**Math 161: Category theory in context**

Problem Set 4

due: March 10, 2015

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**Exercise 1.** Explain how duality can be used to convert the proof that a covariant functor is representable if and only if its category of elements has an initial object into a proof that a contravariant functor is representable if and only if its category of elements has a terminal object.

**Exercise 2.** Given functors $F\colon \mathcal{D} \to \mathcal{C}$ and $G\colon \mathcal{E} \to \mathcal{C}$, the **comma category** $F \downarrow G$ has as objects, triples $(d \in \mathcal{D}, e \in \mathcal{E}, f\colon Fd \to Ge \in \mathcal{C})$, and as morphisms $(d, e, f) \to (d', e', f')$, a pair of morphism $(h\colon d \to d', k\colon e \to e')$ so that the square

$$\begin{array}{ccc} Fd & \xrightarrow{f} & Ge \\ {\scriptstyle Fh}\downarrow & & \downarrow{\scriptstyle Gk} \\ Fd' & \xrightarrow[f']{} & Ge' \end{array}$$

commutes in $\mathcal{C}$. For example, $c \downarrow 1_{\mathcal{C}} \cong c/\mathcal{C}$ and $1_{\mathcal{C}} \downarrow c \cong \mathcal{C}/c$.

Show that for $F\colon \mathcal{C}^{\mathrm{op}} \to \mathrm{Set}$, the category of elements $\int F$ is isomorphic to the comma category $y \downarrow F$ of the Yoneda embedding $y\colon \mathcal{C} \to \mathrm{Set}^{\mathcal{C}^{\mathrm{op}}}$ sliced over the object $F\colon \mathbb{1} \to \mathrm{Set}^{\mathcal{C}^{\mathrm{op}}}$.

**Exercise 3.** Given $F\colon \mathcal{C} \to \mathrm{Set}$, show that $\int F$ is isomorphic to the comma category $* \downarrow F$ of the singleton set $*\colon \mathbb{1} \to \mathrm{Set}$ over the functor $F\colon \mathcal{C} \to \mathrm{Set}$.

**Exercise 4.** Suppose $F\colon \mathcal{C} \to \mathrm{Set}$ is equivalent to $G\colon \mathcal{D} \to \mathrm{Set}$ in the sense that there is an equivalence of categories $H\colon \mathcal{C} \to \mathcal{D}$ so that $GH$ and $F$ are naturally isomorphic.

(i) If $F$ is representable, then is $G$ representable?
(ii) If $G$ is representable, then is $F$ representable?

**Exercise 5.** Do there exist any non-identity natural endomorphisms of the category of spaces? I.e., does there exist any family of continuous maps $X \to X$, defined for all spaces $X$ and not all of which are identities, that are natural in all maps in the category Top?

**Exercise 6.** Fixing two objects $A, B$ in a locally small category $\mathcal{C}$, we define a functor

$$\mathcal{C}(-, A) \times \mathcal{C}(-, B)\colon \mathcal{C}^{\mathrm{op}} \to \mathrm{Set}$$

that carries an object $X$ to the set $\mathcal{C}(X, A) \times \mathcal{C}(X, B)$ whose elements are pairs of map $a\colon X \to A$ and $b\colon X \to B$ in $\mathcal{C}$. What would it mean for this functor to be representable?

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