301 PS2 Answers

Exercise 1

(a) (i) $P \leftrightarrow Q \equiv (P \to Q) \wedge (Q \to P)$. On the other hand we know that $A \to B \overset{\text{l.e.}}{=\!=} \neg A \vee B$ so that the contrapositive $(\neg P \to \neg Q) \overset{\text{l.e.}}{=\!=} \neg\neg P \vee \neg Q \overset{\text{l.e.}}{=\!=} P \vee \neg Q \overset{\text{l.e.}}{=\!=} \neg Q \vee P \overset{\text{l.e.}}{=\!=} (Q \to P)$ by Double Negation Elimination and commutativity. Thus $(P \leftrightarrow Q) \equiv (P \to Q) \wedge (Q \to P) \overset{\text{l.e.}}{=\!=} (P \to Q) \wedge (\neg P \to \neg Q)$.

(ii)

P	Q	$P \to Q$	$\neg P \to \neg Q$	$(P \to Q) \wedge (\neg P \to \neg Q)$	$P \leftrightarrow Q$
T	T	T	T	T	T
T	F	F	T	F	F
F	T	T	F	F	F
F	F	T	T	T	T

.

(b) (i) For this part we make use of the identity $P \to Q \overset{\text{l.e.}}{=\!=} \neg P \vee Q$ so that, using our above work, $(P \leftrightarrow Q) \overset{\text{l.e.}}{=\!=} (P \to Q) \wedge (\neg P \to \neg Q) \overset{\text{l.e.}}{=\!=} (\neg P \vee Q) \wedge (\neg\neg P \wedge \neg Q) \overset{\text{l.e.}}{=\!=} (\neg P \vee Q) \wedge (\neg Q \vee P)$, where the last equivalence follows from Double Negation Elimination and commutativity. Now we expand our statement using distrubitivity, $(\neg P \vee Q) \wedge (\neg Q \vee P) = ((\neg P \vee Q) \wedge \neg Q) \vee ((\neg P \vee Q) \wedge P)$. By further applications of distributivity we expand to

$$((\neg P \wedge \neg Q) \vee (Q \wedge \neg Q)) \vee ((\neg P \wedge P) \vee (Q \wedge P))$$

Now both $P \wedge \neg P$ and $Q \wedge \neg Q$ are false, but are introduced as disjuncts, so we may disregard them as $A \vee \mathrm{F} \overset{\text{l.e.}}{=\!=} A$ and arrive at our desired expression $(\neg P \wedge \neg Q) \vee (P \wedge Q)$ (applying commutativity as necessary).

(ii)

P	Q	$P \wedge Q$	$\neg P \wedge \neg Q$	$(P \wedge Q) \vee (\neg P \wedge \neg Q)$	$P \leftrightarrow Q$
T	T	T	F	T	T
T	F	F	F	F	F
F	T	F	F	F	F
F	F	F	T	T	T

.

§1.5

3 Let P be “it is raining”, Q be “it is windy” and R be ”the sun is shining” then we can evaluate the sentences as

(a) $P \to (Q \wedge \neg R)$

(b) $P \to (Q \wedge \neg R)$

(c) $P \to (Q \wedge \neg R)$

(d) $(Q \wedge \neg R) \to P$

(e) $(\neg Q \vee R) \to \neg P$

(f) $P \to (Q \wedge \neg R)$

(g) $(P \to Q) \vee (P \to \neg R)$

From these we can see that only (d) is the converse of (a), and all the others are equivalent with (e) being the contrapositive.

10. (a) and (b) are equivalent, as they are both logically equivalent to $\neg(P \wedge Q) \vee R$. We also have that (c) and (e) are equivalent, as they are both logically equivalent to $\neg P \vee (Q \wedge R)$. Finally (d) is not equivalent to any of the others. This may be shown using truth tables, Venn diagrams, or logical equivalence of formulae and laws.

§2.1

3 (a) Both x and y are free, and the statement is of the form $\forall z \in \mathbb{R}, (z > x) \to (z > y)$.

