A COURSE ON HYPERBOLIC PDE

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Contents

These notes are taken from [Tay96], [Tay], [Eva98], and [Sog95].

1. The wave equation on bounded domains

Formally, the solution to the wave equation on a bounded domain is quite similar to the solution to the heat equation on a bounded domain, although qualitatively the solutions are quite different. Let $\bar{\mathcal{M}}$ be a compact Riemannian manifold with boundary. The wave equation is given by

$$\frac{\partial^2 u}{\partial t^2} - \Delta u = 0, \tag{1.1}$$

for $u = u(t,x)$, $t \in \mathbb{R}$, $x \in \mathcal{M}$. The initial conditions are given by

$$u(0,x) = f(x), \qquad u_t(0,x) = g(x). \tag{1.2}$$

Then if $\partial\mathcal{M}$ is nonempty, impose the Dirichlet boundary condition

$$u(0,x) = 0, \qquad x \in \partial\mathcal{M}. \tag{1.3}$$

Now let $u_j(x)$ again refer to the orthonormal basis of $L^2(\mathcal{M})$,

$$u_j \in H_0^1(\mathcal{M}) \cap C^\infty(\bar{\mathcal{M}}), \qquad \Delta u_j = -\lambda_j u_j, \qquad 0 \leq \lambda_j \nearrow \infty. \tag{1.4}$$

Then write

$$u(t,x) = \sum_j a_j(t) u_j(x). \tag{1.5}$$

Then the coefficients $a_j(t)$ satisfy
(1.6)
$$a_j''(t) + \lambda_j a_j(t) = 0, \qquad a_j(0) = \hat{f}(j), \qquad a_j'(0) = \hat{g}(j), \qquad \hat{f}(j) = (f, u_j), \qquad \hat{g}(j) = (g, u_j).$$

Therefore,

$$a_j(t) = \hat{f}(j)\cos(\lambda_j^{1/2} t) + \hat{g}(j)\lambda_j^{-1/2}\sin(\lambda_j^{1/2} t). \tag{1.7}$$

If $\partial\mathcal{M} = \emptyset$ and $\mathcal{M}$ is connected, then 0 is an eigenvalue of multiplicity one. In that case,

$$a_0(t) = \hat{f}(0) + \hat{g}(0)t. \tag{1.8}$$

Remark 1. *Notice that for any $t \in \mathbb{R}$,*

$$\lim_{\lambda \searrow 0} \lambda^{-1/2} \sin(\lambda^{1/2} t) = t. \tag{1.9}$$

Suppose for simplicity that all λ_j are nonzero. Then a solution to (1.1)–(1.3) is given by

$$u(t,x) = \sum_j [\hat{f}(j)\cos(\lambda_j^{1/2} t) + \hat{g}(j)\lambda_j^{-1/2}\sin(\lambda_j^{1/2} t)] u_j(x), \tag{1.10}$$

which is equivalent to the operator expression

$$u(t,x) = \cos(t\sqrt{-\Delta}) f + \frac{\sin(t\sqrt{-\Delta})}{\sqrt{-\Delta}} g. \tag{1.11}$$

Then we have

$$f \in \mathcal{D}_s, \qquad g \in \mathcal{D}_{s-1} \qquad \text{implies} \qquad u \in C(\mathbb{R}, \mathcal{D}_s), \qquad \partial_t^j \in C(\mathbb{R}, \mathcal{D}_{s-j}). \tag{1.12}$$

Recall that

$$\mathcal{D}_s = \{ v \in L^2(\mathcal{M}) : \sum_{j \geq 0} |\hat{v}(j)|^2 \lambda_j^s < \infty \}. \tag{1.13}$$

According to the definition $\mathcal{D}_0 = L^2$, $\mathcal{D}_1 = H_0^1$, $\mathcal{D}_2 = H^2(\mathcal{M}) \cap H_0^1(\mathcal{M})$, and

$$\mathcal{D}_{2k} \subset H^{2k}(\mathcal{M}). \tag{1.14}$$

Then if $s > \frac{n}{2}$, $u \in C(\mathbb{R} \times \bar{\mathcal{M}})$ and then the boundary condition (1.2) is satisfied in the ordinary sense.

Now define the energy norm

$$E_s(t) = \|u(t)\|^2_{\mathcal{D}_s} + \|u_t(t)\|^2_{\mathcal{D}_{s-1}}, \tag{1.15}$$

where $\|v\|_{\mathcal{D}_s} = \|(-\Delta)^{s/2} v\|_{L^2(\mathcal{M})}$. Therefore if

$$u \in C^1(\mathbb{R}, \mathcal{D}_s) \cap C^2(\mathbb{R}, \mathcal{D}_{s-1}), \tag{1.16}$$

(1.17)

$$\frac{d}{dt} E_s(t) = 2Re(u_t(t), u(t))_{\mathcal{D}_s} + 2Re(u_t(t), u_{tt}(t))_{\mathcal{D}_{s-1}} = 2Re(u_t(t), (-\Delta)^s u(t)) + 2Re(u_t(t), \Delta(-\Delta)^{s-1} u(t)) = 0.$$

Therefore, we have the energy identity

$$E_s(t) = E_s(0). \tag{1.18}$$

In the case that $\lambda_0 = 0$, (1.15) annihilates constants, so we don't quite get a norm. We now prove that wave equations satisfy the finite propagation speed.

Consider

$$\mathcal{D}_\infty = \cap_j \mathcal{D}_j. \tag{1.19}$$

Notice that $\mathcal{D}_\infty \subset C^\infty(\bar{\mathcal{M}})$. If $K \subset \bar{\mathcal{M}}$ is closed, $s \in \mathbb{R}$, we say that $f \in \mathcal{D}_s$ is $\mathcal{D}$-supported in K if and only if

$$(v, f) = 0, \qquad \text{for all } v \in \mathcal{D}_\infty \qquad \text{such that} \qquad supp(v) \subset \overline{\mathcal{M}} \setminus K. \tag{1.20}$$

This notion coincides with the familiar notion of support when $s \geq 0$.

Lemma 1. *Let $K \subset \overline{\mathcal{M}}$ be closed, $s \in [0, \infty)$, $v \in \mathcal{D}_s \subset L^2(\mathcal{M})$. Then v is $\mathcal{D}$-supported in K if and only if v is supported in K in the usual sense, that is, $v(x) = 0$ for almost all $x \in \overline{\mathcal{M}} \setminus K$.*

Proof. Let $w \in \mathcal{D}_\infty$ have support in the usual sense on a closed set $L \subset \overline{\mathcal{M}} \setminus K$. If $v \in \mathcal{D}_0$ vanishes point wise almost everywhere on $\overline{\mathcal{M}} \setminus K$, then certainly $(v, w) = \int_{\mathcal{M}} v(x)\overline{w(x)} dV(x) = 0$. This proves ($\Leftarrow$).

Now suppose conversely that $(v, w) = 0$ for all $w \in \mathcal{D}_\infty$ that vanish point wise on a neighborhood of K. In particular, $(v, w) = 0$ for all $w \in C_0^\infty(\mathcal{M} \setminus K)$, so v vanishes point wise almost everywhere on the open set $U = \mathcal{M} \setminus K \subset \mathcal{M}$. Therefore, the closure of U lies in $\overline{\mathcal{M}} \setminus K$, which completes the proof. □

For $s \leq 0$, $C_0^\infty(\mathcal{M})$ is dense in $\mathcal{D}_s$. For $s < 0$, given $p \in \partial\mathcal{M}$, there is a nonzero $\nu_p \in \mathcal{D}_s$, for any $s < -\frac{n}{2} - 1$, defined by $(u, \nu_p) = \frac{\partial u(p)}{\partial \nu}$, and ν_p is $\mathcal{D}$-supported on $\{p\}$.

Proposition 1. *If $K \subset \overline{\mathcal{M}}$ is closed, and*

$$K_d = \{x \in \overline{\mathcal{M}} : dist(x, K) \leq d\}, \tag{1.21}$$

then if $f \in \mathcal{D}_s$, $g \in \mathcal{D}_{s-1}$, are $\mathcal{D}$–supported in K, it follows that

$$\cos(t\sqrt{-\Delta}) f, \tag{1.22}$$

and

$$\frac{\sin(t\sqrt{-\Delta})}{\sqrt{-\Delta}}g, \tag{1.23}$$

are $\mathcal{D}$–supported in K_d for $|t| \leq d$.

Proof. Let $v \in \mathcal{D}_\infty$ be supported on $\overline{\mathcal{M}} \setminus K_d$. Then,

$$(\cos(t\sqrt{-\Delta})f, v) = (f, \cos(t\sqrt{-\Delta})v). \tag{1.24}$$

Assuming for a moment that $\cos(t\sqrt{-\Delta})v$ has finite propagation speed when $v \in \mathcal{D}_\infty$, since v is smooth, so the right hand side vanishes for $|t| \leq d$. The same sort of analysis applies to $(-\Delta)^{1/2}\sin(t\sqrt{-\Delta})g$, to complete the proof. □

To show finite propagation speed for smooth functions, suppose Ω does not intersect $\mathbb{R} \times \partial\mathcal{M}$, suppose that $\partial\Omega$ consists of two smooth surfaces Σ_1 and Σ_2, and let Ω_t denote the intersection of Ω with $\{t\} \times \mathcal{M}$. If u solves (1.1),

$$0 = \int_\Omega u_t(u_{tt} - \Delta u)dVdt = \frac{1}{2}\int_\Omega \frac{\partial}{\partial t}[u_t^2 + |\nabla_x u|^2]dVdt - \int_\Omega div_x(u_t \nabla_x u)dVdt. \tag{1.25}$$

Therefore,

$$0 = \frac{1}{2}\int_{\partial\Omega}[u_t^2 + |\nabla_x u|^2]\omega - \int\int_{\partial\Omega_t} u_t \frac{\partial u}{\partial \nu_x} dS_t dt. \tag{1.26}$$

Here dS_t is the natural surface measure on $\partial\Omega_t$. If $N = (N_t, N_x)$ is the outward pointing normal to $\partial\Omega \subset \mathbb{R} \times \mathcal{M}$,

$$\omega = N_t dS, \qquad dS_t dt = |N_x| dS. \tag{1.27}$$

Thus, if u satisfies the wave equation on Ω,

$$\int_{\Sigma_2}\{[u_t^2 + |\nabla_x u|^2]|N_t| - 2u_t \frac{\partial u}{\partial \nu_x}|N_x|\}dS = \int_{\Sigma_1}\{[|u_t|^2 + |\nabla_x u|^2]|N_t| + 2u_t\frac{\partial u}{\partial \nu_x}|N_x|\}dS. \tag{1.28}$$

Since

$$2|u_t \frac{\partial u}{\partial \nu_x}| \leq u_t^2 + |\nabla_x u|^2, \tag{1.29}$$

if

$$|N_x| \leq |N_t|, \tag{1.30}$$

then

$$[u_t^2 + |\nabla_x u|^2]|N_t| - 2u_t \frac{\partial u}{\partial \nu_x}|N_x| \geq 0, \tag{1.31}$$

point wise. This implies finite propagation speed.

Proposition 2. *If $s \in \mathbb{R}$ and $f \in \mathcal{D}_s$ is $\mathcal{D}$–supported in a closed set $K \subset \overline{\mathcal{M}}$, then for any neighborhood K_d of K, there exists a sequence $f_j \in \mathcal{D}_\infty$, all supported in K_d, such that $f_j \to f$ in $\mathcal{D}_s$.*

Proof. Choose $\varphi \in C_0^\infty((-d,d))$, $\int \varphi(t)dt = 1$, and consider

$$f_j = \int \varphi_j(t)\cos(t\sqrt{-\Delta})f\,dt, \qquad \varphi_j(t) = j\varphi(jt). \tag{1.32}$$

Integrating by parts,

$$(-\Delta)^k f_j = \int \varphi_j^{(2k)}(t)\cos(t\sqrt{-\Delta})dt \in \mathcal{D}_s, \tag{1.33}$$

for each k, so $f_j \in \mathcal{D}_\infty$. It is clear that $f_j \to f$, and by Proposition 1, each f_j is $\mathcal{D}$–supported in K_d. Therefore, by Lemma 1, each f_j is supported in K_d. □

2. Wave equation on unbounded domains

Now consider the wave equation on $\mathbb{R} \times \mathcal{M}$, where $\mathcal{M}$ is a noncompact Riemannian manifold. Assume that $\mathcal{M}$ is complete and without boundary. Construct the solution to the wave equation

$$\frac{\partial^2 u}{\partial t^2} - \Delta u = 0, \qquad \text{on} \qquad \mathbb{R} \times \mathcal{M}, \qquad u(0,x) = f(x), \qquad u_t(0,x) = g(x), \tag{2.1}$$

under the hypothesis

$$f \in H_0^1(\mathcal{M}), \qquad g \in L^2(\mathcal{M}), \qquad supp(f,g) \subset K, \tag{2.2}$$

where $K \subset \mathcal{M}$ is compact. Then produce the unique solution

$$u \in C(\mathbb{R}, H^1(\mathcal{M})) \cap C^1(\mathbb{R}, L^2(\mathcal{M})), \tag{2.3}$$

with the property that

$$supp(u(t)) \qquad \text{is compact in} \qquad \mathcal{M}, \qquad \forall t \in \mathbb{R}. \tag{2.4}$$

Let $\bar{\mathcal{O}}_j \subset \mathcal{M}$ be compact subsets with smooth boundary, such that $\mathcal{O}_1 \subset\subset \mathcal{O}_2 \subset\subset ... \subset\subset \mathcal{O}_j \subset\subset\nearrow \mathcal{M}$. Given $f, g \in K$ and $s > 0$, choose N sufficiently large so that $K_s \subset \mathcal{O}_N$, where $K_s = \{x \in \mathcal{M} : dist(x,K) \le s\}$.

