301 PS1 Answers

§1.2

3 (a) The truth table is

P	Q	$P+Q$
T	T	F
T	F	T
F	T	T
F	F	F

.

(b) $P+Q$ is logically equivalent to $(P\vee Q)\wedge(\neg(P\wedge Q))$, read as"(either P or Q) and (not (both P and Q))". The truth table may be derived as

P	Q	$P\vee Q$	$\neg(P\wedge Q)$	$(P\vee Q)\wedge(\neg(P\wedge Q))$
T	T	T	F	F
T	F	T	T	T
F	T	T	T	T
F	F	F	T	F

.

5(b) $P\downarrow Q$ is logically equivalent to $\neg(P\vee Q)$, "not (either P or Q)".

6(b) $P\mid Q$ is logically equivalent to $\neg(P\wedge Q)$, "not (both P and Q)".

12 (a) We first apply a DeMorgan Law and then Double Negation Elimination to the left term to find the logically equivalent $\neg(\neg(P\vee Q))\overset{\text{l.e.}}{=\!=}(\neg\neg P)\wedge\neg Q\overset{\text{l.e.}}{=\!=}P\wedge\neg Q$. Thus $\neg(\neg P\vee Q)\vee(P\wedge\neg R)\overset{\text{l.e.}}{=\!=}(P\wedge\neg Q)\vee(P\wedge\neg R)$ which we recognise as the result of applying distributivity to $P\wedge(\neg Q\vee\neg R)$. The right conjunct is itself amenable to another DeMorgan Law so overall we have found the logically equivalent reduction $P\wedge\neg(Q\wedge R)$.

(b) A similar exercise.

(c) We begin by applying distributivity in the right disjunct, followed by associativity and commutativity

$$(P\wedge R)\vee(\neg R\wedge(P\vee Q))\overset{\text{l.e.}}{=\!=}(P\wedge R)\vee((\neg R\wedge P)\vee(\neg R\wedge Q))\overset{\text{l.e.}}{=\!=}((P\wedge R)\vee(P\wedge\neg R))\vee(\neg R\wedge Q)$$

We then recognise $(P\wedge R)\vee(P\wedge\neg R)\overset{\text{l.e.}}{=\!=}P\wedge(R\vee\neg R)$ by distributivity and further $R\vee\neg R$ as true by the excluded middle. Thus we have reduced our expression to $P\vee(\neg R\wedge Q)$.

15. We will adopt the convention that a statement containing no letters has a single line in its truth table. For example, the statement "T" has the following truth table

T
T

.

Now suppose our statement had n letters in it and we added another letter to it; for example our statement was $P\vee Q$ and we moved to $(P\vee Q)\vee R$. The addition of this letter doubles the amount of rows we had before, for every row must now be reconsidered in two new contexts: one in which the new letter (R) is true, and one in which it is false. We must still consider all the previous rows in the n letter table, and so the total number of rows for $n+1$ letters is precisely double that for n.

Starting with our base case of zero letters having one row, we see that 1 letter has $2 = 2*1$ rows, 2 letters have $4 = 2*2$ rows, 3 letters have $8 = 2*4$ rows, and so on, so that n letters have 2^n rows.

Pierce's Law

(a) We apply a DeMorgan Law to see that $\neg(\neg P \vee Q) \vee P \overset{\text{l.e.}}{=\!=} (\neg\neg P \wedge \neg Q) \vee P$ and then a Double Negation Elimination to find $(\neg\neg P \wedge \neg Q) \vee P \overset{\text{l.e.}}{=\!=} (P \wedge \neg Q) \vee P$.

(b) First the truth table,

P	Q	$P \wedge \neg Q$	$(P \wedge \neg Q) \vee P$
T	T	F	T
T	F	T	T
F	T	F	F
F	F	F	F

.

From this table we see that it doesn't seem to matter whether Q is true. Later on we will be able to understand this in terms of implications, $(P \wedge \neg Q) \to P$ so that we have $(P \wedge \neg Q) \vee P \to P \vee P \overset{\text{l.e.}}{=\!=} P$ and conversely $P \to (P \wedge \neg Q) \vee P$ so that $P \leftrightarrow (P \wedge \neg Q) \vee P$.

§1.3

1. (a) Let $D(n,m)$ be the statement "n divides m", then our statement is $D(3,6) \wedge D(3,9) \wedge D(3,15)$.
   (b) Let $D(n,m)$ be as above, then our statement is $D(2,x) \wedge D(3,x) \wedge \neg D(4,x)$.
   (c) Let $N(n)$ be the statement "n is a natural number" and $P(n)$ be "p is prime". Then our statement is $N(x) \wedge N(y) \wedge (P(x) + P(y))$ (where + is exclusive or).

