A COURSE ON PARABOLIC PDE

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Contents

1. Fundamental solution to the heat equation

Let $\bar{\mathcal{M}}$ be a compact, Riemannian manifold with boundary. The heat equation is given by

$$\frac{\partial u}{\partial t} = \Delta u, \qquad u(0,x) = f(x). \tag{1.1}$$

If $\partial \mathcal{M} \neq \emptyset$, then impose the Dirichlet condition,

$$u(t,x) = 0, \qquad x \in \partial \mathcal{M}. \tag{1.2}$$

We could also impose the Neumann boundary condition $\frac{\partial u}{\partial \nu} = 0$ for $x \in \partial \mathcal{M}$.

It is possible to construct solutions to (1.1)–(1.2) using eigenfunctions of Δ. Indeed, let $\{u_j\}$ be the orthonormal basis of Δ in $L^2(\mathcal{M})$,

$$u_j \in H_0^1(\mathcal{M}) \cap C^{\infty}(\bar{\mathcal{M}}), \qquad \Delta u_j = -\lambda_j u_j, \qquad 0 \leq \lambda_j < \infty. \tag{1.3}$$

Given $f \in L^2(\mathcal{M})$, we can write

$$f = \sum_j \hat{f}(j) u_j, \qquad \hat{f}(j) = (f, u_j). \tag{1.4}$$

Then set

$$u(t,x) = \sum_j \hat{f}(j) e^{-t\lambda_j} u_j(x). \tag{1.5}$$

Define the function space

$$\mathcal{D}_s = \{v \in L^2(\mathcal{M}) : \sum_{j \geq 0} |\hat{v}(j)|^2 \lambda_j^s < \infty\} = \{v \in L^2(\mathcal{M}) : \sum_{j \geq 0} \hat{v}(j) \lambda_j^{s/2} u_j \in L^2(\mathcal{M})\}. \tag{1.6}$$

Now then, since

$$u_j \in H_0^1(\mathcal{M}) \cap C^{\infty}(\bar{\mathcal{M}}), \qquad T u_j = -\mu_j u_j, \qquad \Delta u_j = -\lambda_j u_j, \qquad \lambda_j = \frac{1}{\mu_j}, \tag{1.7}$$

so an equivalent characterization of $\mathcal{D}_s$ is

$$\mathcal{D}_s = (-T)^{s/2} L^2(\mathcal{M}). \tag{1.8}$$

Clearly, $\mathcal{D}_0 = L^2(\mathcal{M})$ and $\mathcal{D}_2 = T L^2(\mathcal{M})$. By the elliptic regularity theorem,

$$\mathcal{D}_2 = H^2(\mathcal{M}) \cap H_0^1(\mathcal{M}). \tag{1.9}$$

In general, $\mathcal{D}_{s+2} = T\mathcal{D}_s$, so by induction,

$$\mathcal{D}_{2k} \subset H^{2k}(\mathcal{M}), \qquad k = 1, 2, 3, ... \tag{1.10}$$

Lemma 1.

$$\mathcal{D}_1 = H_0^1(\mathcal{M}). \tag{1.11}$$

Proof. Observe that $\mathcal{D}_s$ is the completion of the space of finite linear combinations of eigenfunctions $\{u_j\}$, call it $\mathcal{F}$, with respect to the $\mathcal{D}_s$ norm, defined by

$$\|v\|_{\mathcal{D}_s}^2 = \sum_j |\hat{v}(j)|^2 \lambda_j^s. \tag{1.12}$$

Now then, if $v \in \mathcal{F}$,

$$(dv, dv) = (v, -\Delta v) = \sum_j (v, u_j)(u_j, -\Delta v) = \sum_j |\hat{v}(j)|^2 \lambda_j. \tag{1.13}$$

Therefore, for $v \in \mathcal{F}$,

$$\|v\|_{\mathcal{D}_1}^2 = \|dv\|_{L^2(\mathcal{M})}^2. \tag{1.14}$$

In fact, $\mathcal{D}_s$ is the completion of $\mathcal{D}_\sigma$ for any $\sigma > s$. Since (1.14) holds for all $v \in \mathcal{D}_2$ and $\mathcal{D}_2 = H^2(\mathcal{M}) \cap H_0^1(\mathcal{M})$, which implies (1.11). □

Now then, by (1.5),

$$f \in \mathcal{D}_s \Rightarrow u \in C(\mathbb{R}^+, \mathcal{D}_s); \qquad \partial_t^j u \in C(\mathbb{R}^+, \mathcal{D}_{s-2j}). \tag{1.15}$$

It is clear from (1.5) that $\partial_t u = \Delta u$ for $t > 0$. If $f \in \mathcal{D}_s$ with $s > \frac{n}{2}$, then $u \in C([0,\infty) \times \bar{\mathcal{M}})$ and u satisfies (1.1) and (1.2) in the ordinary sense.

Uniqueness for solutions to (1.1) and (1.2) within the class

$$C(\mathbb{R}^+, \mathcal{D}_s) \cap C^1(\mathbb{R}^+, \mathcal{D}_{s-2}), \tag{1.16}$$

follows from the simple energy estimate

$$\frac{d}{dt}\|u(t)\|_{\mathcal{D}_{s-2}}^2 = 2Re(\frac{\partial u}{\partial t}, u(t))_{\mathcal{D}_{s-2}} = -2\|u(t)\|_{\mathcal{D}_{s-1}}^2 \leq 0. \tag{1.17}$$

Denote the solution to (1.1)–(1.2) by

$$u(t,x) = e^{t\Delta} f(x). \tag{1.18}$$

Now, by (1.5),

$$u \in C^\infty((0,\infty), \mathcal{D}_\sigma), \qquad \text{for all} \qquad \sigma \in \mathbb{R}. \tag{1.19}$$

In particular, for any $f \in \mathcal{D}_s$,

$$u \in C^\infty((0,\infty) \times \overline{\mathcal{M}}). \tag{1.20}$$

The heat equation satisfies the maximum principle.

Proposition 1. *If $u \in C([0,a) \times \overline{\mathcal{M}}) \cap C^2((0,a) \times \mathcal{M})$ and u solves (1.1) in $(0,a) \times \mathcal{M}$, then*

$$\sup_{[0,a)\times\bar{\mathcal{M}}} u(t,x) = \max\{\sup_{x\in\overline{\mathcal{M}}} u(0,x), \sup_{x\in\partial\mathcal{M}, t\in[0,a)} u(t,x)\}. \tag{1.21}$$

In particular, if (1.2) and (1.3) hold, then

$$\sup_{[0,a)\times\overline{\mathcal{M}}} u(t,x) = \sup_{\bar{\mathcal{M}}} f(x). \tag{1.22}$$

Proof. Since u solves (1.1) if and only if $-u$ solves (1.1), to prove (1.21), it suffices to show that

$$u > 0 \qquad \text{on} \qquad \{0\} \times \overline{\mathcal{M}} \cup [0,a) \times \partial\mathcal{M} \qquad \text{implies} \qquad u \geq 0, \qquad \text{on} \qquad [0,a) \times \mathcal{M}. \tag{1.23}$$

Indeed, u solves (1.1) if and only if $-u$ solves (1.1), and (1.23) certainly implies that (1.21) holds for $-u$.

