Math 161: Category theory in context
Problem Set 9
due: April 28, 2015

Emily Riehl

Exercise 1. Verify that the Kleisli category is a category by checking that the composition operation of Definition 5.2.8 is associative and unital.

Exercise 2. Describe the image of the canonical functor from the Kleisli category of a monad to the Eilenberg-Moore category.

Exercise 3. Recall a monoid is a set M together with maps $\eta\colon 1 \to M$ and $\mu\colon M \times M \to M$ so that certain diagrams commute in Set; see Definition 1.6.2. A monoid homomorphism is a function $f\colon M \to M'$ so that the diagrams

$$\begin{array}{ccc} 1 & & \\ {\scriptstyle\eta}\downarrow & \searrow^{\eta'} & \\ M & \xrightarrow[f]{} & M' \end{array} \qquad\qquad \begin{array}{ccc} M \times M & \xrightarrow{f \times f} & M' \times M' \\ {\scriptstyle\mu}\downarrow & & \downarrow{\scriptstyle\mu'} \\ M & \xrightarrow[f]{} & M' \end{array}$$

commute in Set. Prove that the functor $U\colon \mathrm{Monoid} \to \mathrm{Set}$ is monadic by appealing to the monadicity theorem.

Exercise 4. For any group G, the forgetful functor $\mathrm{Set}^G \to \mathrm{Set}$ admits a left adjoint that sends a set X to the G-set $G \times X$, with G acting on the left. Prove that this adjunction is monadic by appealing to the monadicity theorem.

Exercise 5. Generalizing Exercise 4, for any small category J and any cocomplete category C the forgetful functor $\mathsf{C}^{\mathsf{J}} \to \mathsf{C}^{\mathrm{ob}\,\mathsf{J}}$ admits a left adjoint $\mathrm{Lan}\colon \mathsf{C}^{\mathrm{ob}\,\mathsf{J}} \to \mathsf{C}^{\mathsf{J}}$ that sends a functor $F \in \mathsf{C}^{\mathrm{ob}\,\mathsf{J}}$ to the functor $\mathrm{Lan}F \in \mathsf{C}^{\mathsf{J}}$ defined by

$$\mathrm{Lan}F(j) = \coprod_{x \in \mathsf{J}} \coprod_{\mathsf{J}(x,j)} Fx.$$

(i) Define $\mathrm{Lan}F$ on morphisms in J.
(ii) Define Lan on morphisms in $\mathsf{C}^{\mathrm{ob}\,\mathsf{J}}$.
(iii) Use the Yoneda lemma to show that Lan is left adjoint to the forgetful (restriction) functor $\mathsf{C}^{\mathsf{J}} \to \mathsf{C}^{\mathrm{ob}\,\mathsf{J}}$.
(iv) Prove that this adjunction is monadic by appealing the monadicity theorem.

Exercise 6.* Describe a more general class of functors $K\colon \mathsf{I} \to \mathsf{J}$ between small categories so that for any cocomplete C the restriction functor $\mathrm{res}_K\colon \mathsf{C}^{\mathsf{J}} \to \mathsf{C}^{\mathsf{I}}$ strictly creates colimits of res_K-split parallel pairs. All such functors admit left adjoints and are thus monadic. As a further challenge, describe the left adjoint.

Dept. of Mathematics, Harvard University, 1 Oxford Street, Cambridge, MA 02138
E-mail address: `eriehl@math.harvard.edu`