Now let Δ_j be the Laplace operator on $\mathcal{O}_j$, with Dirichlet boundary condition, so that $\cos(t\sqrt{-\Delta_j})$ and $(-\Delta_j)^{-1/2}\sin(t\sqrt{-\Delta_j})$ are defined on $L^2(\mathcal{O}_j)$, $H_0^1(\mathcal{O}_j)$, and so forth. By finite propagation speed,

$$u(t) = \cos(t\sqrt{-\Delta_j})f + \frac{\sin(t\sqrt{-\Delta_j})}{\sqrt{-\Delta_j}}g, \qquad \text{for} \qquad |t| < s, j \ge N, \tag{2.5}$$

which has support on $\mathcal{O}_N$ and is independent of $j \ge N$. This specifies the solution to (2.1), given (2.2). Define

$$U(t)\{f,g\} = \{u(t), \partial_t u(t)\}, \tag{2.6}$$

obtaining a one–parameter family of maps

$$U(t) : C_0^\infty(\mathcal{M}) \oplus C_0^\infty(\mathcal{M}), \tag{2.7}$$

which satisfies the group property

$$U(0) = I, \qquad U(t_1 + t_2) = U(t_1)U(t_2). \tag{2.8}$$

Moreover, if $f, g \in C_0^\infty(\mathcal{M})$, the proof of energy conservation implies

$$\|df\|^2_{L^2(\mathcal{M})} + \|g\|^2_{L^2(\mathcal{M})} = \|d_x u(t)\|^2_{L^2(\mathcal{M})} + \|\partial_t u(t)\|^2_{L^2(\mathcal{M})}, \tag{2.9}$$

for each $t \in \mathbb{R}$. Set $\mathcal{H}$ to be the completion of $C_0^\infty(\mathcal{M})$ in the norm

$$\|f\|_{\mathcal{H}} = \|df\|_{L^2(\mathcal{M})}. \tag{2.10}$$

Proposition 3. *The family of maps $U(t)$ in (2.6) has a unique extension to a unitary group*

$$U(t) : \mathcal{H} \oplus L^2(\mathcal{M}) \to \mathcal{H} \oplus L^2(\mathcal{M}). \tag{2.11}$$

The wave equation solution may be used to solve the heat equation,

$$\frac{\partial u}{\partial t} = \Delta u, \qquad u(0,x) = f(x). \tag{2.12}$$

Suppose $f \in L^2(\mathcal{M})$ is supported in a compact set K. Now then, if $K \subset \mathcal{O}_j$, $e^{t\Delta_j} f$ is defined by

$$e^{t\Delta_j} f = \sum_j e^{-t\lambda_j} \hat{f}(j) u_j(x). \tag{2.13}$$

Completing the square,

$$\sum_j \frac{1}{\sqrt{4\pi t}} \int_{-\infty}^{\infty} e^{-s^2/4t} \cos(s\sqrt{\lambda_j}) \hat{f}(j) ds = \sum_j e^{-t\lambda_j} \hat{f}(j) = e^{t\Delta_j} f. \tag{2.14}$$

Therefore, consider

$$H(t)f(x) = \frac{1}{\sqrt{4\pi t}} \int_{-\infty}^{\infty} e^{-s^2/4t} W(s) f(x) ds, \tag{2.15}$$

where $W(s)f(x) = v(t,x)$ solves (2.1) with $g = 0$. Then if f is supported on K,

$$W(s)f(x) = \cos(s\sqrt{-\Delta_j}) f(x), \qquad \text{if} \qquad K_{|s|} \subset \mathcal{O}_j. \tag{2.16}$$

Then,

$$H(t)f(x) = e^{t\Delta_j} f(x) + \frac{1}{\sqrt{4\pi t}} \int_{T_j} e^{-s^2/4t} [W(s)f(x) - \cos(s\sqrt{-\Delta_j}) f(x)] ds, \tag{2.17}$$

where if $K \subset \mathcal{O}_j$,

$$T_j = \{s \in \mathbb{R} : dist(K, \partial\mathcal{O}_j) < |s|\}. \tag{2.18}$$

Since $\cos(s\sqrt{-\Delta_j})$ and $W(s)$ have L^2–operator norms ≤ 1, we have

$$H(t)f = \lim_{j \to \infty} e^{t\Delta_j} f, \qquad \text{in} \qquad L^2(\mathcal{M}), \tag{2.19}$$

for $f \in L^2(\mathcal{M})$ with compact support. Here $e^{t\Delta_j} f(x)$ is set equal to zero on $\mathcal{M} \setminus \mathcal{O}_j$. Thus, $H(t)$ extends uniquely to an operator on $L^2(\mathcal{M})$ of norm ≤ 1, and we have

$$H(t)f = \lim_{j \to \infty} e^{t\Delta_j} P_j f, \qquad \text{in} \qquad L^2(\mathcal{M}), \qquad \forall f \in L^2(\mathcal{M}), \tag{2.20}$$

where $P_j f(x) = \chi_{\mathcal{O}_j}(x) f(x)$.

Proposition 4. *If $\mathcal{M}$ is a complete Riemannian manifold of dimension n, the operator $H(t)$ has integral kernel $h(t,x,y)$, smooth on $(0,\infty) \times \mathcal{M} \times \mathcal{M}$ and satisfying the estimate*

$$0 \leq h(t,x,y) \leq C\kappa(x,\delta)\kappa(y,\delta)(1 + t^{-k}\langle t^{-1}\rho^2\rangle^k)^2 e^{-\rho^2/4t}, \tag{2.21}$$

where $dist(x,y) = \rho + 2\delta$, $\kappa(x,\delta) = C(U)$, for U the ball of radius $\delta > 0$ centered at x and $k > \frac{n}{4}$.

$$H(t)f(x) = \int_{\mathcal{M}} h(t,x,y) f(y) dV(y). \tag{2.22}$$

Furthermore, under certain hypotheses on $\mathcal{M}$, $h(t,x,y)$ will decrease rapidly as $dist(x,y) \to \infty$ for fixed $t > 0$.

Proof. Let U_j be open sets in $\mathcal{M}$ and let $\rho = dist(U_1, U_2) = inf\{dist(y_1, y_2) : y_j \in U_j\}$. Assume f is supported in U_1. Then by finite propagation speed,

$$H(t)f(x) = \frac{1}{\sqrt{4\pi t}} \int_{|x| \geq \rho} e^{-s^2/4t} W(s) f(x) ds, \qquad \text{for} \qquad x \in U_2. \tag{2.23}$$

Now if $R_j f(x) = \chi_{U_j}(x) f(x)$,

$$\|R_2 H(t) R_1\|_{\mathcal{L}(L^2)} \leq \frac{1}{\sqrt{4\pi t}} \int_{|s| \geq \rho} e^{-s^2/4t} ds \leq e^{-\rho^2/4t}. \tag{2.24}$$

To estimate derivatives, use the equation $\partial_s^2 W(s) = \Delta W(s)$. Integrating by parts,

$$\Delta^k H(t) f(x) = \frac{1}{\sqrt{4\pi t}} \int_{|s| \geq \rho} (\partial_s^{2k} e^{-s^2/4t}) W(s) f(x) ds, \tag{2.25}$$

given $x \in U_2$, $supp(f) \subset U_1$. Making the estimate

$$|\partial_s^{2k} e^{-s^2/4t}| \leq C_k t^{-k} \langle (4t)^{-1} s^2 \rangle^k e^{-s^2/4t}. \tag{2.26}$$

Therefore,

$$\|R_2 \Delta^k H(t) R_1\|_{\mathcal{L}(L^2)} \leq C_k t^{-k} \int_{\rho/\sqrt{t}}^{\infty} (1 + s^2)^k e^{-s^2/4} ds \leq C_k t^{-k} \langle t^{-1} \rho^2 \rangle^k e^{-\rho^2/4t}. \tag{2.27}$$

For $k > \frac{n}{4}$, $n = dim(\mathcal{M})$, there is a Sobolev estimate of the form

$$|f(x_2)| \leq C(U_2)[\|\Delta^k f\|_{L^2(U_2)} + \|f\|_{L^2(U_2)}]. \tag{2.28}$$

Therefore,

$$\|h(t, x_2, \cdot)\|_{L^2(U_1)} \leq C' C(U_2)(1 + t^{-k} \langle t^{-1} \rho^2 \rangle^k) e^{-\rho^2/4t}. \tag{2.29}$$

By symmetry and another application of the argument,

$$|h(t, x_2, x_1)| \leq C' C(U_1) C(U_2)(1 + t^{-k} \langle t^{-1} \rho^2 \rangle^k)^2 e^{-\rho^2/4t}. \tag{2.30}$$

Positivity follows from the positivity of heat kernels $h_j(t,x,y)$ of $e^{t\Delta_j}$. In fact, by the maximum principle for the heat equation,

$$0 \leq h_j(t,x,y) \nearrow h(t,x,y), \qquad \text{as} \qquad j \to \infty. \tag{2.31}$$

Therefore the proof is complete. □

3. The linear wave equation

Now let $u(t,x)$ be the solution to the linear wave equation

$$\partial_{tt} u - \Delta u = 0, \qquad u(0,x) = f(x), \qquad u_t(0,x) = g(x). \tag{3.1}$$

The finite propagation speed computations easily imply uniqueness for a solution to (3.1). Now then, in one dimension, the solution may be given by

$$u(t,x) = \frac{1}{2} f(x - t) + \frac{1}{2} f(x + t) + \frac{1}{2} \int_{x-t}^{x+t} g(s) ds. \tag{3.2}$$

It is possible to generalize (3.2) to $\mathbb{R}^n$ with n odd. Let A_r denote the spherical means of a function,

$$(A_r h)(x) = \frac{1}{4\pi} \int_{S^2} h(x + ry) d\sigma(y). \tag{3.3}$$

By the divergence theorem,

$$\partial_r (A_r h)(x) = \frac{1}{4\pi} \int_{S^2} \langle \nabla_x h(x + ry), y \rangle d\sigma(y) = \frac{r}{4\pi} \int_{|y|<1} \Delta_x h(x + ry) dy = \frac{r^{-2}}{4\pi} \Delta_x \int_{|x-y|<r} h(y) dy. \tag{3.4}$$

Rewriting the last integral in polar coordinates,

$$\frac{1}{4\pi} \int_{|y-x|<r} h(y) dy = \int_0^r \rho^2 A_\rho h(x) d\rho. \tag{3.5}$$

Therefore,

$$\partial_r (A_r h(x)) = r^{-2} \Delta_x \int_0^r \rho^2 A_\rho h(x) d\rho. \tag{3.6}$$

Therefore,

$$\frac{\partial}{\partial r} (r^2 \frac{\partial}{\partial r} A_r h(x)) = \Delta_x r^2 A_r h(x). \tag{3.7}$$

Therefore, $H(r, x) = A_r h(x)$ solves Darboux's equation

$$(\frac{\partial^2}{\partial r^2} + \frac{2}{r} \frac{\partial}{\partial r}) H(r, x) = \Delta_x H(r, x). \tag{3.8}$$

Now then, $r \to A_r h(x)$ is even, so

$$H(0, x) = h(x), \qquad \partial_r H(0, x) = 0. \tag{3.9}$$

Now then, suppose $u(t, x)$ is C^2 and that u solves (3.1) in $\mathbb{R}^{1+3}$. Now set

$$U(r; t, x) = (A_r u(t, \cdot))(x) = \frac{1}{4\pi} \int_{S^2} u(t, x + ry) d\sigma(y). \tag{3.10}$$

Therefore,

$$\Delta_x U = (\frac{\partial^2}{\partial r^2} + \frac{2}{r} \frac{\partial}{\partial r}) U = r^{-1} \frac{\partial^2}{\partial r^2} (rU). \tag{3.11}$$

Since $\partial_t^2 u(t, x) = \Delta_x u(t, x)$,

$$\Delta_x U = \frac{1}{4\pi} \int_{S^2} \Delta_x u(t, x + ry) d\sigma(y) = \frac{1}{4\pi} \frac{\partial^2}{\partial t^2} \int_{S^2} u(t, x + ry) d\sigma(y) = \frac{\partial^2}{\partial t^2} U. \tag{3.12}$$

Therefore,

$$v(t, r) = rU(r; t, x), \tag{3.13}$$

solves the one dimensional wave equation

$$\partial_t^2 v = \partial_r^2 v, \qquad v(0, x) = rA_r f(x), \qquad \partial_t v(0, x) = rA_r g(x). \tag{3.14}$$

Plugging in (3.2) to (3.14),

$$v(t, r) = \frac{1}{2} [(r + t) A_{r+t} f(x) + (r - t) A_{r-t} f(x)] + \frac{1}{2} \int_{r-t}^{r+t} \rho A_\rho g(x) d\rho. \tag{3.15}$$

Since $A_r f$ and $A_r g$ are even functions of r and since $v = rU$,

$$U = \frac{1}{2r}[(t+r)A_{t+r}f(x) - (t-r)A_{t-r}f(x)] + \frac{1}{2r}\int_{t-r}^{t+r} \rho A_\rho g(x) d\rho. \tag{3.16}$$

Now then, since $u(t,x) = U(0;t,x)$, and letting $r \searrow 0$,

$$u(t,x) = \partial_t(tA_t f(x)) + tA_t g(x) = \frac{1}{4\pi t^2}\int_{|x-y|=t} [tg(y) + f(y) - \langle \nabla_y f(y), x-y\rangle] d\sigma(y). \tag{3.17}$$

Proposition 5. *Any C^2 solution of the Cauchy problem* (3.1) *in $\mathbb{R}\times\mathbb{R}^3$ must be given by* (3.17) *and therefore must be unique. Conversely, if $f \in C^3(\mathbb{R}^3)$ and $g \in C^2(\mathbb{R}^3)$, then if u is given by* (3.17)*, then u solves* (3.1)*. Also observe that u satisfies the sharp Huygens principle.*

If $n > 3$ is odd, let

$$A_r h(x) = \frac{1}{\omega_{n-1}}\int_{S^{n-1}} h(x+ty) d\sigma(y), \tag{3.18}$$

where ω_{n-1} is the area of the unit sphere $S^{n-1} \subset \mathbb{R}^n$. If $n = 2k+1$, let

$$v = (\frac{1}{r}\frac{\partial}{\partial r})^{k-1} r^{2k-1} A_r u(t,x). \tag{3.19}$$

In this case, $\partial_t^2 v = \partial_r^2 v$ and

$$v(0,r) = (\frac{1}{r}\frac{\partial}{\partial r})^{k-1} r^{2k-1} A_r f(x) = \phi(r), \qquad \partial_t v(0,r) = (\frac{1}{r}\frac{\partial}{\partial r})^{k-1} r^{2k-1} A_r g(x) = \psi(r). \tag{3.20}$$

Then $v_{tt} - v_{rr} = 0$, so therefore v solves the one dimensional wave equation. This fact follows from the identity

Lemma 2. *Let $\phi : \mathbb{R} \to \mathbb{R}$ be C^{k+1}. Then*

$$(\frac{d^2}{dr^2})(\frac{1}{r}\frac{d}{dr})^{k-1}(r^{2k-1}\phi(r)) = (\frac{1}{r}\frac{d}{dr})^k (r^{2k}\frac{d\phi}{dr}). \tag{3.21}$$

Proof. Prove this by induction. When $k = 1$,

$$(\frac{d^2}{dr^2})(r\phi) = 2\phi'(r) + r\phi''(r) = (\frac{1}{r}\frac{d}{dr})(r^2\phi). \tag{3.22}$$

Now show that (3.21) implies that the same result holds with k replaced by $k+1$.

$$\begin{aligned} (\frac{d^2}{dr^2})(\frac{1}{r}\frac{d}{dr})^k (r^{2k+1}\phi(r)) &= (\frac{d^2}{dr^2})(\frac{1}{r}\frac{d}{dr})^{k-1}(\frac{1}{r}\frac{d}{dr})(r^{2k+1}\phi(r)) \\ &= (\frac{d^2}{dr^2})(\frac{1}{r}\frac{d}{dr})^{k-1}((2k+1)r^{2k-1}\phi(r) + r^{2k}\phi'(r)) \\ = (\frac{1}{r}\frac{d}{dr})^k((2k+1)r^{2k}\phi'(r) + r^{2k}\frac{d}{dr}(r\phi'(r))) &= (\frac{1}{r}\frac{d}{dr})^k((2k+2)r^{2k}\phi'(r) + r^{2k+1}\phi''(r)) \\ &= (\frac{1}{r}\frac{d}{dr})^{k+1}(r^{2k+2}\phi'(r)). \end{aligned} \tag{3.23}$$

□

Remark 2. *This proof was showed in class by Zhexing Zhang.*

$$v(r,t) = \frac{1}{2}[\phi(r+t) - \phi(r-t)] + \frac{1}{2}\int_{r-t}^{r+t} \psi(s)ds. \tag{3.24}$$

There are constants c_j with

$$c_0 = 1 \cdot 3 \cdot 5 \cdots (2k-1) = 1 \cdot 3 \cdot 5 \cdots (n-2). \tag{3.25}$$

Therefore,

$$(\frac{1}{r}\frac{\partial}{\partial r})^{k-1}(r^{2k-1}\phi(r)) = \sum_{j=0}^{k-1} c_j r^{j+1} \frac{\partial^j}{\partial r^j}\phi(r). \tag{3.26}$$

Taking $r \searrow 0$,

$$u(t,x) = \lim_{r \searrow 0} A_r u(t,x) = \lim_{r \searrow 0} \frac{1}{c_0 r} v(r,t) = \frac{1}{c_0}\partial_r \phi|_{r=t} + \frac{1}{c_0}\psi(t). \tag{3.27}$$

Proposition 6. *If n is odd,*

$$u(t,x) = \frac{1}{1 \cdot 3 \cdot 5 \cdots (n-2)}[\frac{\partial}{\partial t}(\frac{1}{t}\frac{\partial}{\partial t})^{\frac{n-3}{2}} t^{n-2} A_t f(x) + (\frac{1}{t}\frac{\partial}{\partial t})^{\frac{n-3}{2}} t^{n-2} A_t g(x)]. \tag{3.28}$$

Therefore, u satisfies the sharp Huygens principle. Also, u is a C^2 solution if $f \in C^{(n+3)/2}(\mathbb{R}^n)$ and $g \in C^{(n+1)/2}(\mathbb{R}^n)$.