Set $u_\epsilon(t,x) = u(t,x) + \epsilon t$. For any $\epsilon > 0$, $u_\epsilon > 0$ on $[0,a) \times \mathcal{M}$. Indeed, if this implication is false, then since $\overline{\mathcal{M}}$ is compact, there is a smallest $t_0 \in (0,a)$ such that $u_\epsilon(t_0, x_0) = 0$ for some $x_0 \in \mathcal{M}$. Therefore, $\partial_t u_\epsilon(t_0,x_0) \leq 0$ and $\Delta u_\epsilon(t_0,x_0) \geq 0$. However, since $\partial_t u_\epsilon = \Delta u_\epsilon + \epsilon$, there is a contradiction. □

Corollary 1. *For any $f \in C_0(\mathcal{M})$, $u \in C([0,\infty) \times \overline{\mathcal{M}})$.*

For $\delta_p \in \mathcal{D}_{-n/2-\epsilon}$ for all $\epsilon > 0$, the fundamental solution to the heat equation is

$$H(t,x,p) = e^{t\Delta}\delta_p(x). \tag{1.24}$$

By (1.20), $H(t,x,p)$ is smooth in (t,x) for $t \geq 0$. Since δ_p is a limit in $\mathcal{D}_{-n/2-\epsilon}$ of elements of $C_0^\infty(\mathcal{M})$ that are ≥ 0, it follows that

$$H(t,x,p) \geq 0, \qquad \text{for} \qquad t \in (0,\infty), \qquad x \in \overline{\mathcal{M}}, \qquad p \in \mathcal{M}. \tag{1.25}$$

In fact, there is a variant of the strong maximum principle. that strengthens (1.25) to $H(t,x,p) > 0$ for $t > 0$, $x, p \in \mathcal{M}$.

For the heat equation on $\mathbb{R}^n$, then by the Fourier inversion formula,

$$e^{t\Delta}f = (2\pi)^{-n/2}\int e^{-t|\xi|^2}e^{ix\cdot\xi}\hat{f}(\xi)d\xi = (2\pi)^{-n}\int e^{-t|\xi|^2}e^{ix\cdot\xi}\int e^{-iy\cdot\xi}f(y)dyd\xi. \tag{1.26}$$

By Fubini's theorem, for $t > 0$ and $f \in L^1(\mathbb{R}^n)$,

$$e^{t\Delta}f = (2\pi)^{-n}\int f(y)\int e^{-t|\xi|^2}e^{i(x-y)\cdot\xi}d\xi dy. \tag{1.27}$$

Completing the square,

$$-t|\xi|^2 + i(x-y)\cdot\xi = -t|\xi - \frac{i}{2t}(x-y)|^2 - \frac{|x-y|^2}{4t}. \tag{1.28}$$

Now then, computing an integral in radial coordinates and making a change of variables,

$$\int e^{-|x|^2}dx = A_{n-1}\int_0^\infty e^{-r^2}r^{n-1}dr = \frac{A_{n-1}}{2}\int_0^\infty e^{-u}u^{\frac{n-2}{2}}du = \frac{A_{n-1}}{2}\Gamma(\frac{n}{2}). \tag{1.29}$$

Using the identity that

$$\pi^{n/2} = \int e^{-|x|^2}dx = \frac{A_{n-1}}{2}\Gamma(\frac{n}{2}), \tag{1.30}$$

which gives the identity,

$$A_{n-1} = \frac{2\pi^{n/2}}{\Gamma(\frac{n}{2})}. \tag{1.31}$$

Using (1.28) and contour integration,

$$\int e^{-t|\xi|^2}e^{i(x-y)\cdot\xi}d\xi = \frac{\pi^{n/2}}{t^{n/2}}. \tag{1.32}$$

Plugging (1.32) into (1.27),

$$e^{t\Delta}f = \frac{1}{(4\pi t)^{n/2}}\int e^{-\frac{|x-y|^2}{4t}}f(y)dy. \tag{1.33}$$

Remark 1. *Since $e^{-\frac{|x-y|^2}{4t}} > 0$ for all $x, y \in \mathbb{R}^n$, $e^{t\Delta}f > 0$ for all $x \in \mathbb{R}^n$ and $t > 0$.*

It is straightforward to verify that

$$\frac{1}{(4\pi t)^{n/2}}\int e^{-\frac{|x-y|^2}{4t}}dy = 1. \tag{1.34}$$

Therefore,

$$e^{t\Delta} : L^p \to L^p, \qquad 1 \leq p \leq \infty. \tag{1.35}$$

Next, since $|x|^N e^{-|x|^2} \lesssim_N e^{\frac{|x|^2}{2}}$,

$$\|\nabla^N e^{\frac{|x|^2}{4t}}\|_{L^1} \lesssim_N t^{-\frac{N}{2}}. \tag{1.36}$$

Therefore,

$$\|\nabla^N e^{t\Delta}\|_{L^p \to L^p} \lesssim_N t^{-\frac{N}{2}}. \tag{1.37}$$

Furthermore, since

$$\|\frac{1}{(4\pi t)^{n/2}} e^{-\frac{|x-y|^2}{4t}}\|_{L^\infty} \leq \frac{1}{(4\pi t)^{n/2}}, \tag{1.38}$$

$$\|e^{t\Delta}\|_{L^1 \to L^\infty} \leq \frac{1}{(4\pi t)^{n/2}}. \tag{1.39}$$

Interpolating (1.35) and (1.39), for $1 \leq p \leq q \leq \infty$,

$$\|e^{t\Delta}\|_{L^p \to L^q} \leq \frac{1}{(4\pi t)^{\frac{n}{2}(\frac{1}{p} - \frac{1}{q})}}. \tag{1.40}$$

Moreover, by (1.36),

$$\|\nabla^N e^{t\Delta}\|_{L^p \to L^q} \lesssim_N t^{-\frac{N}{2}} \frac{1}{(4\pi t)^{\frac{n}{2}(\frac{1}{p} - \frac{1}{q})}}. \tag{1.41}$$

2. Semigroups

Definition 1 (Semigroup)**.** *If V is a Banach space, a one–parameter semigroup of operators on V is a set of bounded operators*

$$P(t) : V \to V, \qquad t \in [0, \infty), \tag{2.1}$$

satisfying

$$P(s+t) = P(s)P(t), \qquad \text{for all} \qquad s, t \in \mathbb{R}^+, \tag{2.2}$$

and

$$P(0) = I. \tag{2.3}$$

We also require strong continuity, that is,

$$t_j \to t, \qquad P(t_j)v \to P(t)v, \qquad \text{for each} \qquad v \in V. \tag{2.4}$$

If $P(t)$ is defined for all $t \in \mathbb{R}$ and satisfies the former conditions, we say that $P(t)$ is a one–parameter group of operators.

For example, consider the translation group

$$T_p(t) : L^p(\mathbb{R}) \to L^p(\mathbb{R}), \qquad 1 \leq p < \infty, \tag{2.5}$$

defined by

$$T_p(t)f(x) = f(x-t). \tag{2.6}$$

It is clear that (2.1)–(2.3) hold for each t. Indeed, $\|T_p(t')\| = 1$ for each t, and $\|T_p(t) - T_p(t')\| = 2$ if $t \neq t'$. Indeed, apply the difference to a function supported on an interval of length $\frac{|t-t'|}{2}$. To verify strong continuity, observe that the space $C_0(\mathbb{R})$ is dense in $L^p(\mathbb{R})$ for $1 \leq p < \infty$. If $f \in C_0(\mathbb{R})$, then $T_p(t_j)f(x) = f(x - t_j)$ has support in a fixed compact set and converges uniformly to $f(x-t)$, which implies convergence in L^p norm. Convergence in a dense set implies convergence in V.

Lemma 2. *Let $T_j \in \mathcal{L}(V, W)$ be uniformly bounded. Let L be a dense, linear subspace of V, and suppose*

$$T_j v \to T_0 v, \qquad as \qquad j \to \infty, \tag{2.7}$$

in the W-norm, for each $v \in L$. Then (2.7) holds for each $v \in V$.

