Math 161: Category theory in context

Problem Set 2[1]

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Exercise 1. Show that any map from a terminal object in a category to an initial one is an isomorphism.

Exercise 2. Prove that a bifunctor $F\colon \mathcal{C}\times\mathcal{D}\to\mathcal{E}$ is uniquely determined by:

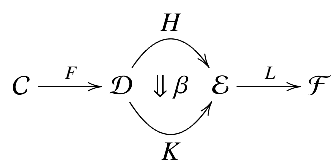
(i) A functor $F(c,-)\colon \mathcal{D}\to\mathcal{E}$ for each $c\in\mathcal{C}$.

(ii) A natural transformation $F(f,-)\colon F(c,-)\Rightarrow F(c',-)$ for each $f\colon c\to c'$ in $\mathcal{C}$, defined functorially in $\mathcal{C}$.

In other words, prove that there is a bijection between functors $\mathcal{C}\times\mathcal{D}\to\mathcal{E}$ and functors $\mathcal{C}\to\mathcal{E}^{\mathcal{D}}$.

Exercise 3. Prove that if $\mathcal{C}\simeq\mathcal{D}$ and $\mathcal{D}\simeq\mathcal{E}$ then $\mathcal{C}\simeq\mathcal{E}$. Conclude that equivalence of categories is an equivalence relation.

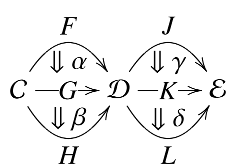
Exercise 4. Given a natural transformation $\beta\colon H\Rightarrow K$ and functors F and L as displayed



define a natural transformation $L\beta F\colon LHF\Rightarrow LKF$ by $(L\beta F)_c = L\beta_{Fc}$. This is the **whiskered composite** of β with L and F. Prove that $L\beta F$ is natural.

Exercise 5. Redefine the horizontal composition of natural transformations using vertical composition and whiskering.

Exercise 6. Given functors and natural transformations



prove that $(\delta\gamma)*(\beta\alpha) = (\delta*\beta)(\gamma*\alpha)$. That is, prove that the natural transformation $JF\Rightarrow LH$ defined by first composing vertically and then composing horizontally equals the natural transformation defined by first composing horizontally and then composing vertically. This is the rule of **middle four interchange**.

Exercise 7. Consider the functors Ab $\to$ Group (inclusion), Ring $\to$ Ab (forgetting the multiplication), $(-)^{\times}\colon$ Ring $\to$ Group (taking the group of units), Ring $\to$ Rng (dropping the multiplicative unit), Field $\to$ Ring (inclusion), $\text{Mod}_R\to$ Ab (forgetful). Determine which functors are full, which are faithful, and which are essentially surjective. Do any define an equivalence of categories?

[1]Problems labelled n^* are optional (fun!) challenge exercises that will not be graded. If you solve this one, please come tell me. I'd like to add an example of this kind to the notes, if it exists.

Exercise 8. Prove that the monomorphisms in any category define a subcategory of that category. Apply duality to prove that the epimorphisms also define a subcategory.

Exercise 9. Show that any faithful functor reflects monomorphisms. That is, if $F\colon \mathcal{C} \to \mathcal{D}$ is faithful, prove that if Ff is a monomorphism in $\mathcal{D}$, then f is a monomorphism in $\mathcal{C}$. Argue by duality that faithful functors also reflect epimorphisms. Conclude that in any concrete category, any injection of underlying sets defines a monomorphism and any surjection of underlying sets defines an epimorphism.

Exercise 10. Find a concrete category that contains a monomorphism whose underlying function is not injective. Find a concrete category that contains an epimorphism whose underlying function is not surjective.

Exercise 11*. For any group G, we may define other groups:

- the **center** $Z(G) = \{h \in G \mid hg = gh \,\forall g \in G\}$, a subgroup of G,
- the **commutator subgroup** $C(G)$, the subgroup of G generated by the elements $ghg^{-1}h^{-1}$ for any $g, h \in G$, and
- the **automorphism group** $\mathrm{Aut}(G)$, the group of isomorphisms $\phi\colon G \to G$ in Group.

Trivially all three constructions define a functor from the discrete category of groups (with only identity morphisms) to Group. Are these constructions functorial in

- the isomorphisms of groups? That is, do they extend to functors $\text{Group}_{\text{iso}} \to$ Group?
- the epimorphisms of groups? That is, do they extend to funtors $\text{Group}_{\text{epi}} \to$ Group?
- all homomorphisms of groups? That is, do they extend to functors Group $\to$ Group?

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