When n is even, use Hadamard's method of descent. If u solves a wave equation in $\mathbb{R}^{1+n}$, then u is also a solution on $\mathbb{R} \times \mathbb{R}^{n+1}$ that is independent of the last variable x_{n+1}. Therefore,

$$\begin{aligned} &\frac{1}{1 \cdot 3 \cdot 5 \cdots (n-1)\omega_n}[\frac{\partial}{\partial t}(\frac{1}{t}\frac{\partial}{\partial t})^{\frac{n-2}{2}} t^{n-1} \int_{|y|^2 + y_{n+1}^2 = 1} f(x+ty) d\sigma(y, y_{n+1}) \\ &\quad + (\frac{1}{t}\frac{\partial}{\partial t})^{\frac{n-2}{2}} t^{n-1} \int_{|y|^2 + y_{n+1}^2 = 1} g(x+ty) d\sigma(y, y_{n+1})]. \end{aligned} \tag{3.29}$$

Projecting the upper and lower hemispheres of S^{n-1} onto $|y| < 1$, where $dy = \sqrt{1 - |y|^2} d\sigma(y, y_{n+1})$,

$$\begin{aligned} &\frac{1}{1 \cdot 3 \cdot 5 \cdots (n-1)\omega_n}[\frac{\partial}{\partial t}(\frac{1}{t}\frac{\partial}{\partial t})^{\frac{n-2}{2}} t^{n-1} \int_{|y|^2 + y_{n+1}^2 = 1} f(x+ty) \frac{dy}{\sqrt{1-|y|^2}} \\ &\quad + (\frac{1}{t}\frac{\partial}{\partial t})^{\frac{n-2}{2}} t^{n-1} \int_{|y|^2 + y_{n+1}^2 = 1} g(x+ty) \frac{dy}{\sqrt{1-|y|^2}}]. \end{aligned} \tag{3.30}$$

Theorem 1. *If $k = 2, 3, ...$ $f \in C^{[n/2]+k}(\mathbb{R}^n)$ and $g \in C^{[n/2]+k-1}(\mathbb{R}^n)$, then the Cauchy problem (3.1) has a unique solution $u \in C^k(\mathbb{R}_+^{n+1})$. Also, if f and g are supported in $\{x : |x| < R\}$ and if n is odd then $u(t,x) = 0$, unless $|t - |x|| < R$ and $u(t,x) = O((1+t)^{-\frac{n-1}{2}})$. For such data and even n, $|x| \leq t + R$ in the support of u and $u(t,x) = O((1+t)^{-\frac{n-1}{2}}(1 + |t - |x||)^{\frac{-n-1}{2}}$.*

4. The Cauchy–Kowalewsky theorem

The Cauchy–Kowalewsky theorem asserts the local existence of a real analytic solution to the Cauchy problem

$$\frac{\partial^m u}{\partial t^m}+\sum_{j=0}^{m-1}\sum_{|\alpha|\le m-j}A_{j\alpha}(t,x)\frac{\partial^\alpha}{\partial x^\alpha}\frac{\partial^j u}{\partial t^j}=f(t,x),\qquad (4.1)$$
$$u(t_0,x)=g_0(x),\cdots,\partial_t^{m-1}u(t_0,x)=g_{m-1}(x).$$

Suppose that $A_{j\alpha}(t,x)$ and $f(t,x)$ are real analytic on a neighborhood of (t_0,x_0) in $\mathbb{R}^{n+1}$ and g_0, ..., g_{m-1} are real analytic in a neighborhood of x_0 in $\mathbb{R}^n$. Without loss of generality suppose $t_0=0$ and $x_0=0$.

As in the case of ordinary differential equations, it is possible to convert (4.1) into a first–order system.

$$\frac{\partial}{\partial t}\begin{pmatrix} u\\ \frac{\partial u}{\partial t}\\ \vdots\\ \frac{\partial^{m-1}u}{\partial t^m}\end{pmatrix}=\begin{pmatrix}0&1&0&\cdots&0\\ 0&0&1&\cdots&0\\ 0&\vdots&\vdots&\vdots&\vdots\\ 0&0&\cdots&0&(4.1).\end{pmatrix}\qquad (4.2)$$

Rewriting (4.2),

$$\frac{\partial u}{\partial t}=L(t,x)\partial_x u+L_0(t,x)u+f,\qquad u(0,x)=g(x),\qquad (4.3)$$

where

$$L(t,x)\partial_x=\sum_{j=1}^{n}L_j(t,x)\frac{\partial}{\partial x_j}.\qquad (4.4)$$

Suppose that $L_j(t,x)$ are real analytic, $K\times K$ matrix–valued functions, and f and g are real analytic, with values in $\mathbb{C}^K$. Then

$$\partial_t^{j+1}u=\sum_{l=0}^{j}\tbinom{j}{l}[(\partial_t^{j-l}L)\partial_x\partial_t^l u+(\partial_t^{j-l}L_0)\partial_t^l u]+\partial_t^j f.\qquad (4.5)$$

Then by induction, $\partial_t^{j+1}u(0,x)$ is uniquely determined. Therefore, (4.3) has at most one real analytic, local solution u.

On the other hand, if we can use (4.5) to get sufficiently good estimates on $\partial_t^{j+1}u|_{t=0}=u_{j+1}(x)$, that the power series

$$u(t,x)=\sum_{j=0}^{\infty}\frac{1}{j!}u_j(x)t^j,\qquad (4.6)$$

converges for t in some neighborhood of 0, then (4.6) furnishes the solution to (4.3). Set $u_0(x)=g(x)$ and define $u_{j+1}(x)$ inductively by

$$u_{j+1}(x)=\sum_{l=0}^{j}\sum_{\nu}\tbinom{j}{l}\partial_t^{j-l}L_\nu(0,x)\cdot\partial_\nu u_l(x)+\partial_t^{j-1}f(0,x).\qquad (4.7)$$

It is useful to extend the real analytic coefficients and other data to holomorphic functions defined on a neighborhood U in $\mathbb{C}^n$. Similarly, extend $L(t,x)$, $f(t,x)$, $g(x)$ as functions holomorphic in x in a neighborhood of $0 \in \mathbb{C}^n$. Suppose $L(t,z)$, $f(t,z)$, and $g(z)$ are all holomorphic for z in a neighborhood of the closed unit ball $\bar{B} \subset \mathbb{C}^n$ with real analytic dependence on t for $|t| \leq 1$.

Define the Banach spaces h_j of functions f, holomorphic on B, and having the property that

$$N_j(f) = \sup_{z \in B} \delta(z)^j |f(z)|, \tag{4.8}$$

is finite, where $\delta(z) = 1 - |z|$ is the distance of z from ∂B. Now then, from (4.7),

$$N_{j+1}(u_{j+1}) \leq \sum_{l=0}^{j} \sum_{\nu} \|\partial_t^{j-l} L_\nu(0)\|_{L^\infty(B)} N_{j+1}(\partial_\nu u_l) + N_{j+1}(\partial_t^j f). \tag{4.9}$$

Claim 1. *There exists a constant γ, depending only on n, such that*

$$N_{j+1}(\partial_{x_\nu} u_l) \leq \gamma(j+1) N_j(u_l). \tag{4.10}$$

Since $N_j(v) \leq N_l(v)$ for $l \leq j$,

$$N_{j+1}(u_{j+1}) \leq \gamma(j+1) \sum_{l=0}^{j} \sum_{\nu} \binom{j}{l} \|\partial_t^{j-l} L_\nu(0)\|_{L^\infty} N_l(u_l) + N_{j+1}(\partial_t^j f). \tag{4.11}$$

Given the hypothesis on L, namely that L is real analytic in t for $|t| \leq 1$, we can assume there are estimates of the form

$$\sum_{\nu} \|\partial_t^m L_\nu(0)\|_{L^\infty(B)} \leq C_1 \lambda^m m!. \tag{4.12}$$

Now make the inductive hypothesis on u_l that there exist constants C_2 and μ such that

$$N_l(u_l) \leq C_2 \mu^l l!, \qquad 0 \leq l \leq j. \tag{4.13}$$

The case when $l = 0$ follows from the hypothesis on $g(x)$. Also assume that for all j,

$$N_{j+1}(\partial_t^j f) \leq C_2 \mu^j (j+1)!. \tag{4.14}$$

Plugging (4.13) and (4.14) into (4.11), yields

$$N_{j+1}(u_{j+1}) \leq \gamma C_1 C_2 (j+1)! \sum_{l=0}^{j} \lambda^{j-l} \mu^l + C_2 \mu^j (j+1)!. \tag{4.15}$$

Suppose without loss of generality that $\mu \geq 2\lambda$ and $\mu \geq 2\gamma C_1 + 1$. Then $\sum_{l=0}^{j} \lambda^{j-l} \mu^l \leq 2\mu^j$, so

$$N_{j+1}(u_{j+1}) \leq C_2 (j+1)! (2\gamma C_1) \mu^j + C_2 \mu^j (j+1)! \leq C_2 \mu^{j+1} (j+1)!. \tag{4.16}$$

This completes the induction,

$$N_j(u_j) \leq C_2 \mu^j j!, \qquad \text{for all} \qquad j. \tag{4.17}$$

Proposition 7. *Given the real analyticity hypothesis on* (4.1)*, there is a unique real analytic solution $u(t,x)$ on a neighborhood of (t_0, x_0) in $\mathbb{R}^{n+1}$. The size of the region on which $u(t,x)$ is defined and analytic depends on the size of the regions to which the coefficients and data of* (4.1) *have holomorphic extension, determined by* (4.12)*,* (4.13)*, and* (4.17)*.*

It is possible to restate the Cauchy–Kowalewsky theorem in coordinate–invariant fashion. Let S be a smooth hypersurface in an open set $\mathcal{O} \subset \mathbb{R}^n$. If S is noncharacteristic for a differential operator $P = p(x, D)$ of order m if for each $x \in S$, $\sigma_P(x, \nu) = p_m(x, \nu)$ is invertible, where ν is a nonvanishing normal to S at x.

Remark 3. *Let* $Pu(x) = \sum_{|\alpha| \leq m} p_\alpha D^\alpha u(x)$*, where* $D^\alpha = D_1^{\alpha_1} \cdots D_n^{\alpha_n}$*, where* $D_j = \frac{1}{i} \frac{\partial}{\partial x_j}$*. The coefficients* $p_\alpha(x)$ *could be matrix valued. Then,*

$$p_m(x, \xi) = \sum_{|\alpha| = m} p_\alpha(x) \xi^\alpha, \tag{4.18}$$

is called the principal symbol.

Consider the following Cauchy problem,

$$p(x, D)u = f, \qquad u|_S = g_0, \qquad Yu|_S = g_1, \qquad \cdots \qquad Y^{m-1}u|_S = g_{m-1}. \tag{4.19}$$

Then on any neighborhood of $x_0 \in S$, we can make an analytic change of variables for some real analytic invertible $A(x)$, $Q = A(x)^{-1} p(x, D)$ has the form of (4.1), and S is given by $t = 0$. Then $\partial_t^j j|_S$ can be determined inductively from $u|_S$, ..., $Y^j u|_S$.

Proposition 8. *If* $p(x, D)$ *is a differential operator of order* m *with real analytic coefficients on* $\mathcal{O}$*,* S *is a real analytic hypersurface in* $\mathcal{O}$*,* Y *is a real analytic vector field transverse to* S*, and* f *and* g_j *are real analytic, then there exists a unique real analytic solution to* (4.1) *on some neighborhood of* S*.*

Now to prove a uniqueness result.

Proposition 9. *Let* $P = p(x, D)$ *be a differential operator of order* m*, with real analytic coefficients on an open set* $O \subset \mathbb{R}^n$*, and let* $S \subset \mathcal{O}$ *be a smooth noncharacteristic hypersurface. Suppose that* $u \in H^m(\mathcal{O})$ *solves*

$$p(x, D)u = 0 \qquad \text{on} \qquad \mathcal{O}, \qquad u|_S = 0, Yu|_S = 0, \qquad \cdots \qquad Y^{m-1}u|_S = 0. \tag{4.20}$$

Proof. Suppose $\mathcal{O} \setminus S$ has two connected components, $\mathcal{O}^+$ and $\mathcal{O}^-$. Alter u to produce v so that $v = u(x)$ ofr $x \in \mathcal{O}^+$ and $v = 0$ for $x \in \mathcal{O}^-$. Then by (4.17),

$$v \in H^m(\mathcal{O}), \qquad p(x, D)v = 0, \qquad \text{on} \qquad \mathcal{O}. \tag{4.21}$$

Choose $x_0 \in S$. If S is noncharacteristic at x_0, then there exists a real analytic hypersurface Σ_0, tangent to S at x_0. Make a real analytic change of variable so that $Q = A(x)^{-1} p(x, D)$ has the form (4.1), and Σ_0 is given by $t = 0$, say $x_n = 0$. Choosing Σ_0 appropriately, arrange S so that S is given by $t = \varphi(x')^2 \geq |x'|^2$, where $x' = (x_1, ..., x_{n-1})$. The adjoint operator Q^* also has real analytic coefficients on $\mathcal{O}$. Let $\Sigma_\tau = \mathcal{O} \cap \{t = \tau\}$. Let $\Sigma_\tau = \mathcal{O} \cap \{t = \tau\}$.