Proof. Given $v \in V$ and $\epsilon > 0$. Choose $w \in L$ such that $\|v - w\| < \epsilon$. Suppose $\|T_j\| \leq M$ for all j. Then,

$$\|T_j v - T_0 v\| \leq \|T_j v - T_j w\| + \|T_j w - T_0 w\| + \|T_0 w - T_0 v\| \leq \|T_j w - T_0 w\| + 2M\|v - w\|. \tag{2.8}$$

Therefore,

$$\limsup_{j \to \infty} \|T_j v - T_0 v\| \leq 2M\epsilon. \tag{2.9}$$

Since $\epsilon > 0$ is arbitrary, the proof is complete. $\square$

Definition 2 (Infinitesimal generator)**.** *A one parameter semigroup $P(t)$ of operators on V has an infinitesimal generator A, which is an operator on V, often unbounded, which is defined by*

$$Av = \lim_{h \searrow 0} \frac{1}{h}(P(h)v - v), \tag{2.10}$$

on the domain

$$\mathcal{D}(A) = \{v \in V : \lim_{h \searrow 0} \frac{1}{h}(P(h)v - v) \qquad exists\ in \qquad V\}. \tag{2.11}$$

For example, let A_p be the infinitesimal generator of the group $T_p(t)$ given by (2.5). By definition, $f \in L^p(\mathbb{R})$ belongs to $\mathcal{D}(A_p)$ if and only if

$$\lim_{h \searrow 0} \frac{1}{h}(f(x - h) - f(x)), \tag{2.12}$$

converges in L^p–norm as $h \to 0$. The limit (2.12) always exists in $C_0^\infty(\mathbb{R})$, and the limit is equal to $-\frac{d}{dx}u$. In fact, we have the following.

Proposition 2. *For $1 \leq p < \infty$, the group $T_p(t)$ given by (2.5)–(2.7) has infinitesimal generator A_p given by*

$$A_p f = -\frac{df}{dx}, \tag{2.13}$$

for $f \in \mathcal{D}(A_p)$, with

$$\mathcal{D}(A_p) = \{f \in L^p(\mathbb{R}) : f' \in L^p(\mathbb{R})\}, \tag{2.14}$$

where $f' = \frac{df}{dx}$ is considered a priori as a distribution.

Proof. The argument above shows that $\mathcal{D}(A_p)$ is contained in the right hand side of (2.14). The reverse containment is derived as a consequence of the following result, with $\mathcal{L} = C_0^\infty(\mathbb{R})$.

Proposition 3. *Let $P(t)$ be a one–parameter semigroup on B, with infinitesimal generator A. Let $\mathcal{L}$ be a weak$*$ dense, linear subspace of B', and suppose $P(t)'\mathcal{L} \subset L$. Suppose that $u, v \in B$ and that*

$$\lim_{h \to 0} \frac{1}{h}\langle P(h)u - u, w\rangle = \langle v, w\rangle, \qquad \forall w \in \mathcal{L}. \tag{2.15}$$

Then $w \in \mathcal{D}(A)$ and $Au = v$.

Proof. The hypothesis (2.15) implies that $\langle P(t)u, w\rangle$ is differentiable, and that for any $w \in \mathcal{L}$,

$$\frac{d}{dt}\langle P(t)u, w\rangle = \frac{d}{ds}\langle P(t)P(s)u, w\rangle|_{s=0} = \frac{d}{ds}\langle P(s)u, P(t)'w\rangle|_{s=0} = \langle v, P(t)'w\rangle = \langle P(t)v, w\rangle. \tag{2.16}$$

Therefore,

$$\langle P(t)u - u, w\rangle = \int_0^t \langle P(s)v, w\rangle ds, \tag{2.17}$$

for all $w \in \mathcal{L}$. The weak$*$ denseness of $\mathcal{L}$ implies that $P(t)u - u = \int_0^t P(s)vds$, and the convergence in the B-norm of

$$\frac{1}{h}(P(h)u - u) = \frac{1}{h}\int_0^h P(s)vds, \tag{2.18}$$

to v as $h \to 0$ follows. □

Now then, it is clear that the right hand side of (2.14) is contained in $\mathcal{D}(A_p)$. □

Proposition 4. *The infinitesimal generator A of $P(t)$ is a closed, densely defined operator. We have*

$$P(t)\mathcal{D}(A) \subset \mathcal{D}(A), \tag{2.19}$$

for all $t \in \mathbb{R}^+$, and

$$AP(t)v = P(t)Av = \frac{d}{dt}P(t)v, \qquad \textit{for} \qquad v \in \mathcal{D}(A). \tag{2.20}$$

Proof. Suppose $v \in \mathcal{D}(A)$. Then for $t \geq 0$,

$$h^{-1}(P(h)P(t)v - P(t)v) = P(t)\frac{1}{h}(P(h)v - v), \tag{2.21}$$

which gives (2.19), as well as

$$AP(t)v = P(t)Av. \tag{2.22}$$

Furthermore, as $h \searrow 0$,

$$h^{-1}[P(t+h)v - P(t)v] = P(t)h^{-1}[P(h)v - v] \to P(t)Av. \tag{2.23}$$

For $h \nearrow 0$, observe that for $0 < h < t$,

$$h^{-1}[P(t)v - P(t-h)v] = P(t-h)h^{-1}(P(h)v - v) \to P(t)Av. \tag{2.24}$$

The last equality uses the fact that $w(h) \to w$ in V norm implies $P(t-h)w(h) \to P(t)w$.

To show that $\mathcal{D}(A)$ is dense in V, let $v \in V$ and let

$$v_\epsilon = \epsilon^{-1}\int_0^\epsilon P(t)vdt. \tag{2.25}$$

Then,

(2.26)

$$h^{-1}(P(h)v_\epsilon - v_\epsilon) = \epsilon^{-1}[h^{-1}\int_\epsilon^{\epsilon+h} P(t)vdt - h^{-1}\int_0^h P(t)vdt] \to \epsilon^{-1}(P(\epsilon)v - v), \qquad \text{as} \qquad h \to 0.$$

□

Now then, by the uniform boundedness principle,

$$\|P(t)\| \leq M, \qquad \text{for} \qquad |t| \leq 1. \tag{2.27}$$

Therefore, (2.27) and (2.2) imply that

$$\|P(t)\| \leq Me^{Kt}. \tag{2.28}$$

The infinitesimal generator determines the one–parameter semigroup uniquely, so we are justified in saying that A generates $P(t)$.

Proposition 5. *If $P(t)$ and $Q(t)$ are one–parameter semigroups with the same infinitesimal generator, then $P(t) = Q(t)$ for all $t \geq 0$.*

Proof. If (2.28) holds and $Re(\zeta) > K$, then ζ belongs to the resolvent set of A, and

$$(\zeta - A)^{-1} v = \int_0^\infty e^{-\zeta t} P(t) v dt. \tag{2.29}$$

Let R_ζ denote the right hand side of (2.29), which is clearly a bounded operator on V. First, show that $R_\zeta(\zeta - A)v = v$ for $v \in \mathcal{D}(A)$. Indeed,

$$\begin{aligned} R_\zeta(\zeta - A)v = \int_0^\infty e^{-\zeta t} P(t)(\zeta v - Av) dt = \int_0^\infty \zeta e^{-\zeta t} P(t) v dt - \int_0^\infty e^{-\zeta t} \frac{d}{dt} P(t) v dt \\ = - \int_0^\infty \frac{d}{dt} (e^{-\zeta t} P(t) v) dt = v. \end{aligned} \tag{2.30}$$

A similar argument shows that $(\zeta - A)R_\zeta$ is bounded on V and $(\zeta - A)R_\zeta v = v$ for $v \in \mathcal{D}(A)$. Since $(\zeta - A)R_\zeta$ is bounded on V and $\mathcal{D}(A)$ is dense in V,

$$(\zeta - A)R_\zeta v = R_\zeta(\zeta - A)v = v, \qquad \text{for all} \qquad v \in V. \tag{2.31}$$

Finally, since $(\zeta - A)^{-1}$ is continuous and everywhere defined, $(\zeta - A)^{-1}$ is closed. If an operator is closed and injective, then its inverse is closed, so in particular A is also closed.

