Math 161: Category theory in context

Problem Set 1[1]

due: February 10, 2015

Emily Riehl

Exercise 1. What is a functor between groups, regarded as one-object categories? What is a natural transformation between a parallel pair of such functors?

Exercise 2. What is a functor between preorders, regarded as categories? What is a natural transformation between a parallel pair of such functors?

Exercise 3. What is the difference between a functor $\mathcal{C}^{\text{op}} \to \mathcal{D}$ and a functor $\mathcal{C} \to \mathcal{D}^{\text{op}}$? What is the difference between a functor $\mathcal{C} \to \mathcal{D}$ and a functor $\mathcal{C}^{\text{op}} \to \mathcal{D}^{\text{op}}$?

Exercise 4. Let $\mathcal{C}$ be a category. Show that the collection of isomorphisms in $\mathcal{C}$ defines a subcategory, the **maximal groupoid** inside $\mathcal{C}$.

Exercise 5. Show that any category that is equivalent to a locally small category is locally small.

Exercise 6. Suppose each component of a natural transformation $\alpha\colon F \Rightarrow G$ is an isomorphism. Show that the inverse morphisms define the components of a natural transformation $\alpha^{-1}\colon G \Rightarrow F$.

Exercise 7. Prove that a functor $F\colon \mathcal{C} \to \mathcal{D}$ defining part of an equivalence of categories is full, faithful, and essentially surjective on objects.

Exercise 8. Assuming the axiom of choice, prove that any full, faithful, and essentially surjective functor defines an equivalence of categories.

Exercise 9. Let $\mathcal{G}$ be a connected groupoid and let G be the group of automorphisms at any of its objects. The inclusion $G \hookrightarrow \mathcal{G}$ defines an equivalence of categories. Construct an inverse equivalence $\mathcal{G} \to G$.

Exercise 10. Characterize the categories that are equivalent to discrete categories.

Exercise 11*. Find an example of a (non-finitely generated) abelian group A that can be written as union (or direct limit) of finitely generated abelian groups, but for which A is not isomorphic to $TA \oplus A/TA$.

Dept. of Mathematics, Harvard University, 1 Oxford Street, Cambridge, MA 02138
E-mail address: `eriehl@math.harvard.edu`

[1]Problems labelled n* are optional (fun!) challenge exercises that will not be graded. If you solve this one, please come tell me. I'd like to add an example of this kind to the notes, if it exists.