By the Cauchy–Kowalewski theorem, there exists $\delta > 0$ such that, for $\tau \in (-\delta, \delta)$ and a polynomial a on $\mathbb{R}^n$,

$$Q^* w = a, \qquad w = \partial_t w = ... = \partial_t^{m-1} w = 0, \tag{4.22}$$

on Σ_τ has a solution w that is real analytic on $\{x \in \mathcal{O} : |x - x_0| < \delta + \sqrt{\delta}\}$. If we pick $\tau \in (0, \delta)$ and let $\mathcal{U}_\tau$ be the set bounded by Σ_τ and S,

$$(u, a)_{L^2(\mathcal{U}_\tau)} = (v, Q^* w)_{L^2(\mathcal{U}_\tau)} = (Qv, w) = 0. \tag{4.23}$$

By the Stone–Weierstrass theorem, since the polynomials are dense in $C(\mathcal{U}_\tau)$, $u = 0$ on $\mathcal{U}_\tau$. ∎

4.1. **Some Banach spaces of harmonic functions.** Let B be the unit ball in $\mathbb{R}^k$ and let $\mathcal{X}_j$ be the space of harmonic functions f on B such that

$$N_j(f) = \sup_{x \in B} \delta(x)^j |f(x)|, \tag{4.24}$$

is finite, where $\delta(x) = 1 - |x|$ is the distance of x from ∂B. When $k = 2n$, $\mathbb{R}^{2n} \approx \mathbb{C}^n$ via $z_l = x_l + i x_{n+l}$. Then the space h_j of holomorphic functions on B such that (4.24) is finite is a closed, linear subspace of $\mathcal{X}_j$. Now then,

$$\frac{\partial}{\partial z_l} : h_j \to h_{j+1}, \tag{4.25}$$

and

$$\partial_l = \frac{\partial}{\partial x_l} : \mathcal{X}_j \to \mathcal{X}_{j+1}. \tag{4.26}$$

Now then, recall the Poisson integral formula on $\mathbb{R}^k$.

Lemma 3. *If u is harmonic on $\Omega \subset \mathbb{R}^k$ and $p \in B_r(p) \subset \Omega$, then for any $\omega \in S^{k-1}$,*

$$\omega \cdot \nabla u(p) = \frac{k-1}{r^2} Avg_{\partial_{B_r(p)}} \{\omega \cdot (y - p) u(y)\}. \tag{4.27}$$

Therefore,

$$\frac{\partial}{\partial x_l} u(x) = \frac{k-1}{\rho^2} Avg_{\partial B_\rho(x)} \{(y_l - x_l) u(y)\}. \tag{4.28}$$

Now then, for $y \in \partial B_\rho(x)$, $|y_l - x_l| \leq \rho$ and $\delta(y) \geq \delta(x) - \rho$. Taking $\rho = \beta \delta(x)$, $0 < \beta < 1$,

$$|\partial_l u(x)| \leq \frac{k-1}{\rho^2} \cdot \rho [(1-\beta)\delta(x)]^{-j} N_j(u) = \frac{k-1}{\beta(1-\beta)^j} \delta(x)^{-(j+1)} N_j(u). \tag{4.29}$$

Therefore, for $u \in \mathcal{X}_j$,

$$N_{j+1}(\partial_l u) \leq \frac{k-1}{\beta(1-\beta)^j} N_j(u). \tag{4.30}$$

The factor on the right is minimized at $\beta = \frac{1}{j+1}$. Plugging this into (4.30),

$$(1 - \frac{1}{j+1})^{-j} \leq e. \tag{4.31}$$

Indeed, $\sum_{n=1}^\infty \frac{1}{(j+1)^n} = \frac{1}{j}$, so by Taylor expansion $\log(1 - \frac{1}{j+1}) \geq -\frac{1}{j}$, so (4.31) follows. Therefore,

$$N_{j+1}(\partial_l u) \leq \gamma_k (j+1) N_j(u), \qquad \gamma_k = (k-1)e. \tag{4.32}$$

Also since $\frac{\partial}{\partial z_l} = \frac{1}{2}(\partial_l - i \partial_{n+l})$, for all $j \geq 0$, $u \in h_j$,

$$N_{j+1}(\frac{\partial u}{\partial z_l}) \leq \gamma_{2n}(j+1) N_j(u). \tag{4.33}$$

Therefore, arguing by induction on (4.32), for $u \in \mathcal{X}_0$,

$$N_m(D^\alpha u) \leq \gamma_k^m (m!) N_0(u), \qquad |\alpha| = m. \tag{4.34}$$

Corollary 1. *The estimate (4.34) implies real analyticity of harmonic functions.*

5. Geometrical optics

In this section we look at solutions to the wave equation,

$$\frac{\partial^2 u}{\partial t^2} - \Delta u = 0, \tag{5.1}$$

where $\mathbb{R} \times \mathcal{M}$, where $\mathcal{M}$ is a Riemannian manifold, having initial data with a simple jump across a smooth surface,

$$u(0,x) = a(x)H(\varphi(x)), \tag{5.2}$$

where H is a Heaviside function, $H(s) = 1$ for $s > 0$, $H(s) = 0$ for $s < 0$. Alternatively suppose the initial data is highly oscillatory,

$$u(0,x) = a(x)F(\lambda\varphi(x)), \tag{5.3}$$

where $\lambda > 0$ is large and $F \in C^\infty(\mathbb{R})$ is bounded, together with all its derivatives, as well as an infinite sequence of antiderivatives. Assume $a \in C_0^\infty(\mathcal{M})$ and $\nabla\varphi \neq 0$ on a neighborhood U of $supp(a)$. Also suppose

$$u_t(0,x) = 0. \tag{5.4}$$

We show that for $|t| < T$, for T sufficiently small, $u(t,x)$ has the asymptotic behavior

$$u(t,x) \sim \sum_{j \geq 0} u_j(t,x), \tag{5.5}$$

where in case (5.2),

$$u_j(t,x) = \sum_{\pm} a_j^\pm(t,x) h_j(\varphi^\pm(t,x)), \tag{5.6}$$

for certain functions $h_j \in C^\infty(\mathbb{R} \setminus \{0\})$ whose j-th derivative jumps at 0. In case (5.3),

$$u_j(t,x) = u_j(t,x,\lambda) = \sum_{\pm} \lambda^{-j} a_j^\pm(t,x) F_j(\lambda\varphi^\pm(x)), \tag{5.7}$$

for certain $F_j \in C^\infty(\mathbb{R})$. In both cases, $a_j^\pm$, $\varphi^\pm \in C^\infty((-T,T) \times \mathcal{M})$ with

$$\varphi^\pm(0,x) = \varphi(x), \tag{5.8}$$

and $a_0^+(0,x) + a_0^-(0,x) = a(x)$. The functions $\varphi^\pm$ are called phase functions and $a_j^\pm$ are called amplitudes. Take $h_0 = H$ and $F_0 = F$. Also, $u - \sum_{j \leq N} u_j$ is relatively smooth and relatively small for N large.

Recall the product rule and chain rule,

$$\begin{aligned} \Delta(uv) &= (\Delta u)v + 2\nabla u \cdot \nabla v + u(\Delta v), \\ \Delta F(u) &= F'(u)\Delta u + F''(u)|\nabla u|^2. \end{aligned} \tag{5.9}$$

Plugging (5.9) into the wave equation with $u_j(t,x) = \sum_\pm \lambda^{-j} a_j^\pm(t,x) F_j(\lambda\varphi^\pm(t,x))$,

$$\begin{aligned} (\partial_t^2 - \Delta)u_j(t,x) = \sum_\pm [\lambda^{2-j} a_j^\pm F_j''(\lambda\varphi^\pm)(|\partial_t\varphi^\pm|^2 - |\nabla_x\varphi^\pm|^2) \\ +\lambda^{1-j} F_j'(\lambda\varphi^\pm)(2\varphi_t^\pm \partial_t a_j^\pm - 2\nabla_x\varphi^\pm \cdot \nabla_x a_j^\pm + a_j^\pm \Delta\varphi^\pm) - \lambda^{-j} F_j(\lambda\varphi^\pm)(\Box a_j^\pm)]. \end{aligned} \tag{5.10}$$

Grouping the terms with coefficients λ^{μ},

$$\mu = 2: \qquad \sum_{\pm} a_0^{\pm} F''(\lambda\varphi^{\pm})(|\partial_t \varphi^{\pm}|^2 - |\nabla_x \varphi^{\pm}|^2) = 0, \tag{5.11}$$

$$\mu = 1: \qquad \sum_{\pm}[a_1^{\pm} F_1''(\lambda\varphi^{\pm})(|\partial_t\varphi^{\pm}|^2 - |\nabla_x\varphi^{\pm}|^2) + F'(\lambda\varphi^{\pm})(2\varphi_t^{\pm}\partial_t a_0^{\pm} - 2\nabla_x\varphi^{\pm}\cdot\nabla_x a_0^{\pm} + a_0^{\pm}\Box\varphi^{\pm})] = 0, \tag{5.12}$$

$$\begin{aligned}\mu = 0: \qquad & \sum_{\pm}[a_{j+1}^{\pm} F_{j+1}''(\lambda\varphi^{\pm})(|\partial_t\varphi^{\pm}|^2 - |\nabla_x\varphi^{\pm}|^2) \\ & + F_j'(\lambda\varphi^{\pm})(2\varphi_t^{\pm}\partial_t a_j^{\pm} - 2\nabla_x\varphi^{\pm}\cdot\nabla_x a_j^{\pm} + a_j^{\pm}\Box\varphi^{\pm}) + F_{j-1}(\lambda\varphi^{\pm})(\Box a_{j-1}^{\pm}))] = 0.\end{aligned} \tag{5.13}$$

First observe that (5.11) vanishes provided $\varphi^{\pm}$ satisfies the eikonal equation

$$|\partial_t\varphi^{\pm}|^2 - |\nabla_x\varphi^{\pm}|^2 = 0. \tag{5.14}$$

Lemma 4. *There is a neighborhood U of $K = supp(a)$ and a $T > 0$ such that this initial value problem has a unique pair of solutions $\varphi^{\pm} \in C^{\infty}((-T,T)\times U)$ satisfying*

$$\varphi^{\pm}(0,x) = \varphi(x), \qquad \partial_t\varphi^{\pm}(0,x) = \pm|\nabla_x\varphi(x)|. \tag{5.15}$$

Proof. Consider the general first order partial differential equation

$$F(x,u,\nabla u) = 0, \tag{5.16}$$

where $F(u,x,\xi)$ is smooth on $\Omega\times\mathbb{R}\times\mathbb{R}^n$ and $u|_S = v$, where S is a smooth hypersurface of Ω and $v \in C^{\infty}(S)$. Set $\zeta_0 = (\frac{\partial v}{\partial x_1}, ..., \frac{\partial v}{\partial x_{n-1}})$ at x_0 and assume that

$$F(x_0, v(x_0), (\zeta_0, \tau_0)) = 0, \qquad \frac{\partial F}{\partial \xi_n} \neq 0. \tag{5.17}$$

This is the noncharacteristic hypothesis.

Definition 1 (Eikonal equation)**.** *An eikonal equation is an equation of the form*

$$F(x,\nabla u) = 0. \tag{5.18}$$

Note that in the case of Lemma 4,

$$F(x_1,...,x_{n+1},\xi_1,...,\xi_{n+1}) = \xi_1^2 + ... + \xi_n^2 - \xi_{n+1}^2 = 0, \tag{5.19}$$

where $\xi_{n+1} = \partial_t\varphi$ and $\xi_i = \frac{\partial\varphi}{\partial x_i}$ for $1 \leq i \leq n$. Here we let $S = \mathbb{R}^n$ be a hypersurface in $\mathbb{R}^{n+1}$. Then

$$\frac{\partial F}{\partial \xi_{n+1}} \neq 0, \tag{5.20}$$

on S, since $|\nabla\varphi| \neq 0$ on S.

Returning to the general eikonal equation, we say that Λ is a graph of ξ if and only if $\xi = \Xi(x)$ is a graph of du.

Proposition 10. *The surface is locally a graph if and only if*

$$\frac{\partial \Xi_j}{\partial x_k} = \frac{\partial \Xi_k}{\partial x_j}. \tag{5.21}$$

Proof. The condition (5.21) is equivalent to the condition that $\sum_j \Xi_j dx_j$ is closed. Indeed,

$$d(\sum_j \Xi_j dx_j) = \sum \frac{\partial \Xi_j}{\partial x_k} dx_k \wedge dx_j = 0. \tag{5.22}$$

Therefore, by Poincare's lemma, $\sum \Xi_j dx_j = d\alpha$ for some 0–form α. This implies that $\xi = du$. □

Proposition 11. *The surface Λ is the graph of u locally if and only if $\sigma(X, Y) = 0$ for all X, Y tangent to Λ, where σ is the symplectic form*

$$\sigma = \sum_{j=1}^{n} d\xi_j \wedge dx_j. \tag{5.23}$$

Proof. Take

$$X_j = \frac{\partial}{\partial x_j} + \sum_l \frac{\partial \Xi_l}{\partial x_k} \frac{\partial}{\partial \xi_l}. \tag{5.24}$$

Now then,

$$\sigma(X_j, X_k) = \sigma(\frac{\partial}{\partial x_j} + \sum_l \frac{\partial \Xi_l}{\partial x_j} \frac{\partial}{\partial \xi_l}, \frac{\partial}{\partial x_k} + \sum_{l'} \frac{\partial \Xi_l}{\partial x_j} \frac{\partial}{\partial \xi_l}) = \frac{\partial \Xi_j}{\partial x_k} - \frac{\partial \Xi_k}{\partial x_j}. \tag{5.25}$$

□

Now specify a surface Σ of dimension $n-1$ over $S = \{x_n = 0\}$ by

$$\Sigma = \{(x, \xi) : x_n = 0, \qquad \xi_j = \partial_j v, \qquad 1 \le j \le n-1, \qquad F(x, \xi) = 0\}. \tag{5.26}$$

Since $\frac{\partial F}{\partial \xi_n} \neq 0$, $F(x', 0; \partial_1 v, ..., \partial_{n-1} v, \tau) = 0$ implicitly defines $\tau(x')$. This defines a smooth surface of dimension $n-1$ through $(x_0, (\zeta_0, \tau_0))$.

Now define Λ to be the union of integral curves of the Hamiltonian vector field H_F through Σ,

$$H_F = \sum_{j=1}^{n} \frac{\partial F}{\partial \xi_j} \frac{\partial}{\partial x_j} - \frac{\partial F}{\partial x_j} \frac{\partial}{\partial \xi_j}. \tag{5.27}$$

Since H_F has a nonvanishing $\frac{\partial}{\partial x_n}$ component over S, locally Λ is the graph of a function $\xi = \Xi(x)$. Also, it is straightforward to see from (5.27) that $H_F F = 0$.

