Math 161: Category theory in context
Problem Set 8
due: April 21, 2015

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Exercise 1. Suppose C is a locally small category with coproducts. Show that a functor $F\colon \mathsf{C} \to \textsc{Set}$ is representable if and only if it admits a left adjoint.

Exercise 2. Suppose that $F\colon \mathsf{A} \times \mathsf{B} \to \mathsf{C}$ is a bifunctor so that for each object $a \in \mathsf{A}$, the induced functor $F(a, -)\colon \mathsf{B} \to \mathsf{C}$ admits a right adjoint $G_a\colon \mathsf{C} \to \mathsf{B}$.

(i) Show that these right adjoints assemble into a unique bifunctor $G\colon \mathsf{A}^{\mathrm{op}} \times \mathsf{C} \to \mathsf{B}$, defined so that $G(a, c) = G_a(c)$ so that the isomorphisms
$$\mathsf{C}(F(a, b), c) \cong \mathsf{B}(b, G(a, c))$$
are natural in all three variables.

(ii) Suppose further that for each $b \in \mathsf{B}$, the induced functor $F(-, b)\colon \mathsf{A} \to \mathsf{C}$ admits a right adjoint $H_b\colon \mathsf{C} \to \mathsf{A}$. Conclude that there is a unique bifunctor $H\colon \mathsf{B}^{\mathrm{op}} \times \mathsf{C} \to \mathsf{A}$ so that $H(b, c) = H_b(c)$ and the isomorphism
$$\mathsf{A}(a, H(b, c)) \cong \mathsf{C}(F(a, b), c) \cong \mathsf{B}(b, G(a, c)) \tag{1}$$
is natural in all three variables.

(iii) Conclude that for each $c \in \mathsf{C}$, the functors $G(-, c)\colon \mathsf{A}^{\mathrm{op}} \to \mathsf{B}$ and $H(-, c)\colon \mathsf{B}^{\mathrm{op}} \to \mathsf{A}$ are mutual right adjoints.

A triple of bifunctors F, G, and H equipped with a natural isomorphism (1) is called a **two-variable adjunction**.

Exercise 3. What are some examples of two-variable adjunctions?

Exercise 4. Suppose V is a category with a bifunctor $\otimes\colon \mathsf{V} \times \mathsf{V}$ that is associative up to coherent natural isomorphism.[1] Suppose also that V has countable coproducts and that the bifunctor $\otimes$ preserves them in each variable.[2] Show that $T(X) = \coprod_{n \geq 0} X^{\otimes n}$ defines a on V by defining natural transformations $\eta\colon 1_{\mathsf{V}} \Rightarrow T$ and $\mu\colon T^2 \Rightarrow T$ that satisfy the required conditions.

Exercise 5. Show that the functor $\beta\colon \textsc{Set} \to \textsc{Set}$ that carries a set to the set of ultrafilters on that set is a monad by defining unit and multiplication natural transformations that satisfy the unit and associativity laws.

Exercise 6. The adjunction associated to a reflective subcategory of C induces an **idempotent monad** on C. Prove that the following three characterizations of an idempotent monad (T, η, μ) are equivalent:

(i) The multiplication $\mu\colon T^2 \Rightarrow T$ is a natural isomorphism.
(ii) The natural transformations $\eta T, T\eta\colon T \Rightarrow T^2$ are equal.
(iii) Each component of $\mu\colon T^2 \Rightarrow T$ is a monomorphism.

Exercise 7. An adjunction gives rise to a monad on one category and a comonad on the other. Define a **comonad** on a category C and describe an example.

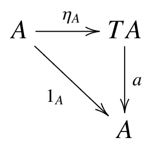
[1] Rather than worry about what this means, feel free to assume that there is a well-defined n-ary functor $\mathsf{V}^{\times n} \xrightarrow{\otimes^n} \mathsf{V}$ built from the bifunctor $\otimes$. Such a functor exists in any monoidal category.

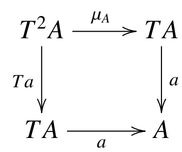
[2] In particular $(v \coprod v') \otimes (w \coprod w') \cong v \otimes w \coprod v' \otimes w \coprod v \otimes w' \coprod v' \otimes w'$.

Exercise 8. Explain how the notion of monad is dual ot the notion of comonad.

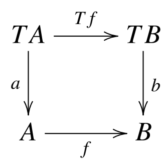
Exercise 9. Consider a monad (T, η, μ) on a category C. Define a category C^T of T-**algebras** in C as follows:

- objects are pairs $(A \in \mathsf{C}, a \colon TA \to A)$ so that the diagrams





  commute in C.
- morphisms $f \colon (A, a) \to (B, b)$ are maps $f \colon A \to B$ in C so that the diagram



  commutes in C.

(i) Define functors $U^T \colon \mathsf{C}^T \to \mathsf{C}$ and $F^T \colon \mathsf{C} \to \mathsf{C}^T$ and prove that they are adjoints.
(ii) What is the induced monad on C?

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