee (\exists i \in I, x \in B_i) \leftrightarrow (x \in \cup_I A_i) \vee (x \in \cup_I B_i) \leftrightarrow x \in (\cup_I A_i) \cup (\cup_I B_i)$.

(b) $x \in (\cap \mathbb{F}) \cap (\cap \mathbb{G}) \leftrightarrow (x \in \cap \mathbb{F}) \wedge (x \in \cap \mathbb{G}) \leftrightarrow (\forall F \in \mathbb{F}, x \in F) \wedge (\forall G \in \mathbb{G}, x \in G) \leftrightarrow (\forall S \in \mathbb{F} \cup \mathbb{G}, x \in S) \leftrightarrow x \in \cap(\mathbb{F} \cup \mathbb{G})$

(c) $x \in \cap_i (A_i \backslash B_i) \leftrightarrow (\forall i \in I, x \in A_i \backslash B_i) \leftrightarrow (\forall i \in I, (x \in A_i) \wedge (x \notin B_i)) \leftrightarrow (\forall i \in I, x \in A_i) \wedge (\forall i \in I, x \notin B_i) \leftrightarrow (\forall i \in I, x \in A_i) \wedge (x \notin \cup_I B_i) \leftrightarrow (x \in \cap_I A_i) \wedge (x \notin \cup_I B_i) \leftrightarrow x \in (\cap_I A_i) \setminus (\cup_I B_i)$

§3.2

Omitted.