Theorem 2. Λ *is locally a graph of du for a solution u to $F(x, du) = 0$, $u|_S = v$.*

Proof. Let X, Y be tangent to Λ at (x, ξ) in $\Lambda \subset \mathbb{R}^{2n}$ and take $\sigma(X, Y)$. Suppose $x \in S$ and $(x, \xi) \in \Sigma$. Decompose $X = X_1 + X_2$ and $Y = Y_1 + Y_2$ where X_1 and Y_1 are tangent to Σ, X_2, Y_2 are multiples of H_F at (x, ξ).

Since Σ is the graph of a gradient,

$$\sigma(X_1, Y_1) = 0. \tag{5.28}$$

Next,

$$\begin{aligned} \sigma(X_1, Y_2) = c\sigma(X_1, H_F) = c\sigma(\frac{\partial}{\partial x_j} + \sum_l \frac{\partial \Xi_l}{\partial x_j} \frac{\partial}{\partial \xi_l}, \sum_{j=1}^{n} \frac{\partial F}{\partial \xi_j} \frac{\partial}{\partial x_j} - \frac{\partial F}{\partial x_j} \frac{\partial}{\partial \xi_j}) \\ = \frac{\partial F}{\partial x_j} + \sum_l \frac{\partial \Xi_l}{\partial x_j} \frac{\partial F}{\partial \xi_l} = X_j(F) = 0. \end{aligned} \tag{5.29}$$

The last equality follows from the fact that $F = 0$ on Σ and X is tangent to Σ.

Now suppose X and Y are tangent to Λ at a point $\mathcal{F}^t(x, \xi)$, where $(x, \xi) \in \Sigma$ and $\mathcal{F}^t$ is the flow generated by H_F. Then

$$\sigma(X, Y) = (\mathcal{F}^{t^*}\sigma)(\mathcal{F}^t_\sharp X, \mathcal{F}^t_\sharp Y). \tag{5.30}$$

Now then, $\mathcal{F}^t_\sharp X$ and $\mathcal{F}^t_\sharp Y$ are tangent to Λ at $(x, \xi) \in \Lambda$. It is a theorem of symplectic geometry that H_F leaves the symplectic form invariant. Therefore,

$$\sigma(X, Y) = \sigma(\mathcal{F}^t_\sharp X, \mathcal{F}^t_\sharp Y) = 0. \tag{5.31}$$

□

This proves the existence of a solution to the eikonal equation. □

Now turn to the $\mu = 1$ term, given by (5.12). The term (5.12) vanishes provided

$$2\varphi_t^\pm \frac{\partial a_0^\pm}{\partial t} = 2\nabla_x \varphi^\pm \cdot \nabla_x a_0^\pm - a_0^\pm (\Box \varphi^\pm). \tag{5.32}$$

By (5.15), $\varphi^\pm \neq 0$ on U for $|t|$ sufficiently small. The linear equations (5.32) for $a_0^\pm$ are called the first transport equation. Now then, using (5.3), (5.4),

$$a_0^+ + a_0^- = a, \qquad \varphi_t^+ a_0^+ + \varphi_t^- a_0^- = 0, \qquad \text{at} \qquad t = 0. \tag{5.33}$$

Therefore,

$$a_0^+(0, x) = a_0^-(0, x) = \frac{1}{2} a(x). \tag{5.34}$$

We have $a_0^\pm \in C^\infty((-T, T) \times U)$, compactly supported in U for each $t \in (-T, T)$ for T sufficiently small.

Now turn to the $\mu = 1 - j \leq 0$ term, $j \geq 1$. This term vanishes provided

$$F_j(s) = \int F_{j-1}(s) ds, \tag{5.35}$$

and

$$2\varphi_t^\pm \frac{\partial a_j^\pm}{\partial t} - 2\nabla_x \varphi^\pm \cdot \nabla_x a_j^\pm + a_j^\pm (\Box \varphi^\pm) = -\Box a_{j-1}^\pm. \tag{5.36}$$

Equation (5.36) is called the higher order transport equations. If $u(t, x)$ is given by (5.5) and (5.7),

$$\partial_t u_j \sim \sum_\pm [\lambda^{1-j} a_j^\pm F_j'(\lambda \varphi^\pm) \varphi_t^\pm + \lambda^{-j} (\partial_t a_j^\pm) F_j(\lambda \varphi^\pm)]. \tag{5.37}$$

Using (5.4) and also requiring that $u_j(0, x) = 0$ for $j \geq 1$, we require that

$$a_j^+ + a_j^- = 0, \qquad \sum_\pm [a_j^\pm F_j'(\lambda \varphi^\pm) \varphi_t^\pm + (\partial_t a_{j-1}^\pm) F_{j-1}(\lambda \varphi^\pm)] = 0, \qquad \text{at} \qquad t = 0. \tag{5.38}$$

Using (5.35) and (5.15),

$$a_j^+ + a_j^- = 0, \qquad \varphi_t^+ (a_j^+ - a_j^-) = -\partial_t (a_{j-1}^+ + a_{j-1}^-), \qquad \text{at} \qquad t = 0. \tag{5.39}$$

Then the transport equations (5.36) have unique solutions $a_j^\pm \in C^\infty((-T, T) \times U)$ that are compactly supported in U for each $t \in (-T, T)$.

Now obtain some estimates on the solutions. Set

$$v_N = \sum_{j=1}^{N} u_j. \tag{5.40}$$

Then v_N satisfies

$$\frac{\partial^2 v_N}{\partial t^2} - \Delta v_N = r_N(t,x), \qquad v_N(0,x) = a(x)F(\lambda\varphi(x)), \qquad \partial_t v_N(0,x) = \rho_N(x), \tag{5.41}$$

where

$$\rho_N(x) = \lambda^{-N} \sum_{\pm} \partial_t a_N^{\pm}(0,x) \cdot F_N(\lambda\varphi), \tag{5.42}$$

and

$$r_N(t,x) = \lambda^{-N} \sum_{\pm} (\Box a_N^{\pm}) F_N(\lambda\varphi^{\pm}). \tag{5.43}$$

Consider the following elementary result.

Proposition 12. *If $\varphi^{\pm} \in C^{\infty}((-T,T) \times \mathcal{M})$ and $b \in C_0^{\infty}(\mathcal{M})$, then*

$$\{\lambda^{-\mu} b(x) F_N(\lambda\varphi^{\pm}) : \lambda > 1\}. \tag{5.44}$$

is bounded in $C^j((-T,T), H^{\mu-j}(\mathcal{M}))$ for each μ, $j \geq 0$ provided $F_N(s)$ and all its derivatives are bounded.

Now, $u - v_N$ satisfies

$$\begin{gathered} (\partial_t^2 - \Delta)(u - v_N) = -r_N, \\ (u - v_N)(0,x) = 0, \qquad \partial_t(u - v_N)(0,x) = -\rho_N(x). \end{gathered} \tag{5.45}$$

Therefore, we have the following.

Proposition 13. *The geometric optics construction of v_N produces an approximation to the solution to* (5.1), (5.3), *and* (5.4) *satisfying*

$$u - v_N \qquad \text{is} \qquad O(\lambda^{-\nu}) \qquad \text{in} \qquad C^j((-T,T), H^{N+1-\nu-j}(\mathcal{M})), \tag{5.46}$$

for $0 \leq \nu \leq N$, $j \geq 0$, as long as, for each N, $F_N(s)$ and all its derivatives are bounded.

6. The formation of caustics

The geometrical optics construction of the previous section breaks down when the eikonal equation does not have a global solution. Take $\mathcal{M} = \mathbb{R}^n$ with the flat metric. Then define

$$\varphi^{\pm}(t,y) = \varphi(x), \qquad y = x \pm tN(x), \qquad N(x) = |\nabla\varphi(x)|^{-1}\nabla\varphi(x). \tag{6.1}$$

It is straightforward to verify that by the implicit function theorem, for t small, $y = x \pm tN(x)$ is $1-1$ and onto for x in a compact set. Moreover, if $\nabla\varphi$ is nowhere zero, then the level sets of φ are $n-1$–dimensional manifolds. Finally, if y is in a level set for $\varphi(t,y)$ for some $t > 0$, and y is the image of x, $\frac{\nabla\varphi(x)}{|\nabla\varphi(x)|}$ is orthogonal to the level set intersecting y at t. Now then, since

$$(\partial_t \varphi(t,y)|_{t=0})^2 = |\nabla_y \varphi(0,y)|^2, \tag{6.2}$$

$\varphi(t,y)$ satisfies the eikonal equation for t small.

Therefore, if $S \subset \mathbb{R}^n$ is a level set of φ, then for fixed t, the level sets of $\varphi^{\pm}(t, \cdot)$ are the images $F_{\pm t}(S)$ under the maps $F_{\pm t}(S)$ on $\mathbb{R}^n$ defined by $F_{\pm}(x) = x \pm tN(x)$. As $|t|$ gets larger, these images can develop singularities or caustics. Compute

$$DN(x) = |\nabla \varphi(x)|^{-1} \nabla_i \nabla_j \varphi(x) - |\nabla \varphi(x)|^{-3} \nabla_j \nabla_k \varphi(x) \nabla_k \varphi(x) \nabla_i \varphi(x). \tag{6.3}$$

Observe that $DN(x)$ annihilates $N(x)$. Indeed,

$$\begin{aligned} DN(x) \cdot N(x) &= |\nabla \varphi(x)|^{-2} \nabla_i \nabla_j \varphi(x) \nabla_i \varphi(x) - |\nabla \varphi(x)|^{-4} \nabla_j \nabla_k \varphi(x) \nabla_k \varphi(x) \nabla_i \varphi(x) \nabla_i \varphi(x) \\ &= |\nabla \varphi(x)|^{-2} \nabla_i \nabla_j \varphi(x) \nabla_i \varphi(x) - |\nabla \varphi(x)|^{-2} \nabla_j \nabla_k \varphi(x) \nabla_k \varphi(x) = 0. \end{aligned} \tag{6.4}$$

If $x \in \Sigma_\beta = \{\varphi(x) = \beta\}$, then $DN(x)$ leaves $T_x \Sigma_\beta$ invariant and acts on it as $-A$, the negative of the Weingarten map. Therefore, the eigenvalues of $DN(x)$ are 0 and the negatives of the principal curvatures of Σ_β at x. Therefore, the derivative

$$DF_t(x) = I + tDN(x), \tag{6.5}$$

is singular if and only if $\frac{1}{t}$ is the value of a principal curvature of Σ_β at x.

Recall the wave equation

$$\begin{aligned} \frac{\partial^2 u}{\partial t^2} - \Delta u = 0, \qquad \text{on} \qquad \mathbb{R} \times \mathbb{R}^2, \\ u(0, x) = a(x) F(\lambda \varphi(x)), \qquad u_t(0, x) = 0. \end{aligned} \tag{6.6}$$

Take $F(s) = e^{is}$. As before, $a \in C_0^\infty(\mathbb{R}^2)$. There is a short–time approximation solution of the form

$$u(t, x) \sim \sum_{\pm} \sum_{j \geq 0} \lambda^{-j} a_j^{\pm}(t, x) e^{i \lambda \varphi^{\pm}(t, x)}. \tag{6.7}$$

Remark 4. *Here we absorb i^{-j} into the amplitudes.*

We want to obtain an asymptotic formula as $\lambda \to \infty$ near the caustics.

Recall that the exact solution to (6.6) is

$$u(t, x) = R'(t) * u_0, \tag{6.8}$$

where $u_0(x) = a(x) e^{i \lambda \varphi(x)}$ and $R'(t)$ is the t–derivative of the Riemann function

$$R(t, x) = c_2 (t^2 - |x|^2)^{-1/2}, \qquad \text{for} \qquad |x| < t, \qquad 0, \qquad \text{for} \qquad |x| > t. \tag{6.9}$$

Therefore, for a fixed $t > 0$, $R'(t)$ is a radial distribution that is singular precisely on a circle of radius t centered at the origin. Therefore, we expect u to have the form

$$v(t, x) = \frac{1}{t} \int_{|y - x| = t} u_0(y) ds(y) = \frac{1}{2\pi} \int_{-\pi}^{\pi} a(x + t(\cos(s), \sin(s))) e^{i \lambda \varphi(x + t(\cos(s), \sin(s)))} ds. \tag{6.10}$$

An integral of the form

$$I(\lambda) = \int_{-\infty}^{\infty} a(s) e^{i \lambda \psi(s)} ds, \qquad a \in C_0^\infty(\mathbb{R}^2), \tag{6.11}$$

can be analyzed by the stationary phase method. If ψ has no critical points,

$$I(\lambda) = \int a(s) (\frac{1}{i \lambda \psi'} \frac{d}{ds})^k e^{i \lambda \psi(s)} ds, \tag{6.12}$$

and integrating by parts.

If ψ has at least one critical point at s_0, and that critical point is nondegenerate, and a is supported near s_0, then $\psi(s)-\psi(s_0)$ or its negative has a smooth, real–valued square root $t(s)$ such that $t(s_0)=0$, $t'(s_0)>0$. Then

$$I(\lambda)=e^{i\lambda\varphi(s_0)}\int b(t)e^{i\alpha\lambda t^2}dt, \qquad b\in C_0^\infty(\mathbb{R}). \tag{6.13}$$

Then if $x=t^2$,

$$I(\lambda)=\frac{1}{2}e^{i\lambda\varphi(s_0)}\int_0^\infty [b(x^{1/2})+b(-x^{1/2})]x^{-1/2}e^{i\alpha\lambda x}dx \sim e^{i\lambda\varphi(s_0)}\lambda^{-1/2}[\alpha_0+\alpha_1\lambda^{-1}+...]. \tag{6.14}$$

If φ has a finite number of critical points,

$$I(\lambda)\sim\sum_j A_j(\lambda)\lambda^{-1/2}e^{i\lambda\psi(s_j)}, \qquad A_j(\lambda)\sim\alpha_{0j}+\alpha_{1j}\lambda^{-1}+... \tag{6.15}$$

If $a(s)=a(y,s)$ and $\psi(s)=\psi(y,s)$ depend smoothly on the parameters y, then we have (6.15) for $I(\lambda)=I(y,\lambda)$ with $\alpha_{kj}=\alpha_{kj}(y)$ and $\psi(s_j)=\psi(y,s_j(y))$ depending smoothly on y as long as the critical points of $\psi(y,s)$ as a function of s are all nondegenerate and depend smoothly on y.

Now then, suppose $\nabla\varphi(y)\neq 0$ for $y\in supp(a)$. Given $x\in\mathbb{R}^2$, $t>0$, denote by $S_t(x)$ the circle of radius t centered at x. The way in which $S_t(x)$ is tangent to various level curves Σ_β of φ determines the nature of the stationary points of the phase in the last integral in (6.10).

