Math 161: Category theory in context
Problem Set 7
due: April 14, 2015

Emily Riehl

Exercise 1. Prove Proposition 4.3.3.

Exercise 2. Use the natural bijection between hom-sets to prove Proposition 4.3.4.

Exercise 3. Use the unit and counit associated to an adjunction to prove Proposition 4.3.4.

Exercise 4. Suppose the diagonal functor $\Delta\colon \mathcal{C} \to \mathcal{C}^{\mathcal{J}}$ admits both left and right adjoints. Describe the units and counits of these adjunctions.

Exercise 5. Prove Lemma 4.4.9.

Exercise 6. Consider a reflective subcategory inclusion $\mathcal{D} \hookrightarrow \mathcal{C}$ with reflector $R\colon \mathcal{C} \to \mathcal{D}$.

(i) Show that $\eta R = R\eta$, and that these natural transformations are isomorphisms.
(ii) Show that an object $c \in \mathcal{C}$ is in the **essential image** of the inclusion $\mathcal{D} \hookrightarrow \mathcal{C}$, i.e., is isomorphic to an object in the subcategory $\mathcal{D}$, if and only if η_c is an isomorphism.
(iii) Show that the essential image of $\mathcal{D}$ consists of those objects c that are **local** for the class of morphisms that is inverted by R. That is, c is in the essential image if and only if the precomposition functions

$$\mathcal{C}(b, c) \xrightarrow{f^*} \mathcal{C}(a, c)$$

are isomorphisms for all $f\colon a \to b$ in $\mathcal{C}$ for which Rf is an isomorphism in $\mathcal{D}$.

Dept. of Mathematics, Harvard University, 1 Oxford Street, Cambridge, MA 02138
E-mail address: `eriehl@math.harvard.edu`