4. (a) $\{n \in \mathbb{N} \mid n \text{ is a square}\}$.
   (b) $\{n \in \mathbb{N} \mid n \text{ is a power of } 2\}$.
   (c) $\{n \in \mathbb{N} \mid 9 < n < 20\}$

8. (a) $\{x \in \mathbb{R} \mid x^2 - 4x + 3 = 0\} = \{1,3\}$.
   (b) $\{x \in \mathbb{R} \mid x^2 - 2x + 3 = 0\} = \emptyset$ as there are no real solutions to this equation.
   (c) $\{x \in \mathbb{R} \mid x^2 < 25\} = (-5,5)$ which we see by elaborating the constraint $5 \in \{y \in \mathbb{R} \mid x^2 < 50 - y^2\}$.

1. §1.4

7. (a) Let us write $A(x)$ for the statement whose truth set has $\{x \in X \mid A(x)\} = A$, and similarly for $B(x)$ and $C(x)$. Thus we see that $(A \cup B) \setminus C = \{x \in X \mid (A(x) \vee B(x)) \wedge \neg C(x)\}$. We may now leverage our knowledge of the calculus of logical statements to see that $(A(x) \vee B(x)) \wedge \neg C(x) \overset{\text{l.e.}}{=\!=} (A(x) \wedge \neg C(x)) \vee (B(x) \wedge \neg C(x))$ via distributivity. Thus we have

$$\begin{aligned}(A \cup B) \setminus C &= \{x \in X \mid (A(x) \vee B(x)) \wedge \neg C(x)\} \\ &= \{x \in X \mid (A(x) \wedge \neg C(x)) \vee (B(x) \wedge \neg C(x))\} \\ &= \{x \in X \mid A(x) \wedge \neg C(x)\} \cup \{x \in X \mid B(x) \wedge \neg C(x)\} \\ &= (A \setminus C) \cup (B \setminus C).\end{aligned}$$

(b) Adopting the same convention as above, $\{x \in X \mid A(x)\} = A$, we see that this time we are working with $A \cup (B \setminus C) = \{x \in X \mid A(x) \vee (B(x) \wedge \neg C(x)\}$. Once more we choose to concentrate on the statement in the elementhood test and reduce it. This time that means $A(x) \vee (B(x) \wedge \neg C(x)) \overset{\text{l.e.}}{=\!=} (A(x) \vee B(x)) \wedge (A(x) \vee \neg C(x))$ via distributivity. But this is not quite the desired form of the answer, we are looking for $(A \cup B) \setminus (C \setminus A)$, and so we apply a DeMorgan Law and Double Negation Elimination to the right conjunct to rewrite this as $(A(x) \vee B(x)) \wedge \neg(\neg A(x) \wedge \neg\neg C(x)) \overset{\text{l.e.}}{=\!=} (A(x) \vee B(x)) \wedge \neg(\neg A(x) \wedge C(x))$. Here we apply commutativity to come up with our final statement, $A(x) \vee (B(x) \wedge \neg C(x)) \overset{\text{l.e.}}{=\!=} (A(x) \vee B(x)) \wedge \neg(C(x) \wedge \neg A(x))$. Putting this together we have

$$\begin{aligned} A \cup (B \setminus C) &= \{x \in X \mid A(x) \vee (B(x) \wedge \neg C(x)\} \\ &= \{x \in X \mid (A(x) \vee B(x)) \wedge \neg(C(x) \wedge \neg A(x))\} \\ &= \{x \in X \mid A(x) \vee B(x)\} \setminus \{x \in X \mid C(x) \wedge \neg A(x)\} \\ &= (A \cup B) \setminus (C \setminus A). \end{aligned}$$

8. We adopt the same convention as Q7, and begin by writing out the sets as truth sets.

(a) $(A \setminus B) \setminus C = \{x \in X \mid (A(x) \wedge \neg B(x)) \wedge \neg C(x)\}$

(b) $A \setminus (B \setminus C) = \{x \in X \mid A(x) \wedge \neg(B(x) \wedge \neg C(x))\}$

(c) $(A \setminus B) \cup (A \cap C) = \{x \in X \mid (A(x) \wedge \neg B(x)) \vee (A(x) \wedge C(x))\}$

(d) $(A \setminus B) \cap (A \setminus C) = \{x \in X \mid (A(x) \wedge \neg B(x)) \wedge (A(x) \wedge \neg C(x))\}$

(e) $A \setminus (B \cup C) = \{x \in X \mid A(x) \wedge \neg(B(x) \vee C(x))\}$

Now the question reduces to determining which of the statements are logically equivalent. In this case: (a),(d), and (e) are equal, and (b) and (c) are equal.