If $\frac{1}{t}$ is bigger than the largest curvature of any Σ_β then $S_t(x)$ will have only simple tangencies with such level curves. Now then, suppose $y\in\Sigma_\beta$ and $\frac{1}{t}=\kappa(y)$, the curvature of Σ_β at y. Let $x=y+tN(y)$. Then $S_t(x)$ has higher order tangency with Σ_β at y. Suppose y is not a stationary point for κ on Σ_β at a nonzero rate at y. In this case,

$$\psi(s_0)=\beta, \qquad \psi'(s_0)=\psi''(s_0)=0, \qquad \psi'''(s_0)\neq 0. \tag{6.16}$$

In this case, $\psi(s)-\beta$ has a smooth cube root near $s=s_0$, call it $t(s)$, $t(s_0)=0$, $t'(s_0)>0$. Then

$$I(\lambda)=e^{i\lambda\varphi(s_0)}\int b(t)e^{i\lambda t^3}dt, \qquad b\in C_0^\infty(\mathbb{R}). \tag{6.17}$$

Setting $x=t^3$,

$$I(\lambda)=\frac{1}{3}e^{i\lambda\varphi(s_0)}\int b(x^{1/3})x^{-2/3}e^{i\lambda x}dx\sim e^{i\lambda\varphi(x_0)}\lambda^{-1/3}[\alpha_0+\alpha_1\lambda^{-1}+...]. \tag{6.18}$$

7. Pseudodifferential operators

Write the Fourier inversion formula as

$$f(x)=\int\hat{f}(\xi)e^{ix\cdot\xi}d\xi, \tag{7.1}$$

where

$$\hat{f}(\xi)=(2\pi)^{-n}\int f(x)e^{-ix\cdot\xi}dx. \tag{7.2}$$

Remark 5. *We customarily write* (7.1) *and* (7.2) *with a coefficient of* $(2\pi)^{-n/2}$. *Of course, it is possible to distribute the* $(2\pi)^{-n}$ *between* (7.1) *and* (7.2) *however we wish.*

If $D_j = \frac{1}{i}\frac{\partial}{\partial x_j}$, $D^{\alpha} = D_1^{\alpha_1} \cdots D_n^{\alpha_n}$, one obtains

$$D^{\alpha} f(x) = \int \xi^{\alpha} \hat{f}(\xi) e^{ix\cdot\xi} d\xi. \tag{7.3}$$

Now suppose $p(x, D)$ is a differential operator,

$$p(x, D) = \sum_{|\alpha| \leq k} a_{\alpha}(x) D^{\alpha}. \tag{7.4}$$

Then,

$$p(x, D) f(x) = \int p(x, \xi) \hat{f}(\xi) e^{ix\cdot\xi} d\xi, \tag{7.5}$$

where

$$p(x, \xi) = \sum_{|\alpha| \leq k} a_{\alpha}(x) \xi^{\alpha}. \tag{7.6}$$

It is possible to generalize (7.5) and (7.6) to belong to a number of different symbol classes.

Definition 2 (Symbol class)**.** *For $\rho, \delta \in [0, 1]$, $m \in \mathbb{R}$, define $S^m_{\rho,\delta}$ to consist of C^{∞} functions $p(x, \xi)$ satisfying*

$$|D_x^{\beta} D_{\xi}^{\alpha} p(x, \xi)| \leq C_{\alpha\beta} \langle \xi \rangle^{m - \rho|\alpha| + \delta|\beta|}. \tag{7.7}$$

We say that the operator defined by (7.5) *belongs to $OPS^m_{\rho,\delta}$. We say that $p(x, \xi)$ is the symbol of $p(x, D)$.*

Remark 6. *When $p(x, D)$ is a differential operator of the form* (7.4) *and $a_{\alpha}(x)$ and all its derivatives are bounded, then $\rho = 1$, $\delta = 0$, and $m = k$.*

Next suppose there are smooth $p_{m-j}(x, \xi)$ that are homogeneous in ξ of degree $m - j$ for $|\xi| \geq 1$, that is, $p_{m-j}(x, r\xi) = r^{m-j} p_{m-j}(x, \xi)$ for $r, |\xi| \geq 1$, and if

$$p(x, \xi) \sim \sum_{j \geq 0} p_{m-j}(x, \xi), \tag{7.8}$$

in the sense that

$$p(x, \xi) - \sum_{j=0}^{N} p_{m-j}(x, \xi) \in S^{m-N-1}_{1,0}, \tag{7.9}$$

for all N, then we say that $p(x, \xi) \in S^m_{cl}$.

Remark 7. *Again observe that if $p(x, D)$ is a differential operator of order m then $p \in S^m_{cl}$.*

Definition 3. *We call $p_m(x, \xi)$ the principal symbol of $p(x, D)$.*

Claim 2. *We have the estimate*

$$p(x, D) : \mathcal{S}(\mathbb{R}^n) \to \mathcal{S}(\mathbb{R}^n). \tag{7.10}$$

Proof. It is straightforward to verify that if $f \in \mathcal{S}(\mathbb{R}^n)$ then since $p \in S^m_{\rho,\delta}$, $\int p(x, \xi) \hat{f}(\xi) e^{ix\cdot\xi} d\xi$ is bounded. Next, since $x^{\alpha} e^{ix\cdot\xi} = (-D_{\xi})^{\alpha} e^{ix\cdot\xi}$, integrating by parts implies $x^{\alpha} \int p(x, \xi) \hat{f}(\xi) e^{ix\cdot\xi} d\xi$ is bounded. Taking a derivative

$$D_j(p(x, \xi) e^{ix\cdot\xi}) = \xi_j p(x, \xi) e^{ix\cdot\xi} + D_j p(x, \xi) e^{ix\cdot\xi}, \tag{7.11}$$

which proves the bound. □

Lemma 5. *If $\delta < 1$, then*

$$p(x,D) : \mathcal{S}'(\mathbb{R}^n) \to \mathcal{S}'(\mathbb{R}^n). \tag{7.12}$$

Proof. Given $u \in \mathcal{S}'$ and $v \in \mathcal{S}$, then formally,

$$\langle v, p(x,D)u\rangle = \langle p_v, \hat{u}\rangle, \tag{7.13}$$

where

$$p_v(\xi) = (2\pi)^{-n} \int v(x)p(x,\xi)e^{ix\cdot\xi}dx. \tag{7.14}$$

Integrating by parts,

$$\xi^\alpha p_v(\xi) = (2\pi)^{-n} \int D_x^\alpha(v(x)p(x,\xi))e^{ix\cdot\xi}dx, \tag{7.15}$$

so

$$|p_v(\xi)| \le C_\alpha\langle\xi\rangle^{m+\delta|\alpha|-|\alpha|}. \tag{7.16}$$

Therefore, if $\delta < 1$, $p_v(\xi)$ is rapidly decreasing. Similarly, we get a rapid decrease of derivatives of $p_v(\xi)$, so $p_v(\xi) \in \mathcal{S}$. Therefore, the right hand side of (7.13) is well–defined. $\square$

7.1. **Adjoints and products.** Given $p(x,\xi) \in S^m_{\rho,\delta}$ the adjoint has the formula

$$p(x,D)^*v = (2\pi)^{-n} \int p(y,\xi)^* e^{i(x-y)\cdot\xi}v(y)dyd\xi. \tag{7.17}$$

The amplitude $p(y,\xi)^*$ is not a function of (x,ξ), so we need to transform (7.17) into such a function. To do this, define a general class of operators

$$Au(x) = (2\pi)^{-n} \int a(x,y,\xi)e^{i(x-y)\cdot\xi}u(y)dyd\xi. \tag{7.18}$$

We say that $a(x,y,\xi) \in S^m_{\rho,\delta_1,\delta_2}$ if

$$|D_y^\gamma D_x^\beta D_\xi^\alpha a(x,y,\xi)| \le C_{\alpha\beta\gamma}\langle\xi\rangle^{m-\rho|\alpha|+\delta_1|\beta|+\delta_2|\gamma|}. \tag{7.19}$$

We can transform (7.18) into

$$(2\pi)^{-n} \int q(x,\xi)e^{i(x-y)\cdot\xi}u(y)dyd\xi, \qquad q(x,\xi) = (2\pi)^{-n} \int a(x,y,\eta)e^{i(x-y)\cdot(\eta-\xi)}dyd\eta = e^{iD_\xi\cdot D_y}a(x,y,\xi)|_{y=x}. \tag{7.20}$$

Indeed, since $(2\pi)^{-n}\int e^{i(y'-y)\cdot\xi}d\xi = \delta(y'-y)$,

$$\begin{aligned} &(2\pi)^{-2n} \int\int a(x,y',\eta)e^{i(x-y')\cdot(\eta-\xi)}e^{i(x-y)\cdot\xi}u(y)dyd\xi d\eta dy' \\ &= (2\pi)^{-n} \int a(x,y,\eta)\delta(y'-y)e^{i(x-y')\cdot\eta}dy'dyd\eta = (2\pi)^{-n} \int a(x,y,\eta)e^{i(x-y)\cdot\eta}dyd\eta. \end{aligned} \tag{7.21}$$

Now then, formally making a Taylor expansion of $a(x,y,\eta)$,

$$(2\pi)^{-n} \int a(x,x,\eta)e^{i(x-y)\cdot(\eta-\xi)}dyd\eta = \int a(x,x,\eta)\delta(\eta-\xi)d\eta = a(x,x,\xi). \tag{7.22}$$

Next, integrating by parts,

(7.23)

$$(2\pi)^{-n} \int \partial_y a(x,y,\eta)|_{y=x}(y-x)e^{i(x-y)\cdot(\eta-\xi)}dyd\eta = (2\pi)^{-n} \int \partial_y a(x,y,\eta)|_{y=x}\frac{-1}{i}\partial_\eta(e^{i(x-y)\cdot(\eta-\xi)})dyd\eta$$

$$= \frac{1}{i}\int \partial_\eta \partial_y a(x,y,\eta)|_{x=y}e^{i(x-y)\cdot(\eta-\xi)}dyd\eta = iD_\xi \cdot D_y a(x,y,\xi)|_{x=y},$$

which gives

$$q(x,\xi) \sim \sum_{\alpha \geq 0} \frac{i^{|\alpha|}}{\alpha!} D_\xi^\alpha D_y^\alpha a(x,y,\xi)|_{y=x}. \tag{7.24}$$

If $a(x,y,\xi) \in S^m_{\rho,\delta_1,\delta_2}$ with $0 \leq \delta_2 < \rho \leq 1$, then the general term in (7.24) belongs to $S^{m-(\rho-\delta_2)|\alpha|}_{\rho,\delta}$ where $\delta = \max\{\delta_1, \delta_2\}$.

Proposition 14. *If $a(x,y,\xi) \in S^m_{|rho,\delta_1,\delta_2}$ with $0 \leq \delta_2 < \rho \leq 1$, then* (7.18) *defines an operator*

$$A \in OPS^m_{\rho,\delta}, \qquad \delta = \max\{\delta_1,\delta_2\}. \tag{7.25}$$

Furthermore, $A = q(x,D)$ where $q(x,\xi)$ has the asymptotic expansion

$$q(x,\xi) = \sum_{|\alpha|<N} \frac{i^{|\alpha|}}{\alpha!} D_\xi^\alpha D_y^\alpha a(x,y,\xi)|_{y=x} = r_N(x,\xi) \in S^{m-N(\rho-\delta_2)}_{\rho,\delta}. \tag{7.26}$$

Applying Proposition 14 to (7.17), we obtain

Proposition 15. *If $p(x,D) \in OPS^m_{\rho,\delta}$, $0 \leq \delta < \rho \leq 1$, then*

$$p(x,D)^* = p^*(x,D) \in OPS^m_{\rho,\delta}, \tag{7.27}$$

with

$$p^*(x,\xi) \sim \sum_{\alpha \geq 0} \frac{i^{|\alpha|}}{\alpha!} D_\xi^\alpha D_x^\alpha p(x,\xi)^*. \tag{7.28}$$

It is possible to utilize this argument for products of pseudodifferential operators.

Proposition 16. *Given $p_j(x,D) \in OPS^{m_j}_{\rho_j,\delta_j}$, suppose*

$$0 \leq \delta_2 < \rho \leq 1, \qquad \rho = \min\{\rho_1,\rho_2\}. \tag{7.29}$$

Then

$$p_1(x,D)p_2(x,D) = q(x,D) \in OPS^{m_1+m_2}_{\rho,\delta}, \tag{7.30}$$

with $\delta = \max\{\delta_1,\delta_2\}$, and

$$q(x,\xi) \sim \sum_{\alpha \geq 0} \frac{i^{|\alpha|}}{\alpha!} D_\xi^\alpha p_1(x,\xi) D_x^\alpha p_2(x,\xi). \tag{7.31}$$

Proof. Indeed, formally computing the product,

(7.32)

$$p_1(x,D)p_2(x,D) = (2\pi)^{-2n} \int p_1(x,\eta)e^{i(x-y)\cdot\eta} \int p_2(y,\xi)e^{i(y-y')\cdot\xi}u(y')dy'dyd\eta d\xi$$

$$= (2\pi)^{-n} \int e^{i(x-y')\cdot\xi}A(x,\xi)u(y')dy'd\xi, \qquad A(x,\xi) = (2\pi)^{-n}\int p_1(x,\eta)p_2(y,\xi)e^{i(x-y)\cdot(\eta-\xi)}dyd\eta.$$

Again make a Taylor expansion of $p_2(y,\xi)$ in y.

$$(2\pi)^{-n}\int p_1(x,\eta)p_2(x,\xi)e^{i(x-y)\cdot(\eta-\xi)}dyd\eta = \int p_1(x,\eta)p_2(x,\xi)\delta(\eta-\xi)d\eta = p_1(x,\xi)p_2(x,\xi). \tag{7.33}$$

Next, integrating by parts,

$$\begin{aligned} &(2\pi)^{-n}\int p_1(x,\eta)\partial_y p_2(y,\xi)|_{y=x}(y-x)e^{i(x-y)\cdot(\eta-\xi)}dyd\eta \\ &= (2\pi)^{-n}\int p_1(x,\eta)\partial_y p_2(y,\xi)|_{y=x}\frac{-1}{i}\partial_\eta(e^{i(x-y)\cdot(\eta-\xi)})dyd\eta = \partial_\eta p_1(x,\eta)\partial_y p_2(y,\xi)|_{y=x,\eta=\xi}. \end{aligned} \tag{7.34}$$

□

Now then, if $P_j = p_j(x,D) \in OPS^{m_j}_{\rho,\delta}$ are scalar and $0 \le \delta < \rho \le 1$, then the leading order terms in the expansions of the symbols of P_1P_2 and P_2P_1 agree. Therefore, if $P_j \in OPS^{m_j}_{\rho,\delta}$ are scalar, $[P_1,P_2] \in OPS^{m_1+m_2-(\rho-\delta)}_{\rho,\delta}$. Moreover, the leading order term in the expansion of the symbol of $[P_1,P_2]$ is given by the Poisson bracket

$$\{p_1,p_2\}(x,\xi) = \sum_j \frac{\partial p_1}{\partial\xi_j}\frac{\partial p_2}{\partial x_j} - \frac{\partial p_1}{\partial x_j}\frac{\partial p_2}{\partial\xi_j}, \tag{7.35}$$

with

$$[P_1,P_2] = q(x,D), \qquad q(x,\xi) = \frac{1}{i}\{p_1,p_2\}(x,\xi) \qquad mod \qquad S^{m_1+m_2-2(\rho-\delta)}_{\rho,\delta}. \tag{7.36}$$