Now let $v \in V$ and $w \in V'$. Then for $Re(\zeta)$ sufficiently large,

$$\int_0^\infty e^{-\zeta t} \langle P(t)v, w \rangle dt = \langle (\zeta - A)^{-1} v, w \rangle = \int_0^\infty e^{-\zeta t} \langle Q(t)v, w \rangle dt. \tag{2.32}$$

Uniqueness of the Laplace transform implies that $\langle P(t)v, w \rangle = \langle Q(t)v, w \rangle$ for any $v \in V$ and $w \in V'$. By the Hahn–Banach theorem, $P(t)v = Q(t)v$. □

Therefore, it makes sense to write

$$P(t) = e^{tA}. \tag{2.33}$$

Proposition 6. *Let A be the infinitesimal generator of a semigroup. If a function $u \in C([0,T), \mathcal{D}(A)) \cap C^1([0,T), V)$ satisfies*

$$\frac{du}{dt} = Au, \qquad u(0) = f, \tag{2.34}$$

then $u(t) = e^{tA} f$ for $t \in [0,T)$.

Proof. We have that $e^{(t-s)A} u(s)$ is differentiable in $s \in (0,t)$, and

$$\frac{\partial}{\partial s} e^{(t-s)A} u(s) = -e^{(t-s)A} Au(s) + e^{(t-s)A} Au(s) = 0. \tag{2.35}$$

Therefore, $e^{(t-s)A} u(s)$ has the same value at $s = t$ and $s = 0$, so $u(t) = e^{tA} f$. □

Given $g \in C([0,T),\mathcal{D}(A))$, $f \in \mathcal{D}(A)$, the equation

$$\frac{\partial u}{\partial t} = Au + g(t), \qquad u(0) = f, \tag{2.36}$$

has a unique solution $u \in C([0,T),\mathcal{D}(A)) \cap C^1([0,T),V)$, and it is given by

$$u(t) = e^{tA}f + \int_0^t e^{(t-s)A}g(s)ds. \tag{2.37}$$

Indeed,

$$\frac{\partial}{\partial s}e^{(t-s)A}u(s) = e^{(t-s)A}g(s), \qquad 0 \le s \le t. \tag{2.38}$$

Therefore,

$$u(t) - e^{tA}f = \int_0^t e^{(t-s)A}g(s)ds. \tag{2.39}$$

3. Semilinear parabolic equations

Consider semilinear equations of the form

$$\frac{\partial u}{\partial t} = Lu + F(t,x,u,\nabla u), \qquad u(0) = f, \tag{3.1}$$

where $u(t,x)$ is a function on $[0,T] \times \mathcal{M}$. For the moment, suppose that $\mathcal{M}$ has no boundary. Also suppose that $L = \nu\Delta$, for some $\nu > 0$.

When $F(t,x,u,\nabla u) = F(t,x)$, the solution to (3.1) is given by

$$u(t,x) = e^{tL}f + \int_0^t e^{(t-s)L}F(s,\cdot)ds. \tag{3.2}$$

Indeed, formally computing (3.2),

$$\frac{\partial u}{\partial t} = Lu + F(t,x). \tag{3.3}$$

It is possible to establish that (3.1) has a solution via the contraction mapping principle.

Proposition 7. *Suppose X and Y are Banach spaces for which*

$$e^{tL} : X \to X \qquad \text{is a strongly continuous semigroup, for} \qquad t \ge 0, \tag{3.4}$$

$$\Phi : X \to Y, \qquad \text{is Lipschitz, uniformly on bounded sets,} \tag{3.5}$$

$$e^{tL} : Y \to X, \qquad \text{for} \qquad t > 0, \tag{3.6}$$

and for some $\gamma < 1$,

$$\|e^{tL}\|_{\mathcal{L}(Y,X)} \le Ct^{-\gamma}, \qquad \text{for} \qquad t \in (0,1]. \tag{3.7}$$

Then the parabolic equation (3.1) with $f \in X$ has a unique solution $u \in C([0,T],X)$, where $T > 0$ is estimable from below in terms of $\|f\|_X$.

Definition 3. *A semigroup $P(t)$ is called strongly continuous if $t_j \to t$ implies $P(t_j)v \to P(t)v$ for each $v \in X$.*

Proof. Convert (3.1) to the integral equation,

$$u(t) = e^{tL} f + \int_0^t e^{(t-s)L} \Phi(u(s)) ds = \Psi u(t). \tag{3.8}$$

Fix $\alpha > 0$ and set

$$\mathcal{Z} = \{u \in C([0,T], X) : u(0) = f, \|u(t) - f\|_X \leq \alpha\}. \tag{3.9}$$

We want to choose T sufficiently small so that $\Psi : \mathcal{Z} \to \mathcal{Z}$ is a contraction. First, observe that by (3.4), for $T_1 > 0$ sufficiently small,

$$\|e^{tL} f - f\|_X \leq \frac{\alpha}{2}, \qquad \text{for} \qquad t \in [0, T_1]. \tag{3.10}$$

Next, by (3.5), for $u \in X$, then by (3.5), we have the estimate

$$\|\Phi(u(s))\|_Y \leq K_1, \qquad \text{for} \qquad s \in [0, T_1]. \tag{3.11}$$

Then by (3.7),

$$\| \int_0^t e^{(t-s)L} \Phi(u(s)) ds \|_X \leq C_\gamma t^{1-\gamma} K_1. \tag{3.12}$$

For $T_2 \leq T_1$ sufficiently small, (3.12) $\leq \frac{\alpha}{2}$ for $t \in [0, T_2]$. Therefore,

$$\Psi : \mathcal{Z} \to \mathcal{Z}, \qquad \text{provided} \qquad T \leq T_2. \tag{3.13}$$

To arrange that Ψ is a contraction, (3.5) implies that for $u, v \in \mathcal{Z}$, there exists $K < \infty$ such that

$$\|\Phi(u(s)) - \Phi(v(s))\|_Y \leq K \|u(s) - v(s)\|_X. \tag{3.14}$$

Therefore, for $t \in [0, T_2]$,

(3.15)

$$\|\Psi(u)(t) - \Psi(v)(t)\|_X = \| \int_0^t e^{(t-s)L} [\Phi(u(s)) - \Phi(v(s))] ds \|_X \leq C_\gamma t^{1-\gamma} K \sup_{s \in [0,t]} \|u(s) - v(s)\|_X.$$

Therefore, for $T \leq T_2$ sufficiently small, $C_\gamma T^{1-\gamma} K < 1$, which makes Ψ a contraction mapping on $\mathcal{Z}$. Therefore, Ψ has a unique fixed point. □

