BASIC CONCEPTS IN CATEGORY THEORY

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1. Basic Definitions

Definition 1.1. A *category* $\mathcal{C}$ consists of:

(i) A collection of *objects* $\mathrm{ob}\,\mathcal{C}$ denoted by $A, B, C, \ldots$
(ii) A collection of *morphisms* $\mathrm{mor}\,\mathcal{C}$ denoted by $f, g, h, \ldots$
(iii) A rule assigning to each $f \in \mathrm{mor}\,\mathcal{C}$ two objects $\mathrm{dom}\, f$ and $\mathrm{cod}\, f$, its *domain* and *codomain*. We write $f : \mathrm{dom}\, f \to \mathrm{cod}\, f$ or $\mathrm{dom}\, f \xrightarrow{f} \mathrm{cod}\, f$.
(iv) For each pair (f, g) of morphisms with $\mathrm{cod}\, f = \mathrm{dom}\, g$ we have a composite morphism $gf : \mathrm{dom}\, f \to \mathrm{cod}\, g$ subject to the axiom $h(gf) = (hg)f$ whenever gf and hg are defined.
(v) For each object A we have an identity morphism $1_A : A \to A$, subject to the axioms $1_B f = f = f 1_A$ for all $f : A \to B$.

Definition 1.2. We say a category $\mathcal{C}$ is *small* if it has only a set of objects and a set of morphisms. We say a category $\mathcal{C}$ is *locally small* if for any two objects A, B of $\mathcal{C}$ the collection of all morphisms $A \to B$ in $\mathcal{C}$ is a set. We denote this set by $\mathcal{C}(A, B)$.

Given a category $\mathcal{C}$, its opposite category $\mathcal{C}^{\mathrm{op}}$ has the same objects but with the domain and codomain operations interchanged (and thus composition is reversed). Hence, when we prove a theorem in category theory, we simultaneously prove a "dual theorem," obtained by replacing all the categories involved with their opposites.

Definition 1.3. Let $\mathcal{C}$ and $\mathcal{D}$ be categories. A *functor* $F : \mathcal{C} \to \mathcal{D}$ consists of

(i) a mapping $A \mapsto FA : \mathrm{ob}\,\mathcal{C} \to \mathrm{ob}\,\mathcal{D}$
(ii) a mapping $f \mapsto Ff : \mathrm{mor}\,\mathcal{C} \to \mathrm{mor}\,\mathcal{D}$

such that $\mathrm{dom}\, Ff = F(\mathrm{dom}\, f)$, $\mathrm{cod}\, Ff = F(\mathrm{cod}\, f)$, $F(1_A) = 1_{FA}$, and $F(gf) = (Fg)(Ff)$ whenever gf is defined in $\mathcal{C}$.

Definition 1.4. Let $\mathcal{C}, \mathcal{D}$ be two categories and $F, G : \mathcal{C} \rightrightarrows \mathcal{D}$ two functors. A *natural transformation* $\alpha : F \to G$ consists of a mapping $A \mapsto \alpha_A$ $\mathrm{ob}\,\mathcal{C} \to \mathrm{mor}\,\mathcal{D}$ such that $\alpha_A : FA \to GA$ for all A and

$$\begin{array}{ccc} FA & \xrightarrow{\alpha_A} & GA \\ {\scriptstyle Ff}\downarrow & & \downarrow{\scriptstyle Gf} \\ FB & \xrightarrow{\alpha_B} & GB \end{array}$$

commutes for any $f : A \to B$ in $\mathcal{C}$.

Note that, given another functor H and another transformation $\beta : G \to H$ we can form the composite $\beta\alpha$ defined by $(\beta\alpha)_A = \beta_A \alpha_A$. The composition is associative and has identities so we have a category $[\mathcal{C}, \mathcal{D}]$ of functors $\mathcal{C} \to \mathcal{D}$ and natural transformations between them.

Definition 1.5. We say a morphism $f : A \to B$ is a *monomorphism* if $fg = fh \Rightarrow g = h$ for all $g, h : C \to A$. Dually, f is an *epimorphism* if $kf = \ell f \Rightarrow k = \ell$ for all $k, \ell : B \to C$.

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2. Limits and Colimits

Definition 2.1. Let J be a category (almost always small or finite). By a *diagram of shape* J we mean a functor $D : J \to \mathcal{C}$. The objects $D(j)$ for $j \in \operatorname{ob} J$ are called *vertices* of D and the morphisms $D(\alpha)$ for $\alpha \in \operatorname{mor} J$ are called *edges* of D.