7.2. **Elliptic operators and parametrices.** We say that $p(x,D) \in OPS^m_{\rho,\delta}$ is elliptic if for some $r < \infty$,

$$|p(x,\xi)^{-1}| \le C\langle\xi\rangle^{-m}, \qquad \text{for} \qquad |\xi| \ge r. \tag{7.37}$$

Therefore, if $\psi(\xi) \in C^\infty(\mathbb{R}^n)$, $\psi = 0$ for $|\xi| \le r$, $\psi = 1$ for $|\xi| \ge 2r$, then by the chain rule,

$$\psi(\xi)p(x,\xi)^{-1} = q_0(x,\xi) \in S^{-m}_{\rho,\delta}. \tag{7.38}$$

Then by (7.31),

$$\begin{aligned} q_0(x,D)p(x,D) &= I + r_0(x,D), \\ p(x,D)q_0(x,D) &= I + \tilde{r}_0(x,D), \end{aligned} \tag{7.39}$$

with

$$r_0(x,\xi), \tilde{r}_0(x,\xi) \in S^{-(\rho-\delta)}_{\rho,\delta}. \tag{7.40}$$

Make the formal expansion

$$I - r_0(x,D) + r_0(x,D)^2 - ... \sim I + s(x,D) \in OPS^0_{\rho,\delta}, \tag{7.41}$$

and setting $q(x,D) = (I + s(x,D))q_0(x,D) \in OPS^{-m}_{\rho,\delta}$, we have

$$q(x,D)p(x,D) = I + r(x,D), \qquad r(x,\xi) \in S^{-\infty}. \tag{7.42}$$

Similarly, let $\tilde{q}(x,D) \in OPS^{-m}_{\rho,\delta}$ satisfy

$$p(x,D)\tilde{q}(x,D) = I + \tilde{r}(x,D), \qquad \tilde{r}(x,\xi) \in S^{-\infty}. \tag{7.43}$$

Now then, evaluating $q(x,D)p(x,D)\tilde{q}(x,D) = q(x,D) = \tilde{q}(x,D)\ mod\ OPS^{-\infty}$. In fact,

$$q(x,D)p(x,D) = I\ mod\ OPS^{-\infty}, \qquad p(x,D)q(x,D) = I\ mod\ OPS^{-\infty}. \tag{7.44}$$

Definition 4. *$q(x, D)$ is the two–sided parametrix for $p(x, D)$.*

8. Hyperbolic evolution equations

Now we turn to examining first order systems of the form

$$\frac{\partial u}{\partial t} = L(t, x, D_x)u + g(t, x), \qquad u(0) = f. \tag{8.1}$$

Assume $L(t, x, \xi) \in S^1_{1,0}$ with smooth dependence on t, so

$$|D^j_t D^\beta_x D^\alpha_\xi L(t, x, \xi)| \leq C_{j\alpha\beta} \langle \xi \rangle^{1-|\alpha|}, \tag{8.2}$$

where $L(t, x, \xi)$ is a $K \times K$ matrix–valued function. Make the hypothesis of symmetric hyperbolicity,

$$L(t, x, \xi)^* + L(t, x, \xi) \in S^0_{1,0}. \tag{8.3}$$

Suppose $f \in H^s(\mathbb{R}^n)$, $s \in \mathbb{R}$, $g \in C(\mathbb{R}, H^s(\mathbb{R}^n))$.

Our strategy is to obtain a solution to (8.1) as a limit of solutions u_ϵ to

$$\frac{\partial u_\epsilon}{\partial t} = J_\epsilon L J_\epsilon u_\epsilon + g, \qquad u_\epsilon(0) = f, \tag{8.4}$$

where

$$J_\epsilon = \varphi(\epsilon D_x), \tag{8.5}$$

for some $\varphi(\xi) \in \mathcal{S}(\mathbb{R}^n)$, $\varphi(0) = 1$. The family of operators J_ϵ is called the Friedrichs mollifier, for $\epsilon \in (0, 1]$, J_ϵ is bounded on $OPS^0_{1,0}$.

For any $\epsilon > 0$, $J_\epsilon L J_\epsilon$ is a bounded linear operator on each H^s and solvability of (8.4) is elementary. The next task is to obtain estimates on u_ϵ independent of $\epsilon \in (0, 1]$. Use the norm $\|u\|_{H^s} = \|\Lambda^s u\|_{L^2}$. Now then,

$$\frac{d}{dt}\|\Lambda^s u_\epsilon(t)\|^2_{L^2} = 2Re(\Lambda^s J_\epsilon L J_\epsilon u_\epsilon, \Lambda^s u_\epsilon) + 2Re(\Lambda^s g, \Lambda^s u_\epsilon). \tag{8.6}$$

Now then,

$$2Re(\Lambda^s J_\epsilon L J_\epsilon u_\epsilon, \Lambda^s u_\epsilon) + 2Re([\Lambda^s, L] J_\epsilon u_\epsilon, \Lambda^s J_\epsilon u_\epsilon). \tag{8.7}$$

By (8.3), $L + L^* = B(t, x, D) \in OPS^0_{1,0}$,

$$(B(t, x, D)\Lambda^s J_\epsilon u_\epsilon, \Lambda^s J_\epsilon u_\epsilon) \leq C\|J_\epsilon u_\epsilon\|^2_{H^s}. \tag{8.8}$$

Proposition 17. *If $p(x, \xi) \in S^0_{1,0}$, then $p : L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$.*

Proof. If $a \in S^{-m}_{\rho,\delta}$ for m sufficiently large, $\rho > \delta$, a has a kernel $K(x, x - y) \lesssim \frac{1}{(1+|x-y|)^{n+1}}$, and thus, $p : L^p \to L^p$ for any $1 \leq p \leq \infty$. Therefore, for any $\sigma > 0$, if $p \in S^{-\sigma}_{\rho,\delta}$, $p : L^2 \to L^2$, since $(P^*P)^k \in S^{-k\sigma}_{\rho,\delta}$, which implies $(P^*P)^k : L^2 \to L^2$, so $P : L^2 \to L^2$.

Now let $q(x, D) = p(x, D)^* p(x, D) \in OPS^0_{\rho,\delta}$. Then suppose $|q(x, \xi)| \leq M - b$ for $b > 0$, so $A(x, \xi) = (M - Re(q(x, \xi)))^{1/2} \in S^0_{\rho,\delta}$, and therefore,

$$A(x, D)^* A(x, D) = M - q(x, D) + r(x, D), \qquad r(x, D) \in OPS^{-(\rho-\delta)}_{\rho,\delta}. \tag{8.9}$$

Since $r(x, D)$ is bounded on $L^2(\mathbb{R}^n)$,

$$M\|u\|^2_{L^2} - \|p(x, D)u\|^2_{L^2} = \|A(x, D)u\|^2_{L^2} - (r(x, D)u, u) \geq -C\|u\|^2_{L^2}. \tag{8.10}$$

Therefore,

$$\|p(x,D)u\|_{L^2}^2 \leq (M+C)\|u\|_{L^2}^2. \tag{8.11}$$

□

Furthermore, by (7.36), $[\Lambda^s, L] \in OPS^s_{1,0}$, so the second term in (8.7) is also bounded by the right hand side of (8.8). Likewise,

$$2(\Lambda^s g, \Lambda^s u_\epsilon) \leq \frac{1}{2}\|\Lambda^s g\|_{L^2}^2 + \frac{1}{2}\|\Lambda^s u_\epsilon\|_{L^2}^2. \tag{8.12}$$

Therefore,

$$\frac{d}{dt}\|\Lambda^s u_\epsilon\|_{L^2}^2 \leq C\|\Lambda^s u_\epsilon(t)\|_{L^2}^2 + C\|g(t)\|_{H^s}^2. \tag{8.13}$$

Therefore, by Gronwall's inequality,

$$\|u_\epsilon(t)\|_{H^s}^2 \leq C(t)[\|f\|_{H^s}^2 + \|g\|_{C([0,t],H^s)}^2], \tag{8.14}$$

independent of $\epsilon \in (0,1]$. Now we can prove the following existence result.

Proposition 18. *If* (8.1) *is symmetric hyperbolic and*

$$f \in H^s(\mathbb{R}^n), \qquad g \in C(\mathbb{R}, H^s(\mathbb{R}^n)), \qquad s \in \mathbb{R}, \tag{8.15}$$

then there is a solution u to (8.1)*, satisfying*

$$u \in L^\infty_{loc}(\mathbb{R}, H^s(\mathbb{R}^n)) \cap Lip(\mathbb{R}, H^{s-1}(\mathbb{R}^n)). \tag{8.16}$$

Proof. Fix $I = [-T, T]$. The bounded family

$$u_\epsilon \in C(I, H^s) \cap C^1(I, H^{s-1}), \tag{8.17}$$

will have a weak limit point satisfying (8.16). Furthermore, u satisfies (8.1). □

This result can be improved to

$$u \in C(\mathbb{R}, H^s(\mathbb{R}^n)) \cap C^1(\mathbb{R}, H^{s-1}(\mathbb{R}^n)). \tag{8.18}$$

Let $f_j \in H^{s+1}$, $f_j \to f$ in H^s, and let u_j solve (8.1) with $u_j(0) = f_j$. Then each $u_j \in L^\infty_{loc}(\mathbb{R}, H^{s+1}) \cap Lip(\mathbb{R}, H^s)$, so in particular each $u_j \in C(\mathbb{R}, H^s)$. Now, $v_j = u - u_j$ solves (8.1) with $v_j(0) = f - f_j$, and $\|f - f_j\|_{H^s} \to 0$ as $j \to \infty$. Using the estimates proving Propsition 18, $\|v_j(t)\|_{H^s} \to 0$ locally uniformly in t, giving $u \in C(\mathbb{R}, H^s)$.

There are other notions of hyperbolicity. In particular, (8.1) is said to be symmetrizable hyperbolic if there is a $K \times K$ matrix valued $S(t,x,\xi) \in S^0_{1,0}$ that is positive definite and such that $S(t,x,\xi)L(t,x,\xi) = \tilde{L}(t,x,\xi)$ satisfies (8.3). In this case, construct $S(t) \in OPS^0_{1,0}$, positive definite, with symbol equal to $S(t,x,\xi) mod S^{-1}_{1,0}$. Then replace the left hand side of (8.6) by

$$\frac{d}{dt}(\Lambda^s u_\epsilon(t), S(t)\Lambda^s u_\epsilon(t))_{L^2}. \tag{8.19}$$

A $K \times K$ system with $L(t,x,\xi) \in S^1_{cl}$ is said to be strictly hyperbolic if its principal symbol $L_1(t,x,\xi)$, homogeneous of degree 1 in ξ has K distinct, purely imaginary eigenvalues, for each x and each $\xi \neq 0$.

Proposition 19. *Whenever* (8.1) *is strictly hyperbolic, it is symmetrizable.*

Proof. If we denote the eigenvalues of $L_1(t,x,\xi)$ by $i\lambda_\nu(t,x,\xi)$, ordered so that $\lambda_1(t,x,\xi) < ... < \lambda_K(t,x,\xi)$, then λ_ν are well-defined C^∞ functions of (t,x,ξ) homogeneous of degree 1 in ξ. If $P_\nu(t,x,\xi)$ are the projections onto the $i\lambda_\nu$–eigenspaces of L_1,

$$P_\nu(t,x,\xi) = \frac{1}{2\pi i}\int_{\gamma_\nu} (\zeta - L_1(t,x,\xi))^{-1} d\zeta, \tag{8.20}$$

where γ_ν is a small circle about $i\lambda_\nu(t,x,\xi)$, then P_ν is smooth and homogeneous of degree 0 in ξ. Then,

$$S(t,x,\xi) = \sum_j P_j(t,x,\xi)^* P_j(t,x,\xi), \tag{8.21}$$

gives the desired symmetrizer. □

Higher order, strictly hyperbolic PDE can be reduced to strictly hyperbolic, first order systems of this nature. Therefore, the first order results can be extended to higher–order hyperbolic equations.

9. Egorov's theorem

Now examine the behavior of operators obtained by conjugating a pseudodifferential operator $P_0 \in OPS^m_{1,0}$ by a solution operator to a scalar hyperbolic equation of the form

$$\frac{\partial u}{\partial t} = iA(t,x,D_x)u, \tag{9.1}$$

where $A = A_1 + A_0$,

$$A_1(t,x,\xi) \in S^1_{cl} \qquad \text{real}, \qquad A_0(t,x,\xi) \in S^0_{cl}. \tag{9.2}$$

Also suppose that $A_1(t,x,\xi)$ is homogeneous in ξ for $|\xi| \geq 1$. Then let $S(t,s)$ be the solution operator to (9.1) taking $u(s)$ to $u(t)$. This is a bounded operator on each Sobolev space H^σ with inverse $S(s,t)$. Then set

$$P(t) = S(t,0)P_0S(0,t). \tag{9.3}$$

Theorem 3 (Egorov's theorem)**.** *If $P_0 = p_0(x,D) \in OPS^m_{1,0}$, then for each t, $P(t) \in OPS^m_{1,0}$ modulo a smoothing operator. The principal symbol of $P(t)$ mod $S^{m-1}_{1,0}$ at a point (x_0,ξ_0) is equal to $p_0(y_0,\eta_0)$, where (y_0,η_0) is obtained from (x_0,ξ_0) by following the flow $\mathcal{C}(t)$ generated by the (time–dependent) Hamiltonian vector field*

$$H_{A_1(t,x,\xi)} = \sum_{j=1}^n (\frac{\partial A}{\partial \xi_j}\frac{\partial}{\partial x_j} - \frac{\partial A_1}{\partial x_j}\frac{\partial}{\partial \xi_j}). \tag{9.4}$$

Proof. To start the proof, differentiating (9.3) gives

(9.5)

$$P'(t) = S'(t,0)P_0S(0,t) + S(t,0)P_0S'(0,t) = iA(t,x,D_x)S(t,0)P_0S(0,t) - iS(t,0)P_0S(0,t)A(t,x,D_x)$$

$$P'(t) = i[A(t,x,D_x),P(t)], \qquad P(0) = P_0.$$

Now then, construct an approximate solution $Q(t)$ to (9.5) and show that $Q(t) - P(t)$ is a smoothing operator. That is, construct $Q(t)$ such that

$$Q'(t) = i[A(t,x,D_x),Q(t)] + R(t), \qquad Q(0) = P_0, \tag{9.6}$$

where $R(t)$ is a smooth family of operators in $OPS^{-\infty}$, where

$$q(t,x,\xi) \sim q_0(t,x,\xi) + q_1(t,x,\xi) + ... \tag{9.7}$$

The symbol of $i[A, Q(t)]$ is of the form

$$H_{A_1} q + \{A_0, q\} + i \sum_{|\alpha|\geq 2} \frac{i^{|\alpha|}}{\alpha!} (A^{(\alpha)} q_{(\alpha)} - q^{(\alpha)} A_{(\alpha)}), \tag{9.8}$$

where $A^{(\alpha)} = D_\xi^\alpha A$ and $A_{(\alpha)} = D_x^\alpha A$. Since we want the difference between this and $\frac{\partial q}{\partial t}$ to have order $-\infty$, define $q_0(t, x, \xi)$ by

$$(\frac{\partial}{\partial t} - H_{A_1}) q_0(t, x, \xi) = 0, \qquad q_0(0, x, \xi) = p_0(x, \xi). \tag{9.9}$$

Therefore, $q_0(t, x_0, \xi_0) = p_0(y_0, \eta_0)$ and $q_0(t, x, \xi) \in S^m_{1,0}$. Equation (9.9) is called a transport equation.