There are a number of function spaces X and Y which satisfy (3.4)–(3.7). For example, suppose $\mathcal{M}$ is a compact Riemannian manifold and let

$$X = C^1(\mathcal{M}), \qquad Y = C(\mathcal{M}). \tag{3.16}$$

By the maximum principle, (3.4) clearly holds. Since $\Phi(u) = F(u, \nabla u)$, (3.5) also holds. Finally, since

$$\|e^{t\Delta}\|_{\mathcal{L}(C, C^1)} \leq C t^{-1/2}, \qquad \text{for} \qquad t \in (0, 1], \tag{3.17}$$

so we have short–time solutions to (3.1) with $f \in C^1(\mathcal{M})$. In fact, we have

Proposition 8. *Given $f \in C^1(\mathcal{M})$, $L = \Delta$, the equation has, for some* (3.1)*, a unique solution*

$$u \in C([0,T], C^1(\mathcal{M})) \cap C^\infty((0,T] \times \mathcal{M}). \tag{3.18}$$

It is possible to weaken the hypothesis (3.4). Suppose X and Z are Banach spaces such that

$$C_0^\infty(\mathcal{M}) \subset X \subset Z \subset \mathcal{D}'(\mathcal{M}). \tag{3.19}$$

We say that $u(t)$ taking values in X, for $t \in I$, belongs to $C(I, X)$ provided $u(t)$ is locally bounded in X and $u \in C(I, Z)$. Then we say that e^{tL} is an almost continuous semigroup on X provided e^{tL} is uniformly bounded on X for $t \in [0, T]$, given $T < \infty$, $e^{(s+t)L}u = e^{sL}e^{tL}u$ for each $u \in X$, $s, t \in [0, \infty)$, and

$$u \in X, \qquad \text{implies} \qquad e^{tL}u \in C([0, \infty), X). \tag{3.20}$$

For example, if $\mathcal{M}$ is compact, we can take $X = L^\infty(\mathcal{M})$ and $Z = L^p(\mathcal{M})$ with $p < \infty$. We can also choose $e^{t\Delta}$ on $L^\infty(\mathcal{M})$ and on the Hölder spaces $C^r(\mathcal{M})$, $r \in \mathbb{R}^+ \setminus \mathbb{Z}^+$.

Proposition 9. *Let X and Y be Banach spaces for which* (3.5)–(3.7) *hold. In place of* (3.4)*, suppose that e^{tL} is an almost continuous semigroup on X. Also, augment* (3.5) *with the condition that $\Phi : C(I, X) \to C(I, Y)$. Then the initial value problem* (3.1)*, given $f \in X$, has a unique solution $u \in C([0, T], X)$, where $T > 0$ is estimable from below in terms of $\|f\|_X$.*

For example, consider $X = C^{r+1}(\mathcal{M})$ and $Y = C^r(\mathcal{M})$, $r \geq 0$. If r is not an integer, these are Hölder spaces. Then, for any $s > 0$,

$$\|e^{t\Delta}\|_{\mathcal{L}(C^r, C^{r+s})} \leq C_s t^{-s/2}, \qquad 0 < t \leq 1. \tag{3.21}$$

If $f \in C^{r+1}$, one has a solution $u \in C([0, T], C^{r+1})$, and for each $t > 0$, $u(t) \in C^{r+s}$ for every $s < 2$.

Proposition 10. *Given $f \in C^1(\mathcal{M})$, $L = \Delta$, the equation* (3.1) *has, for some $T > 0$, a unique solution*

$$u \in C([0, T], C^1(\mathcal{M})) \cap C^\infty((0, T] \times \mathcal{M}). \tag{3.22}$$

Using the estimates in (1.35)–(1.41), it is possible to take the sets Y and X and the bound on $\|e^{t\Delta}\|_{\mathcal{L}(Y,X)}$.

$$Y = L^q(\mathcal{M}), \qquad X = L^p(\mathcal{M}), \qquad \|e^{t\Delta}\|_{\mathcal{L}(Y,X)} \leq Ct^{-\frac{n}{2}(\frac{1}{q} - \frac{1}{p})}, \tag{3.23}$$

$$Y = H^{r,p}(\mathcal{M}), \qquad X = H^{s,p}(\mathcal{M}), \qquad \|e^{t\Delta}\|_{\mathcal{L}(Y,X)} \leq Ct^{-\frac{1}{2}(s-r)}, \tag{3.24}$$

and

$$Y = H^{r,q}(\mathcal{M}), \qquad X = H^{s,p}(\mathcal{M}), \qquad \|e^{t\Delta}\|_{\mathcal{L}(Y,X)} \leq Ct^{-\frac{n}{2}(\frac{1}{q} - \frac{1}{p}) - \frac{1}{2}(s-r)}. \tag{3.25}$$

Take the case $F(u, \nabla u) = \sum_j \partial_j F_j(u)$ with $L = \nu\Delta$,

$$\frac{\partial u}{\partial t} = \nu\Delta u + \sum_j \partial_j F(u), \qquad u(0) = f. \tag{3.26}$$

For example, take $\mathcal{M} = \mathbb{T}^n$. Now then, suppose

$$|F_j(u)| \leq C\langle u\rangle^p, \qquad |\nabla F_j(u)| \leq C\langle u\rangle^{p-1}. \tag{3.27}$$

Proposition 11. *Under the hypotheses in* (3.27)*, if $f \in L^q(\mathcal{M})$, the partial differential equation* (3.26) *has a unique solution $u \in C([0, T], L^q(\mathcal{M}))$, provided*

$$q \geq p, \qquad \text{and} \qquad q > n(p-1). \tag{3.28}$$

Furthermore, $u \in C^\infty((0, T] \times \mathcal{M})$.

Proof. Take the Banach spaces

$$X = L^q(\mathcal{M}), \qquad H^{-1,\frac{q}{p}}(\mathcal{M}). \tag{3.29}$$

We need $q \geq p$, so that $\frac{q}{p} \geq 1$, and $F_j : L^q \to L^{q/p}$ is locally Lipschitz. Indeed,

$$F_j(u) - F_j(v) = G_j(u,v)(u-v), \qquad G_j(u,v) = \int_0^1 F_j'(su + (1-s)v) ds. \tag{3.30}$$

By the generalized Hölder inequality,

$$\|F_j(u) - F_j(v)\|_{L^{q/p}} \leq \|G_j\|_{L^{q/(p-1)}} \|u - v\|_{L^q}, \tag{3.31}$$

so we have (3.5). Next,

$$\|e^{t\Delta}\|_{\mathcal{L}(H^{-1,q/p}, L^q)} \leq C t^{-\frac{n}{2}(\frac{p}{q} - \frac{1}{p}) - \frac{1}{2}}. \tag{3.32}$$

Therefore, we have (3.7) when $\frac{n(p-1)}{q} < 1$.

It suffices to establish smoothness. First, replacing L^q by L^{q_1} in (3.2), for any $t \in (0, T]$, $u(t) \in L^{q_1}$ for all $q_1 < \frac{q}{p - q/n}$. Since $p - \frac{q}{n} < 1$, this means that q_1 exceeds q by a factor > 1. Iterating, $u(t) \in L^{q_j}$, where q_j exceeds q_{j-1} by a factor > 1. When $q_j > np$, the next iteration gives $u(t) \in C^r(\mathcal{M})$.

Now consider the spaces

$$X = C^r(\mathcal{M}), \qquad Y = H^{r-1-\epsilon,q}(\mathcal{M}), \tag{3.33}$$

for some $\epsilon > 0$ very small and q very large. Then, $u \mapsto F_j(u)$ is locally Lipschitz from $C^r(\mathcal{M})$ to $C^r(\mathcal{M})$, hence to $H^{r-\epsilon,p}(\mathcal{M})$. Then by (3.25), for any $t > 0$, $u(t) \in C^{r_1}(\mathcal{M})$, $r_1 - r > 0$, which is estimable from below. Making a finite number of iterations, $u \in C^1(\mathcal{M})$, and then by Proposition 8, the proof is complete. □

We can establish a global existence theorem for solutions to (3.26).

