Math 161: Category theory in context
Problem Set 3[1]
due: March 3, 2015

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Exercise 1. In the presence of a full and faithful functor $F\colon C \to \mathcal{D}$ prove that objects $x, y \in C$ are isomorphic if and only if Fx and Fy are isomorphic in $\mathcal{D}$.

Exercise 2. For each of the three functors

$$\mathbb{1} \underset{1}{\overset{0}{\rightrightarrows}} \mathbb{2}, \qquad \mathbb{2} \xrightarrow{!} \mathbb{1}$$

between the categories $\mathbb{1}$ and $\mathbb{2}$, describe the corresponding natural transformations between the covariant functors $\textsc{Cat} \to \textsc{Set}$ represented by these categories.

Exercise 3. A functor F defines a **subfunctor** of G if there is a natural transformation $\alpha\colon F \Rightarrow G$ whose components are monomorphisms. In the case of $G\colon C^{\mathrm{op}} \to \textsc{Set}$ a subfunctor is given by a collection of subsets $Fc \subset Gc$ so that each $Gf\colon Gc \to Gc'$ restricts to define a function $Ff\colon Fc \to Fc'$. Provide an elementary characterization of the elements of a subfunctor of the representable functor $C(-, c)$.

Exercise 4. As discussed in Section 2.2, diagrams of shape ω are determined by a countably infinite family of objects and a countable infinite sequence of morphisms. Describe the Yoneda embedding $y\colon \omega \hookrightarrow \textsc{Set}^{\omega^{\mathrm{op}}}$ in this manner (as a family of ω^{op}-indexed functors and natural transformations). Prove, without appealing to the Yoneda lemma, that y is full and faithful.

Exercise 5. Use the defining universal property of the tensor product to prove that

(i) $\Bbbk \otimes_{\Bbbk} V \cong V$ for any $\Bbbk$-vector space V
(ii) $U \otimes_{\Bbbk} (V \otimes_{\Bbbk} W) \cong (U \otimes_{\Bbbk} V) \otimes_{\Bbbk} W$ for any $\Bbbk$-vector spaces U, V, W.

Exercise 6. There is a natural automorphism ι of the contravariant power-set functor $P\colon \textsc{Set}^{\mathrm{op}} \to \textsc{Set}$ whose component functions $\iota_A\colon P(A) \to P(A)$ send a subset $A' \subset A$ to its complement. The Yoneda lemma tells us that ι is induced by an endomorphism of the representing object. What is it? Does this function induce a natural automorphism of the covariant power-set functor?

Exercise 7. The set B^A of functions from a set A to a set B represents the contravariant functor $\textsc{Set}(- \times A, B)\colon \textsc{Set}^{\mathrm{op}} \to \textsc{Set}$. In analogy with the universal bilinear map $\otimes$ of Example 2.3.6, describe the universal property of the element

$$\mathrm{ev}\colon B^A \times A \to B$$

in $\textsc{Set}(B^A \times A, B) \cong \textsc{Set}(B^A, B^A)$ that classifies the natural isomorphism.

Exercise 8*. Leaving your phone, computer, book, and notes at home, barricade yourself in a room in the Science Center until you can state and prove the Yoneda lemma.[2]

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[1]Problems labelled n^* are optional (fun!) challenge exercises that will not be graded.
[2]For safety's sake, inform a reliable friend who would be able to arrange a search and rescue operation.