(b) There are no free variables, and the statement is $\forall a \in \mathbb{R}, (\exists x \in \mathbb{R}, ax^2 + 4x - 2 = 0) \leftrightarrow (a \geq -2)$.

(c) There are no free variables, and the statement is $\forall x \in \mathbb{R}, (x^3 - 3x < 3) \to (x < 10)$.

(d) Only w is free, and the statement is $(\exists x \in \mathbb{R}, x^2 + 5x = w) \wedge (\exists y \in \mathbb{R}, 4 - y^2 = w) \to (-10 < w < 10)$.

7 (a) This is true, we may take $y = 2x$.

(b) This is false, there is no single y which is equal to every non-negative even number. If there was, then $2 = 4$ as a result.

(c) This is false too, $1 = 2y$ has no solution for $y \in \mathbb{N}$.

(d) This is true, if $y < x < 10$ then in $\mathbb{N}$ $y < 9$ in particular (y could be less still, but we don't mind).

(e) This is true, we may take $y = z = 50$.

(f) This is false, if we take $x = 100$ then $y \geq 101$ and there is no $z \in \mathbb{N}$ for which $y + z = 100$.

8 (a) This is still true, take $y = 2x$.

(b) This is still false, for the same reasons.

(c) This is now true, take $y = x/2$.

(d) This is now false, take $x = 9.5$ and $y = 9.4$.

(e) This is still true, for the same reason as before.

(f) This is now true, we can always take $y = x + 1$ and $z = 100 - y$.

9 (a) This is still true, take $y = 2x$.

(b) This is still false, for the same reasons.

(c) This is now false again, $1 = 2y$ has no solution for $y \in \mathbb{Z}$.

(d) This is now true again, for the same reason as it was originally.

(e) This is still true, for the same reason as before.

(f) This is now true, for the same reason as the case for $\mathbb{R}$.

§2.2

3 (a) To see that whether this is true, we will assume that $x < 7$ (if it's not the implication is automatically true) and see if we can find a, b, and c for which $a^2 + b^2 + c^2 = x$. Fortunately there are only seven cases to check: $0 = 0^2 + 0^2 + 0^2$, $1 = 1^2 + 0^2 + 0^2$, $2 = 1^2 + 1^2 + 0^2$, $3 = 1^2 + 1^2 + 1^2$, $4 = 2^2 + 0^2 + 0^2$, $5 = 2^2 + 1^2 + 0^2$, and $6 = 2^2 + 1^2 + 1^2$. So it's true, but had we instead asked for $x \leq 7$ then it would have been false!

(b) This is false. While there certainly exists a natural number x such that $(x-4)^2 = 9$, it is not unique as both 1 and 7 work.

(c) This is true. There exists a natural x solving this equation, namely 9, and it is the only natural solution to this equation: the only other solution is -1, but this is not a natural number. Thus if there is another natural number which solves the equation, it must be 9 and so equal to x .

(d) This is true, we may take $x = y = 9$. Note that we are not asked for distinct x and y, otherwise this would not be possible by (c).

9. These are not the same. The truth $\forall x \in X, (P(x) \vee Q(x))$ means that for every x at least one of $P(x)$ and $Q(x)$ holds for the same value of x. On the other hand, $(\forall x \in X, P(x)) \vee (\forall x \in X, Q(x))$ is now claim that at least one of "for every x, $P(x)$ is true" or "for every x, $Q(x)$ is true" is true.

The difference is that in the second case it's not good enough for $P(x)$ to be true for some x and $Q(x)$ to be true for the others, or visa versa. For it to be true at least one of them must be true for every x.

To illustrate this, consider the case of $P(x)$ as "$x < 163$" and $Q(x)$ as "$x \geq 163$", over $x \in \mathbb{N}$. Thus the claim $\forall x \in \mathbb{N}, (x < 163) \vee (x \geq 163)$ is true, but in fact both $\forall x \in \mathbb{N}, x < 163$ and $\forall x \in \mathbb{N}, x \geq 163$ are false individually.