Remark 8. *Indeed, observe that*

$$\frac{\partial}{\partial t} q_0(t, x_0, \xi_0) = \frac{\partial}{\partial t} p_0(y_0, \eta_0) = \frac{\partial p_0}{\partial x} \cdot \dot{y}_0 + \frac{\partial p}{\partial \xi} \cdot \dot{\eta}_0 = \frac{\partial p_0}{\partial x} \cdot H_{A_1} y_0 + \frac{\partial p_0}{\partial \xi} \cdot H_{A_1} \eta_0 = H_{A_1} p_0(y_0, \eta_0). \tag{9.10}$$

Now recursively obtain transport equations

$$(\frac{\partial}{\partial t} - H_{A_1}) q_j(t, x, \xi) = b_j(t, x, \xi), \qquad q_j(0, x, \xi) = 0. \tag{9.11}$$

Remark 9. *Set*

$$b_1(t, x, \xi) = \{A_0, q_0\} + i \sum_{|\alpha|\geq 2} \frac{i^\alpha}{\alpha!} (A^{(\alpha)} q_{0,(\alpha)} - q_0^{(\alpha)} A_{(\alpha)}) \in S^{m-1}_{1,0}, \tag{9.12}$$

and suppose that $q_1(t, x, \xi)$ *solves* (9.11). *Then,*

$$\begin{aligned} \frac{\partial}{\partial t}(q_0 + q_1) = H_{A_1} q_0 + H_{A_1} q_1 + \{A_0, q_0\} + i \sum_{|\alpha|\geq 2} \frac{i^\alpha}{\alpha!} (A^{(\alpha)} q_{0,(\alpha)} - q_0^{(\alpha)} A_{(\alpha)}) \\ = i[A(t, x, D), Q_0 + Q_1] + R(t), \end{aligned} \tag{9.13}$$

where $R(t)$ *with symbol* $-b_1(t, x, \xi) \in OPS^{m-2}_{1,0}$,

$$b_1(t, x, \xi) = \{A_0, q_1\} + i \sum_{|\alpha|\geq 2} \frac{i^\alpha}{\alpha!} (A^{(\alpha)} q_{1,(\alpha)} - q_1^{(\alpha)} A_{(\alpha)}). \tag{9.14}$$

Finally, we show that $P(t) - Q(t)$ is a smoothing operator. This is equivalent to showing that for any $f \in H^\sigma(\mathbb{R}^n)$,

$$v(t) - w(t) = P(t)S(t, 0)f - Q(t)S(t, 0)f = S(t, 0)P_0 f - Q(t)S(t, 0)f \in H^\infty(\mathbb{R}^n), \tag{9.15}$$

where $H^\infty(\mathbb{R}^n) = \cap_s H^s(\mathbb{R}^n)$. Now then,

$$\frac{\partial v}{\partial t} = iA(t, x, D_x)v, \qquad v(0) = P_0 f, \tag{9.16}$$

while by (9.6),

$$\frac{\partial w}{\partial t} = iA(t, x, D_x)w + g, \qquad w(0) = P_0 f, \tag{9.17}$$

where

$$g = R(t)S(t, 0)w \in C^\infty(\mathbb{R}, H^\infty(\mathbb{R}^n)). \tag{9.18}$$

Therefore,

$$\frac{\partial}{\partial t}(v-w) = iA(t,x,D_x)(v-w) - g, \qquad v(0)-w(0)=0. \tag{9.19}$$

Therefore, by energy estimates, $v(t) - w(t) \in H^{\infty}$ for any $f \in H^{\sigma}(\mathbb{R}^n)$, which completes the proof. □

Remark 10. *A check on the proof shows that*

$$P_0 \in OPS^m_{cl} \Rightarrow P(t) \in OPS^m_{cl}. \tag{9.20}$$

Indeed, since $A_1 \in S^1_{cl}$,

$$\frac{\partial P}{\partial t} = H_{A_1} P \in S^1_{cl}, \tag{9.21}$$

since $\frac{\partial A}{\partial \xi} \in S^0_{cl}$, $\frac{\partial P}{\partial x} \in S^1_{cl}$, $\frac{\partial A}{\partial x} \in S^1_{cl}$, *and* $\frac{\partial A}{\partial \xi} \in S^0_{cl}$.

Using the same argument,

Proposition 20. *With* $A(t,x,D_x)$ *as before,*

$$P_0 \in OPS^m_{\rho,\delta} \Rightarrow P(t) \in OPS^m_{\rho,\delta}, \tag{9.22}$$

provided

$$\rho > \frac{1}{2}, \qquad \delta = 1-\rho. \tag{9.23}$$

Proof. We need $\delta = 1-\rho$ to ensure that $p(\mathcal{C}(t)(x,\xi)) \in S^m_{\rho,\delta}$ and $\rho > \delta$ to ensure that the transport equations generate $q_j(t,x,\xi)$ of progressively lower order. □

10. Microlocal regularity

Now define the notion of the wave front set of a distribution $u \in H^{-\infty}(\mathbb{R}^n) = \cup_s H^s(\mathbb{R}^n)$, which refines the notion of singular support. If $p(x,\xi) \in S^m$ has principal symbol $p_m(x,\xi)$, scalar and homogeneous in ξ, then the characteristic set of $P = p(x,D)$ is given by

$$Char(P) = \{(x,\xi) \in \mathbb{R}^n \times (\mathbb{R}^n \setminus \{0\}) : p_m(x,\xi) = 0\}. \tag{10.1}$$

If $p_m(x,\xi)$ is a $K \times K$ matrix, take the determinant. Equivalently, (x_0,ξ_0) is noncharacterstic for P, or P is elliptic at (x_0,ξ_0), if $|p(x,\xi)^{-1}| \leq C|\xi|^{-m}$, for (x,ξ) in a small conic neighborhood of (x_0,ξ_0) and $|\xi|$ large. A conic set is invariant under the dilations $(x,\xi) \mapsto (x, r\xi)$, $r \in (0,\infty)$. The wave front set is defined by

$$WF(u) = \cap\{Char(P) : P \in OPS^0, \qquad Pu \in C^{\infty}\}. \tag{10.2}$$

Remark 11. *$WF(u)$ is a closed conic subset of* $\mathbb{R}^n \times (\mathbb{R}^n \setminus 0)$.

Proposition 21. *If* π *is the projection* $\pi : (x,\xi) \mapsto x$, *then*

$$\pi(WF(u)) = singsupp(u). \tag{10.3}$$

Proof. First show that $\pi(WF(u)) \subset singsupp(u)$. If $x_0 \notin singsupp(u)$ then there exists $\varphi \in C_0^{\infty}(\mathbb{R}^n)$, $\varphi = 1$ near x_0, such that $\varphi u \in C_0^{\infty}(\mathbb{R}^n)$. Since $(x_0,\xi) \notin Char(\varphi)$ for any $\xi \neq 0$, so $\pi(WF(u)) \subset singsupp(u)$.

Now suppose $x_0 \notin \pi(WF(u))$. Then for any $\xi \neq 0$, there is $Q \in OPS^0$ such that $(x_0,\xi) \notin Char(Q)$ and $Qu \in C^{\infty}$. Therefore, we can construct finitely many $Q_j \in OPS^0$ such that $Q_j u \in$

C^∞ and each (x_0, ξ) with $|\xi| = 1$ is noncharacteristic for some Q_j. Let $Q = \sum Q_j^* Q_j \in OPS^0$. Then Q is elliptic near x_0 and $Qu \in C^\infty$, so u is C^∞ near x_0. □

Now define the associated notion of $ES(P)$ for a pseudodifferential operator. Let U be an open conic subset of $\mathbb{R}^n \times (\mathbb{R}^n \setminus 0)$. We say that $p(x,\xi) \in S^m_{\rho,\delta}$ has order $-\infty$ on U if for each closed conic set V of U, for each N,

$$|D_x^\beta D_\xi^\alpha p(x,\xi)| \le C_{\alpha\beta N V} \langle \xi \rangle^{-N}, \qquad (x,\xi) \in V. \tag{10.4}$$

Definition 5 (Essential support)**.** *The essential support of P (and of $p(x,\xi)$) is the smallest closed conic set on the complement of which $p(x,\xi)$ has order $-\infty$.*

It follows from symbolic calculus that

$$ES(P_1 P_2) \subset ES(P_1) \cap ES(P_2), \tag{10.5}$$

provided $P_j \in OPS^{m_j}_{\rho_j,\delta_j}$ and $\rho_1 > \delta_2$. Indeed, recall that the symbol of $P_1 P_2$ is given by

$$\sum_\alpha \frac{i^\alpha}{\alpha!} D_\xi^\alpha p_1(x,\xi) D_x^\alpha p_2(x,\xi). \tag{10.6}$$

If p_1 or p_2 satisfies (10.4) at (x,ξ), (10.4) also holds for (10.6).

To relate $WF(Pu)$ to $WF(u)$ and $ES(P)$, we begin with the following.

Lemma 6. *Let $u \in H^{-\infty}(\mathbb{R}^n)$ and suppose that U is a conic open set satisfying*

$$WF(u) \cap U = \emptyset. \tag{10.7}$$

If $P \in OPS^m_{\rho,\delta}$, $\rho > 0$, $\delta < 1$, and $ES(P) \subset U$, then $Pu \in C^\infty$.

Proof. Take $P_0 \in OPS^0$ with symbol identically 1 on a conic neighborhood of $ES(P)$ so that $P = PP_0 \ mod\ OPS^{-\infty}$, it suffices to conclude that $P_0 u \in C^\infty$, so we can specialize the hypothesis to $P \in OPS^0$.

By hypothesis, we can find $Q_j \in OPS^0$ such that $Q_j u \in C^\infty$, and each $(x,\xi) \in ES(P)$ is noncharacteristic for some Q_j, and if $Q = \sum_j Q_j^* Q_j$, then $Qu \in C^\infty$ and $Char(Q) \cap ES(P) = \emptyset$. Then there exists an operator $A \in OPS^0$ so that $AQ = P\ mod\ OPS^{-\infty}$. Indeed, let $\tilde{Q}$ be an elliptic operator whose symbol equals that of Q on a conic neighborhood of $ES(P)$ and let $\tilde{Q}^{-1}$ denote a parametrix for $\tilde{Q}$. Then set $A = P\tilde{Q}^{-1}$, and $(modC^\infty)$, $Pu = AQu \in C^\infty$. □

Now state a basic result on the preservation of wave front sets by a pseudodifferential operator.

Proposition 22. *If $u \in H^{-\infty}$ and $P \in OPS^m_{\rho,\delta}$ with $\rho > 0$, $\delta < 1$, then*

$$WF(Pu) \subset WF(u) \cap ES(P). \tag{10.8}$$

Proof. First show that $WF(Pu) \subset ES(P)$. Suppose $(x_0,\xi_0) \notin ES(P)$. Choose $Q = q(x,D) \in OPS^0$ such that $q(x,\xi) = 1$ on a conic neighborhood of (x_0,ξ_0) and $ES(Q) \cap ES(P) = \emptyset$. Therefore, $QP \in OPS^{-\infty}$, so $QPu \in C^\infty$. Therefore, $(x_0,\xi_0) \notin WF(Pu)$.

To show that $WF(Pu) \subset WF(u)$, let Γ be a conic neighborhood of $WF(u)$ and write $P = P_1 + P_2$, where $P_j \in OPS^m_{\rho,\delta}$ with $ES(P_1) \subset \Gamma$ and $ES(P_2) \cap WF(u) = \emptyset$. By Lemma 6, $P_2 u \in C^\infty$. Thus, $WF(u) = WF(P_1 u) \subset \Gamma$, which shows that $WF(Pu) \subset WF(u)$. □

Definition 6. *A pseudodifferential operator of type (ρ,δ) with $\rho > 0$ and $\delta < 1$ is microlocal.*

Corollary 2. *If $P \in OPS^m_{\rho,\delta}$ is elliptic, $0 \leq \delta < \rho \leq 1$, then*

$$WF(Pu) = WF(u). \tag{10.9}$$

Proof. We have seen that $WF(Pu) \subset WF(u)$. On the other hand, if $E \in OPS^{-m}_{\rho,\delta}$ is the parametrix of P, $WF(u) = WF(EPu) \subset WF(Pu)$. In fact, for a general P, $WF(u) \subset WF(Pu) \cup Char(P)$. □

Now let e^{itA} be the solution operator to the scalar hyperbolic equation $\frac{\partial u}{\partial t} = iA(x, D)u$. Suppose $A(x, \xi) \in S^1_{cl}$ with real principal symbol and $WF(u) = \Sigma$. Then there is a countable family of symbols that vanishes in a neighborhood of Σ, but such that

$$\Sigma = \cap_j \{(x, \xi) : p_j(x, \xi) = 0\}. \tag{10.10}$$

We know that $p_j(x, D)u \in C^\infty$ for each j. By Egorov's theorem, we want to construct a family of pseudodifferential operators $q_j(x, D) \in OPS^0$ such that $q_j(x, D)e^{itA}u \in C^\infty$. Let $q_j(x, D) = e^{itA} p_j(x, D)e^{-itA}$. By Egorov's theorem, $q_j(x, D) \in OPS^0$ modulo a smoothing operator and gives the principal symbol for $q_j(x, D)$. Since $p_j(x, D)u \in C^\infty$, $e^{itA} p_j(x, D)u \in C^\infty$, which implies that $q_j(x, D)e^{itA}u \in C^\infty$. Therefore, $WF(e^{itA}u)$ is contained in the intersection of characteristics of the $q_j(x, D)$, which is precisely equal to $\mathcal{C}(t)\Sigma$,

$$WF(e^{itA}u) \subset \mathcal{C}(t)WF(u). \tag{10.11}$$

Since the argument is reversible, $u = e^{-itA}(e^{itA}u)$, the wave front sets are identical.

Proposition 23. *If $A = A(x, D) \in OPS^1$ is scalar with real principal symbol, then for $u \in H^{-\infty}$,*

$$WF(e^{itA}u) = \mathcal{C}(t)WF(u). \tag{10.12}$$

The same holds for the solution operator $S(t, 0)$ to a time–dependent scalar hyperbolic equation.

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