Proposition 12. *Suppose F_j satisfy (3.27) with $p = 1$. Then given $f \in L^2(\mathcal{M})$, the equation (3.26) has a unique solution*

$$u \in C([0,\infty), L^2(\mathcal{M})) \cap C^\infty((0,\infty) \times \mathcal{M}), \tag{3.34}$$

provided when u takes values in $\mathbb{R}^K$, $F_j(u) = (F_j^1(u), ..., F_j^K(u))$, that

$$\frac{\partial F_j^k}{\partial u_i} = \frac{\partial F_j^i}{\partial u_k}, \qquad 1 \leq i, k \leq K. \tag{3.35}$$

Proof. When $p = 1$, we can take $q = 2$ and $n(p-1)/q < 1$, $q > n(p-1)$. Therefore, a local solution exists,

$$u \in C([0,T], L^2) \cap C^\infty((0,T) \times \mathcal{M}). \tag{3.36}$$

To get global existence, it suffices to bound $\|u(t)\|_{L^2}$. Indeed,

$$\frac{d}{dt}\|u(t)\|_{L^2}^2 = 2(u(t), \sum_j \partial_j F_j(u(t))) - 2\nu \|\nabla u(t)\|_{L^2}^2 \leq 2(u(t), \sum_j \partial_j F_j(u)). \tag{3.37}$$

By (3.35), there exist smooth G_j such that $F_j^k = \frac{\partial G_j}{\partial u_k}$. Therefore, the right hand side of (3.37) is given by

$$-2 \sum \int \partial_j G_j(u) dx = 0. \tag{3.38}$$

□

For a scalar equation, it is possible to eliminate the restriction $p = 1$ for bounded initial data.

Proposition 13. *If* (3.26) *is scalar and* $f \in L^\infty(\mathcal{M})$*, then there is a unique solution*

$$u \in L^\infty([0,\infty) \times \mathcal{M}) \cap C^\infty((0,\infty) \times \mathcal{M}), \tag{3.39}$$

such that, as $t \searrow 0$*,* $u(t) \to f$ *in* $L^p(\mathcal{M})$ *for all* $p < \infty$*.*

Proof. Suppose $\|f\|_{L^\infty} \le M$, and alter $F_j(u)$ on $|u| \ge M + \frac{1}{2}$ so that $\tilde{F}_j(u)$ is constant on $u \le -M-1$ and on $u \ge M+1$. Then by Proposition 12, this modified PDE has a global solution. This u solves

$$\frac{\partial u}{\partial t} = \nu \Delta u + \sum_j a_j(t,x) \partial_j u, \qquad a_j(t,x) = \tilde{F}_j'(u(t,x)). \tag{3.40}$$

Furthermore, the maximum principle for linear parabolic equations holds, so $\|u(t)\|_{L^\infty} \le M$ for all t, so u solves the original PDE. □

Now suppose that $x \in \bar{\mathcal{M}}$, a compact region with a boundary, and that F is smooth in its arguments. Specifically, take the Dirichlet problem

$$u = 0 \qquad \text{on} \qquad \mathbb{R}^+ \times \partial\mathcal{M}, \tag{3.41}$$

and suppose

$$\frac{\partial u}{\partial t} = \Delta u + F(t,x,u,\nabla u), \qquad u(0) = f. \tag{3.42}$$

Since Propositions 7 and 9 were phrased on a very general level, a number of short–time existence results follow simply by verifying that (3.4)–(3.7) hold for appropriate Banach spaces X and Y on $\bar{\mathcal{M}}$. For example, suppose $X = C_b^1(\overline{\mathcal{M}})$ and $Y = C(\bar{\mathcal{M}})$, where for $j \ge 0$,

$$C_b^j(\bar{\mathcal{M}}) = \{f \in C^j(\bar{\mathcal{M}}) : f = 0 \qquad \text{on} \qquad \partial\mathcal{M}\}. \tag{3.43}$$

We have the following estimate

$$\|e^{t\Delta} f\|_{C^1(\bar{\Omega})} \le C t^{-1/2} \|f\|_{L^\infty(\Omega)}, \qquad 0 < t \le 1, \tag{3.44}$$

as well as the proposition.

Proposition 14. *If* $\bar{\Omega}$ *is a compact Riemannian manifold with boundary, on which the Dirichlet condition is placed, then* $e^{t\Delta}$ *defines a strongly continuous semigroup on the Banach space*

$$C_b^1(\bar{\Omega}) = \{f \in C^1(\bar{\Omega}) : f|_{\partial\Omega} = 0\}. \tag{3.45}$$

Therefore, we have the following.

Proposition 15. *If* $f \in C_b^1(\bar{\mathcal{M}})$*, then* (3.41)–(3.42) *has a unique solution*

$$u \in C([0,T), C^1(\bar{\mathcal{M}})), \tag{3.46}$$

for some $T > 0$*, estimable from below in terms of* $\|f\|_{C^1}$*.*

Now suppose that F is independent of ∇u, that is,

$$\frac{\partial u}{\partial t} = \Delta u + F(t,x,u), \qquad u(0) = f, \tag{3.47}$$

so we can take $X = C_b(\bar{\mathcal{M}})$, $Y = C(\bar{\mathcal{M}})$, and by the above arguments obtain

Proposition 16. *If $f \in C_b(\bar{\mathcal{M}})$, then* (3.47), (3.41) *has a unique solution*

$$u \in C([0,T), C(\bar{\mathcal{M}})), \tag{3.48}$$

for some $T > 0$, estimable from below in terms of $\|f\|_{L^\infty}$.

We can obtain further regularity results on solutions to (3.42) and (3.47) with boundary condition (3.41) by making use of the regularity results for

$$\frac{\partial u}{\partial t} = \Delta u + g(t,x), \qquad u(t,x) = 0, \qquad \text{for} \qquad x \in \partial \mathcal{M}. \tag{3.49}$$

For any $k \in \mathbb{Z}^+$ define the set

$$\mathcal{H}^k(I \times \mathcal{M}) = \{u \in L^2(I \times \mathcal{M}) : \partial_t^j u \in L^2(I, H^{2k-2j}(\mathcal{M})), \qquad 0 \leq j \leq k\}. \tag{3.50}$$

Then if (3.49) holds on $I \times \mathcal{M}$ with $I = [0, T_0]$, then

$$g \in \mathcal{H}^k(I \times \mathcal{M}) \Rightarrow u \in \mathcal{H}^{k+1}(I' \times \mathcal{M}), \tag{3.51}$$

for $I' = [\epsilon, T_0]$, $\epsilon > 0$. Therefore, if $g = F(t,x,u,\nabla u)$ for Proposition 15 and $g = F(t,x,u)$ for Proposition 18, $g \in \mathcal{H}^0(I \times \mathcal{M})$ whenever $T_0 < T$. Therefore,

$$u \in \mathcal{H}^1(I' \times \mathcal{M}). \tag{3.52}$$

Therefore, one also has higher order regularity. Therefore, we have proved

Proposition 17. *Assume F is smooth in its arguments. The solution* (3.46) *of* (3.49), (3.41) *has the property*

$$u \in C^\infty((0,T) \times \bar{\mathcal{M}}). \tag{3.53}$$

Proof. We begin with the implication

$$u \in C(I \times \bar{\mathcal{M}}) \cap \mathcal{H}^1(I' \times \mathcal{M}) \Rightarrow F(t,x,u) \in \mathcal{H}^1(I' \times \mathcal{M}). \tag{3.54}$$

Then by (3.51), $u \in \mathcal{H}^2(I' \times \mathcal{M})$. More generally,

$$u \in C(I \times \bar{\mathcal{M}}) \cap \mathcal{H}^k(I' \times \mathcal{M}) \Rightarrow F(t,x,u) \in \mathcal{H}^k(I' \times \mathcal{M}). \tag{3.55}$$

Arguing by induction proves the Proposition. □

The estimates in (3.54) and (3.55) utilize the Moser estimate.