For any object A of $\mathcal{C}$ and any J we have a *constant diagram* ΔA of shape J all of whose vertices are A and all of whose edges are 1_A. By a cone over $D : J \to \mathcal{C}$ with *summit* A we mean a natural transformation $\lambda : \Delta A \to D$. Equivalently, this is a family $(\lambda_j : A \to D(j) \,|\, j \in \operatorname{ob} J)$ of morphisms (the *legs* of the cone) such that

$$\begin{tikzcd} & A \arrow[dl, "\lambda_j"'] \arrow[dr, "\lambda_{j'}"] & \\ D(j) \arrow[rr, "D(\alpha)"] & & D(j') \end{tikzcd}$$

commutes for any $\alpha : j \to j'$ in J. Note that Δ is a functor $\mathcal{C} \to [J, \mathcal{C}]$.

Definition 2.2. A *limit* of a diagram $D : J \to \mathcal{C}$ is a cone (A, λ) that is universal among cones over D. That is, given another cone (B, δ) over D, there is a unique morphism $f : B \to A$ such that $\delta_j = \lambda_j f$ for each $j \in J$.

The dual notion is a *colimit*. A cone under $D : J \to \mathcal{C}$ with summit A is a natural transformation $\lambda : D \to \Delta A$. A *colimit* of D is a cone (A, λ) under D that is universal among cones under D. That is, given another cone (B, δ) under D, there is a unique morphism $f : A \to B$ such that $\delta_j = f \lambda_j$ for each $j \in J$.

Lemma 2.3. *Limits and colimits, when they exist, are unique up to unique isomorphism. That is, given two limits (A, λ) and (A', λ') of a diagram D, there exists unique isomorphisms $f : A \to A'$ and $g : A' \to A$ that commute with the legs of the cones.*

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Definition 2.5. We say that $\mathcal{C}$ *has limits* of shape J if every diagram $D : J \to \mathcal{C}$ has a limit. This is equivalent to saying that the functor $\Delta : \mathcal{C} \to [J, \mathcal{C}]$ has a right adjoint (see Section 4).

Examples 2.6.

(a) If $J = \emptyset$ then $[J, \mathcal{C}]$ has a unique object and the category of cones over it is isomorphic to $\mathcal{C}$. So a limit for this diagram is a *terminal object* $*$ of $\mathcal{C}$, i.e. one such that there is a unique morphism $A \to *$ for all A. A colimit for it is called an *initial object*.

(b) If J is a discrete category, a diagram of shape J is just a family of objects of $\mathcal{C}$, and a cone over it is a family of morphisms $(\lambda_j : A \to D(j) \,|\, j \in \operatorname{ob} J)$. A limit for it is a *product* $\prod_{j \in \operatorname{ob} J} D(j)$. Similarly a colimit for this diagram is a *coproduct* $\sum_{j \in \operatorname{ob} J} D(j)$.

(c) Let J be the finite category $\cdot \rightrightarrows \cdot$ (so a diagram of shape J is a parallel pair $A \underset{g}{\overset{f}{\rightrightarrows}} B$). A cone over such a digram is of the form $A \xleftarrow{h} C \xrightarrow{k} B$ such that $fh = k = gh$, or equivalently a morphism $h : C \to A$ satisfying $fh = gh$. A limit for the diagram is called an *equalizer* for (f, g) (and a colimit for it is a *coequalizer* for (f, g)).

(d) Let J be the finite category $\cdot \longleftarrow \cdot \longrightarrow \cdot$. Then a diagram of shape J is a pair of morphisms $B \xrightarrow{g} C \xleftarrow{f} A$ with common codomain. A cone over this has the form

$$\begin{tikzcd} D \arrow[r, "i"] \arrow[d, "k"'] \arrow[dr, "\ell"] & A \\ B & C \end{tikzcd}$$

satisfying $fh = \ell = gk$ or equivalently a completion of the diagram to a commutative square. A terminal such completion is called a *pullback* for the pair (f, g). If $\mathcal{C}$ has products and equalizers then it has pullbacks: form the product $A \times B$ and then the equalizer $E \xrightarrow{e} A \times B \overset{f\pi_1}{\underset{g\pi_2}{\rightrightarrows}} C$. Then

$$\begin{tikzcd} E \arrow[r, "\pi_1 e"] \arrow[d, "\pi_2 e"'] & A \\ B & \end{tikzcd} \quad \text{is a limit for} \quad \begin{tikzcd} & A \arrow[d, "f"] \\ B \arrow[r, "g"] & C \end{tikzcd}$$

A colimit of shape J^{op} (i.e. of a diagram $C \xleftarrow{g} A \xrightarrow{f} B$) is called a *pushout* of (f, g).

Theorem 2.7.

(i) *If $\mathcal{C}$ has equalizers and all small (resp. all finite) products, then $\mathcal{C}$ has all small (resp. all finite) limits.*
(ii) *If $\mathcal{C}$ has pullbacks and a terminal object, then $\mathcal{C}$ has all finite limits.*

Definition 2.8. Let $F : \mathcal{C} \to \mathcal{D}$ be a functor, J a (small) category.

