Math 266x: Categorical Homotopy Theory
FINAL

Emily Riehl

1. Prove that any right Quillen functor has a right derived functor.

2. State and prove a formula for the geometric realization of an n-skeletal simplicial object in terms of its n-truncation.

3. Let X be a bisimplicial set. Define a map from its homotopy colimit, constructed using the two-sided bar construction, to its geometric realization and explain why this map is a weak equivalence.

4. Prove that the homotopy category of spaces is a closed symmetric monoidal category.

5. Let G be a topological group and let X be a continuous functor from G to a complete and cocomplete, tensored, cotensored, and topologically enriched category. Define what it would mean for an object to be an enriched homotopy limit and colimit of X and prove that such objects exist.

6. Suppose $\mathscr{M}$ is a tensored and cotensored $\mathscr{V}$-category and let $\mathscr{D}$ be small and unenriched. Show that $\mathscr{M}^{\mathscr{D}}$ is also a tensored and cotensored $\mathscr{V}$-category.

7. Describe the limit of a diagram $\mathbf{3} \to \mathbf{qCat}$ weighted by $N(\mathbf{3}/-)\colon \mathbf{3} \to \mathbf{sSet}$ as an ordinary limit. What are the vertices of this simplicial set?

8. Fix a ring R. Describe the functorial factorization produced by the $\mathbf{Ab}$-enriched algebraic small object argument on the category $\mathbf{Mod}_R$ with respect to the single generating arrow $0 \to R$.

9. Suppose $(\otimes, \{\}, \hom)\colon \mathscr{V} \times \mathscr{M} \to \mathscr{N}$ is a two-variable adjunction between categories with pullbacks and pushouts and let $(\mathcal{L}_1, \mathcal{R}_1)$, $(\mathcal{L}_2, \mathcal{R}_2)$, and $(\mathcal{L}_3, \mathcal{R}_3)$ be weak factorization systems on the categories $\mathscr{V}$, $\mathscr{M}$, and $\mathscr{N}$ respectively. State and prove two conditions that are equivalent to the assertion that $\mathcal{L}_1 \hat{\otimes} \mathcal{L}_2 \subset \mathcal{L}_3$.

10. Describe the simplicial categories $\mathfrak{C}(\Delta^1 \times \Delta^1)$ and $\mathfrak{C}\Delta^1 \times \mathfrak{C}\Delta^1$.

11. Show that right fibrations between ∞-categories reflect equivalences. Show furthermore that given any right fibration $p\colon X \to Y$ and equivalence $p(x) \to y$ in Y, this 1-simplex lifts to an equivalence $x \to x'$ in X.

12. Recall the definition of weak categorical equivalence forced by Joyal's model structure on $\mathbf{sSet}$. Show that weak categorical equivalences are precisely equivalences in the 2-category of quasi-categories. You don't need to prove each result you use to demonstrate this, but you should know which ones depend on which others.

Department of Mathematics, Harvard University, 1 Oxford Street, Cambridge, MA 02138
E-mail address: `eriehl@math.harvard.edu`