Proposition 18. *Let F be smooth and suppose $F(0) = 0$. Then, for $u \in H^k \cap L^\infty$,*

$$\|F(u)\|_{H^k} \leq C_k(\|u\|_{L^\infty})(1 + \|u\|_{H^k}). \tag{3.56}$$

Proof. By the chain rule,

$$D^\alpha F(u) = \sum_{\beta_1 + \ldots + \beta_\mu = \alpha} C_\beta u^{(\beta_1)} \cdots u^{(\beta_\mu)} F^{(\mu)}(u). \tag{3.57}$$

Therefore,

$$\|D^\alpha F(u)\|_{L^2} \leq C_k(\|u\|_{L^\infty}) \sum_{\beta_1 + \ldots + \beta_\mu = \alpha} \|u^{(\beta_1)} \cdots u^{(\beta_\mu)}\|_{L^2}. \tag{3.58}$$

Then by $u \in L^\infty \cap H^k$ and interpolation, the proof is complete. □

4. The L^p spectral theory of the Laplace operator

Suppose Δ is the Laplace operator on the manifold $\mathcal{M}$, where $\mathcal{M}$ is a compact Riemannian manifold without boundary. For any $\lambda > 0$, $(\lambda - \Delta)^{-1}$ is a bijective operator between $L^p(\mathcal{M})$ and $H^{2,p}(\mathcal{M})$ for $1 < p < \infty$.

Proof of claim. To see this in the case when $p = 2$, observe that (1.17) implies that

$$\|e^{t\Delta} f\|_{L^2(\mathcal{M})} \leq \|f\|_{L^2(\mathcal{M})}, \tag{4.1}$$

and therefore by (2.29), $(\lambda - \Delta)^{-1}$ is bijective from $L^2(\mathcal{M})$ to $H^2(\mathcal{M})$. Meanwhile, by the maximum principle,

$$\|e^{t\Delta} f\|_{L^\infty(\mathcal{M})} \leq \|f\|_{L^\infty(\mathcal{M})}. \tag{4.2}$$

By interpolation, for any $2 \leq p \leq \infty$,

$$\|e^{t\Delta} f\|_{L^p(\mathcal{M})} \leq \|f\|_{L^p(\mathcal{M})}. \tag{4.3}$$

Now then, by duality, (4.3) implies that for $1 < p < 2$,

$$\|e^{t\Delta} f\|_{L^p(\mathcal{M})} \leq \|f\|_{L^p(\mathcal{M})}. \tag{4.4}$$

Taking the adjoint of the action of $e^{t\Delta}$ on $C(\mathcal{M})$ implies that $e^{t\Delta}$ acts on finite Borel measures on $\mathcal{M}$, so $e^{t\Delta}$ preserves $L^1(\mathcal{M})$. Since $C_0^\infty(\mathcal{M})$ is dense in $L^p(\mathcal{M})$ for $1 \leq p < \infty$, $e^{t\Delta}$ defines a strongly continuous semigroup on $L^p(\mathcal{M})$. Thus, define Δ_p on $L^p(\mathcal{M})$ to be the operator Δ acting on $H^{2,p}(\mathcal{M})$. Therefore, Δ_p is a closed operator with finite–dimensional eigenspaces consisting of functions in $C^\infty(\mathcal{M})$. Each of these functions are actual eigenfunctions, so the L^p spectrum of Δ coincides with its L^2 spectrum. □

Now define a holomorphic semigroup. Let $\mathcal{K}$ be a closed cone in the right hand plane of $\mathbb{C}$ with vertex at 0. If $P(z) : X \to X$ is a family of bounded operators on a Banach space X, we say that it is a holomorphic semigroup if it satisfies $P(z_1)P(z_2) = P(z_1 + z_2)$ for $z_j \in \mathcal{K}$, is strongly continuous in $z \in \mathcal{K}$, and is holomorphic in the interior of $\mathcal{K}$.

Remark 2. *Strong continuity implies that $\|e^{z\Delta}\|$ is locally uniformly bounded on $\mathcal{K}$.*

The operator $e^{z\Delta} f$ defines a holomorphic semigroup on $L^2(\mathcal{M})$. Indeed, by the spectral decomposition of $L^p(\mathcal{M})$,

$$\|e^{z\Delta} f\|_{L^2(\mathcal{M})} \leq \|f\|_{L^2(\mathcal{M})}. \tag{4.5}$$

Also, $e^{z\Delta}$ is holomorphic in $L^2(\mathcal{M})$ since $\frac{d}{dz} e^{z\Delta} = \Delta e^{z\Delta}$. In fact, we can prove

Proposition 19. *$e^{z\Delta}$ defines a holomorphic semigroup $H_p(z)$ on $L^p(\mathcal{M})$, for each $p \in [1, \infty)$.*

Proof. This follows from the parametrix construction. We do not do this in the general case here, but rather refer the interested reader to Chapter 7, section 13 of [Taya]. However, observe that in the computations in (1.26)–(1.32),

$$e^{at\Delta} f = \frac{1}{(4\pi at)^{n/2}} \int e^{-\frac{|x-y|^2}{4at}} f(y) dy. \tag{4.6}$$

Therefore, when $Re(a) > 0$, the operator $e^{at\Delta}$ retains the properties in (1.34)–(1.41). □

Here is a useful property of semigroups.

Proposition 20. *Let $P(z)$ be a holomorphic semigroup on a Banach space X with generator A. Then,*

$$t > 0, f \in X \to P(t)f \in \mathcal{D}(A), \tag{4.7}$$

and

$$\|AP(t)f\|_X \leq \frac{C}{t}\|f\|_X, \qquad \textit{for} \qquad 0 < t < 1. \tag{4.8}$$

Proof. Using the holomorphicity of $P(z)$ and the structure of $\mathcal{K}$, there exists $a > 0$ such that there exists a circle $\gamma(t)$ of radius $a|t|$ such that $\gamma(t) \in \mathcal{K}$, for all $t \in (0, \infty)$. Thus,

$$AP(t)f = P'(t)f = -\frac{1}{2\pi i}\int_{\gamma(t)} (t - \zeta)^{-2} P(\zeta) f d\zeta. \tag{4.9}$$

Since $\|P(\zeta)f\| \leq C_2\|f\|$ for $\zeta \in \mathcal{K}$, $|\zeta| \leq 1 + a$, we have (4.8). □

In particular, for $1 < p < \infty$, $0 < t \leq 1$,

$$f \in L^p(\mathcal{M}) \Rightarrow \|e^{t\Delta} f\|_{H^{2,p}(\mathcal{M})} \leq \frac{C}{t}\|f\|_{L^p(\mathcal{M})}. \tag{4.10}$$

Then by interpolation,

$$\|e^{t\Delta} f\|_{H^{s,p}(\mathcal{M})} \leq C t^{-s/2}\|f\|_{L^p(\mathcal{M})}, \qquad \text{for} \qquad 0 \leq s \leq 2, \qquad 0 < t \leq 1. \tag{4.11}$$

