Math 161: Category theory in context
Problem Set 5[1]
due: March 24, 2015

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Exercise 1. For a fixed diagram $F \in C^{\mathcal{J}}$, describe the actions of the cone functors $\mathrm{Cone}(-, F) \colon C^{\mathrm{op}} \to \mathrm{Set}$ and $\mathrm{Cone}(F, -) \colon C \to \mathrm{Set}$ on morphisms in C.

Exercise 2. Prove[2] that the category of cones over $F \in C^{\mathcal{J}}$ is isomorphic to the comma category $\Delta \downarrow F$ formed from the constant functor $\Delta \colon C \to C^{\mathcal{J}}$ and the functor $F \colon \mathbb{1} \to C^{\mathcal{J}}$. Argue by duality the category of cones under F is the comma category $F \downarrow \Delta$.

Exercise 3. Prove that if

$$E \xrightarrow{h} A \underset{g}{\overset{f}{\rightrightarrows}} B$$

is an equalizer diagram then h is a monomorphism.

Exercise 4. Prove that if

$$\begin{array}{ccc} P & \overset{k}{\rightarrowtail} & C \\ {\scriptstyle h}\downarrow & \lrcorner & \downarrow{\scriptstyle g} \\ B & \underset{f}{\rightarrowtail} & A \end{array}$$

is a pullback square and f is a monomorphism then k is a monomorphism.

Exercise 5. Show that if $\mathcal{J}$ has an initial object then the limit of any functor indexed by $\mathcal{J}$ is the value of that functor at an initial object.

Exercise 6. Suppose C has all $\mathcal{J}$-shaped (co)limits. Prove that for any small category $\mathcal{A}$, the functor category $C^{\mathcal{A}}$ again has all $\mathcal{J}$-shaped colimits, constructed pointwise. That is, given a diagram $F \colon \mathcal{J} \to C^{\mathcal{A}}$ show that it has a limit $\lim F \in C^{\mathcal{A}}$ whose value at $a \in \mathcal{A}$ is the limit of the diagram

$$\mathcal{J} \xrightarrow{F} C^{\mathcal{A}} \xrightarrow{\mathrm{ev}_a} C$$

Exercise 7. Show that if $K \colon C \to \mathcal{D}$ is an equivalence then for any $\mathcal{D}$-indexed functor F, if the limit of FK exists then it also defines a limit of and limit cone for F.

Exercise 8. Let C be a category admitting coproducts and coequalizers and consider a functor $F \colon \mathcal{J} \to C$. Define a parallel pair between two coproducts whose coequalizer will equal the colimit of F. You should also define the cone under this parallel pair with nadir $\mathrm{colim}\, F$, but you do not have to verify that it defines a coequalizer cone.

Exercise 9.* Prove that the diagram constructed in Exercise 8 is a colimit either by directly verifying the universal property or by appealing the Yoneda lemma and a theorem from class.

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[1]Problems labelled n* are optional (fun!) challenge exercises that will not be graded.
[2]Prove in this context means unpack the definitions and observe that they coincide.