- We say F *preserves limits* of shape J if, given $D : J \to \mathcal{C}$ and a limit cone $(\lambda_j : L \to D(j) \,|\, j \in \mathrm{ob}\, J)$ the cone $(F\lambda_j : FL \to FD(j) \,|\, j \in \mathrm{ob}\, J)$ is a limit cone for FD in $\mathcal{D}$.
- We say F *reflects limits* of shape J if given $D : J \to \mathcal{C}$ and a cone $(\lambda_j : L \to D(j) \,|\, j \in \mathrm{ob}\, J)$ such that $(F\lambda_j : FL \to FD(j) \,|\, j \in \mathrm{ob}\, \mathcal{C})$ is a limit for FD, then the original cone was a limit for D.
- We say that F *creates limits* of shape J if, given $D : J \to \mathcal{C}$ and a limit $(\mu_j : M \to FD(j) \,|\, j \in \mathrm{ob}\, J)$ for FD, there exists a cone $(\lambda_j : L \to D(j) \,|\, j \in \mathrm{ob}\, J)$ over D mapping to a limit for FD, and any such cone is a limit in $\mathcal{C}$. (Note that if we require M to be in the image of F then category equivalences might not create limits, as M may not be in the image of the equivalence. This definition says that if there is a limit for FD in $\mathcal{D}$ then there is a limit for D in $\mathcal{C}$ that maps to a limit of FD in $\mathcal{D}$.)

Corollary 2.9. *Let $F : \mathcal{C} \to \mathcal{D}$ be a functor. In any version of the above theorem 2.7 we may replace "$\mathcal{C}$ has" by either "$\mathcal{C}$ has and F preserves" or "$\mathcal{D}$ has and F creates."*

3. Equivalences of Categories

Definition 3.1. Let $\mathcal{C}$ and $\mathcal{D}$ be two categories. By an *equivalence* between $\mathcal{C}$ and $\mathcal{D}$ we mean a pair of functors $F : \mathcal{C} \to \mathcal{D}$, $G : \mathcal{D} \to \mathcal{C}$ together with natural isomorphisms $\alpha : 1_C \to GF$ and $\beta : FG \to 1_{\mathcal{D}}$. We write $\mathcal{C} \simeq \mathcal{D}$ if there exists an equivalence between $\mathcal{C}$ and $\mathcal{D}$.

Definition 3.2. Let $F : \mathcal{C} \to \mathcal{D}$ be a functor.

(i) We say F is *faithful* if given any two objects $A, B \in \mathcal{C}$ and two morphisms $f, g : A \to B$ $Ff = Fg$ implies $f = g$.
(ii) We say F is *full* if given any two objects $A, B \in \mathcal{C}$ every morphisms $g : FA \to FB$ n $\mathcal{D}$ is of the form Ff for some $f : A \to B$ in $\mathcal{C}$.
(iii) We say a subcategory $\mathcal{C}'$ of $\mathcal{C}$ is *full* if the inclusion $\mathcal{C}' \to \mathcal{C}$ is a full functor.

Lemma 3.3. *(Assuming the axiom of choice.) A functor $F : \mathcal{C} \to \mathcal{D}$ is part of an equivalence iff it is full, faithful and essentially surjective on objects. (i.e. every $B \in \mathrm{ob}\, \mathcal{D}$ is isomorphic to some FA).*

4. Adjunctions

Definition 4.1. Suppose we are given categories $\mathcal{C}, \mathcal{D}$ and functors $F : \mathcal{C} \to \mathcal{D}$, $G : \mathcal{D} \to \mathcal{C}$. We say that F is *left adjoint* to G or G is *right adjoint* to F we're given, for each $A \in \mathrm{ob}\, \mathcal{C}$ and each $B \in \mathrm{ob}\, \mathcal{D}$ a bijection between morphisms $FA \to B$ in $\mathcal{D}$ and morphisms $A \to GB$ in $\mathcal{C}$, which is natural in A and B. (If $\mathcal{C}$ and $\mathcal{D}$ are locally small this means that the functors $\mathcal{C}^{\mathrm{op}} \times \mathcal{D} \to \mathbf{Set}$ sending (A, B) to $\mathcal{D}(FA, B)$ and to $\mathcal{C}(A, GB)$ are naturally isomorphic.) We write $(F \dashv G)$ if F is left adjoint to G.

We call the corresponding morphisms $h : FA \to B$, $\hat{h} : A \to GB$ of an adjunction *adjuncts*. Note that the naturality condition means that

$$\begin{tikzcd} FA \arrow[r, "h"] \arrow[d, "Ff"'] & B \arrow[d, "g"] \\ FC \arrow[r, "j"] & D \end{tikzcd} \quad \text{commutes iff} \quad \begin{tikzcd} A \arrow[r, "\hat{h}"] \arrow[d, "f"'] & GB \arrow[d, "Gg"] \\ C \arrow[r, "\hat{j}"] & GD \end{tikzcd} \quad \text{commutes.}$$

Given functors $F : \mathcal{C} \leftrightarrow \mathcal{D} : G$ with $(F \dashv G)$ we have a natural transformation $\eta : 1_{\mathcal{C}} \to GF$ and dually a natural transformation $\epsilon : FG \to 1_{\mathcal{D}}$ (the *counit* of the adjunction), where η_A is defined to be the adjunct of 1_{FA} and ϵ_B is the adjunct of 1_{GB}.

Theorem 4.2. *Given functors $F : \mathcal{C} \rightleftarrows \mathcal{D} : G$, specifying an adjunction $F(\dashv G)$ is equivalent to specifying natural transformations $\eta : 1_{\mathcal{C}} \to GF$ and $\epsilon : FG \to 1_{\mathcal{D}}$ satisfying the triangular identities:*

$$\begin{tikzcd} F \arrow[r, "F\eta"] \arrow[dr, "1_F"'] & FGF \arrow[d, "\epsilon_F"] \\ & F \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} G \arrow[r, "\eta_G"] \arrow[dr, "1_G"'] & GFG \arrow[d, "G\epsilon"] \\ & G \end{tikzcd}$$

Theorem 4.3. *Left adjoints preserve colimits and right adjoints preserve limits.*

5. The Yoneda Lemma

If $\mathcal{C}$ is locally small then the mapping $B \to \mathcal{C}(A,B)$ becomes a functor $\mathcal{C}(A,-) : \mathcal{C} \to \mathbf{Set}$. Given a morphism $g : B \to C$ in $\mathcal{C}$, $\mathcal{C}(A,g) : \mathcal{C}(A,B) \to \mathcal{C}(A,B)$ sends $f \in \mathcal{C}(A,B)$ to gf. (Associativity of composition implies that this is a functor.) Similarly, $A \mapsto \mathcal{C}(A,B)$ defines a functor $\mathcal{C}(-,B) : \mathcal{C}^{\text{op}} \to \mathbf{Set}$.

Lemma 5.1 (Yoneda Lemma)**.**

(i) *Let $\mathcal{C}$ be a locally small category, $A \in \text{ob}\,\mathcal{C}$ and $F : \mathcal{C} \to \mathbf{Set}$ a functor. Then there is a bijection between natural transformations $\mathcal{C}(A,-) \to F$ and elements of FA.*

(ii) *Moreover, this bijection is natural in A and F.*

Corollary 5.2. *For a locally small category $\mathcal{C}$ there is a full and faithful functor $Y : \mathcal{C}^{\text{op}} \to [\mathcal{C}, \mathbf{Set}]$ (the Yoneda embedding) sending $A \in \text{ob}\,\mathcal{C}$ to $\mathcal{C}(A,-)$.*

To explain Yoneda (ii), suppose that $\mathcal{C}$ is small. Then $[\mathcal{C}, \mathbf{Set}]$ is locally small, since a natural transformation $F \to G$ is a set-indexed family of functions $\alpha_A : FA \to GA$. We have a functor $\mathcal{C} \times [\mathcal{C}, \mathbf{Set}] \to \mathbf{Set}$ sending (A,F) to FA, and another functor which is the composite

$$\mathcal{C} \times [\mathcal{C}, \mathbf{Set}] \xrightarrow{Y \times 1_{[\mathcal{C},\mathbf{Set}]}} [\mathcal{C}, \mathbf{Set}]^{\text{op}} \times [\mathcal{C}, \mathbf{Set}] \xrightarrow{[\mathcal{C},\mathbf{Set}](-,-)} \mathbf{Set}$$

(ii) is saying that these two functors are naturally isomorphic in each variable. Notice, however, that since the existence of a natural isomorphism is a purely "local" condition, we only need to require that the category be locally small.

Definition 5.3. We say that a functor $F : \mathcal{C} \to \mathbf{Set}$ is *representable* if it is naturally isomorphic to $\mathcal{C}(A,-)$ for some A. By a *representation* of F we mean a pair (A,x) where $A \in \text{ob}\,\mathcal{C}$ and $x \in FA$ is such that $\Psi(x) : \mathcal{C}(A,-) \to F$ is an isomorphism. We call x a *universal element* of F. It has the property that any $y \in FB$ is of the form $(Ff)(x)$ for some $f \in \mathcal{C}(A,B)$.