Now let $\bar{\Omega}$ be a compact Riemannian manifold with smooth boundary and let Δ be the Laplacian on $\bar{\Omega}$ with Dirichlet boundary condition. Assume that $\bar{\Omega}$ is connected and $\partial\Omega \neq \emptyset$. For $\lambda \geq 0$,

$$R_\lambda = (\lambda - \Delta)^{-1} : L^2(\Omega) \to L^2(\Omega), \tag{4.12}$$

with range $H^2(\Omega) \cap H_0^1(\Omega)$. For $f \in L^\infty(\Omega)$, we can analyze $R_\lambda f$ by noting that R_λ is positivity preserving,

$$\lambda \geq 0, \qquad g \geq 0, \qquad \text{on} \qquad \Omega \qquad \Rightarrow \qquad R_\lambda g \geq 0, \qquad \text{on} \qquad \Omega. \tag{4.13}$$

This follows from the maximum principle. We can also prove this using the the positivity principle of $e^{t\Delta}$ combined with the resolvent formula. Combining positivity preserving with regularity estimates and estimates on $R_\lambda 1$, if $0 \leq f \leq 1$, $R_\lambda(1 - f) \geq 0$ and $R_\lambda f \geq 0$, so $0 \leq R_\lambda f \leq R_\lambda 1$, so

$$R_\lambda : C(\bar{\Omega}) \to C(\bar{\Omega}), \qquad R_\lambda : L^\infty(\Omega) \to L^\infty(\Omega). \tag{4.14}$$

Taking the adjoint of R_λ acting on $C(\bar{\Omega})$, we have R_λ acting on the finite Borel measures on $\bar{\Omega}$. Since the closure of $L^2(\Omega)$ in the set of finite Borel measures is $L^1(\Omega)$,

$$R_\lambda : L^1(\Omega) \to L^1(\Omega). \tag{4.15}$$

Then by interpolation,

$$R_\lambda : L^p(\Omega) \to L^p(\Omega), \qquad 1 \leq p \leq \infty. \tag{4.16}$$

By a similar argument, we can show that

$$e^{t\Delta} : L^2(\mathcal{M}) \to L^2(\mathcal{M}), \tag{4.17}$$

and by the maximum principle,

$$e^{t\Delta} : L^\infty(\mathcal{M}) \to L^\infty(\mathcal{M}). \tag{4.18}$$

Then by interpolation and duality,

$$e^{t\Delta} : L^p(\Omega) \to L^p(\Omega), \qquad 1 \leq p \leq \infty. \tag{4.19}$$

Proposition 21. *For* $1 < p < \infty$, $e^{z\Delta}$ *defines a holomorphic semigroup on* $L^p(\Omega)$, *on any symmetric cone* $\mathcal{K}$ *about* $\mathbb{R}^+$ *of angle* $< \pi$.

Proof. The remaining parts of the proof are in [Tayb]. □

Making use of Proposition 20, which we know applies to $e^{t\Delta}$ on $L^p(\mathcal{M})$ gives the bound

$$\|v(t)\|_{H^{1,p}(\mathcal{M})} \leq C|t|^{-1/2}\|f\|_{L^p(\Omega)}. \tag{4.20}$$

5. Galerkin's method

Returning to the parabolic PDE

$$\frac{\partial u}{\partial t} = \nu\Delta u + \sum_j \partial_j F_j(u), \qquad u(0) = f, \tag{5.1}$$

suppose that

$$|F_j(u)| \leq C\langle u\rangle^p, \qquad |\nabla F_j(u)| \leq C\langle u\rangle^{p-1}, \tag{5.2}$$

holds with $p = 2$ and that

$$\frac{\partial F_j^k}{\partial u_i} = \frac{\partial F_j^i}{\partial u_k}. \tag{5.3}$$

Take $\mathcal{M} = \mathbb{T}^n$, and we can use the Galerkin method to produce a sequence of approximations, converging to a solution to (5.1).

Now then, for any $\epsilon > 0$, define the projection P_ϵ on $L^2(\mathcal{M})$ by

$$P_\epsilon f(x) = \sum_{|k|\leq \frac{1}{\epsilon}} \hat{f}(k)e^{ik\cdot x}. \tag{5.4}$$

Consider the initial value problem

$$\frac{\partial u_\epsilon}{\partial t} = \nu P_\epsilon \Delta P_\epsilon u_\epsilon + P_\epsilon \sum \partial_j F_j(P_\epsilon u_\epsilon), \qquad u_\epsilon(0) = P_\epsilon f. \tag{5.5}$$

Now take $f \in L^2(\mathcal{M})$. For each $0 < \epsilon \leq 1$, ODE theory gives a unique, short–time solution to (5.5), satisfying $u_\epsilon(t) = P_\epsilon u_\epsilon(t)$. Furthermore,

$$\frac{d}{dt}\|u_\epsilon(t)\|^2_{L^2} = 2\nu(P_\epsilon\Delta P_\epsilon u_\epsilon, u_\epsilon) + 2\sum(P_\epsilon\partial_j F_j(P_\epsilon u_\epsilon), u_\epsilon). \tag{5.6}$$

Integrating by parts, the first term on the right hand side is equal to

$$-2\nu\|\nabla P_\epsilon u_\epsilon(t)\|^2_{L^2} \leq 0. \tag{5.7}$$

The second term is equal to

$$2\sum(\partial_j F_j(P_\epsilon u_\epsilon), P_\epsilon u_\epsilon) = -2\sum(F_j(P_\epsilon u_\epsilon), \partial_j P_\epsilon u_\epsilon) = -2\sum\int \partial_j[G_j(P_\epsilon u_\epsilon)]dx = 0. \tag{5.8}$$

Therefore,

$$\|u_\epsilon(t)\|_{L^2} \leq \|f\|_{L^2}. \tag{5.9}$$

Hence, for each $\epsilon > 0$, (5.5) is solvable for all $t > 0$ and

$$\{u_\epsilon : 0 < \epsilon \leq 1,\} \tag{5.10}$$

is bounded in $L^{\infty}(\mathbb{R}^{+}, L^{2}(\mathcal{M}))$. Furthermore, by (5.6)–(5.9), for any $0 < T < \infty$,

$$2\nu \int_{0}^{T} \|\nabla P_{\epsilon} u_{\epsilon}(t)\|_{L^2}^{2} dt = \|P_{\epsilon} f\|_{L^2}^{2} - \|u_{\epsilon}(T)\|_{L^2}^{2}. \tag{5.11}$$

Therefore, for each bounded interval $I = [0, T]$, since $P_{\epsilon} u_{\epsilon} = u_{\epsilon}$,

$$\{u_{\epsilon}\} \text{ is bounded in } L^{2}(I, H^{1}(\mathcal{M})). \tag{5.12}$$

Given that $|F_j(u)| \leq C\langle u\rangle^{2}$, since u_{ϵ} is bounded in $L^{\infty}(\mathbb{R}^{+}, L^{2}(\mathcal{M}))$, $\{F_j(P_{\epsilon} u_{\epsilon})\}$ is bounded in $L^{\infty}(\mathbb{R}^{+}, L^{1}(\mathcal{M})) \subset L^{\infty}(\mathbb{R}^{+}, H^{-n/2-\delta}(\mathcal{M}))$, for each $\delta > 0$.

Using the evolution equation (5.5),

$$\{\frac{\partial u_{\epsilon}}{\partial t}\} \text{ is bounded in } L^{2}(I, H^{-n/2-1-\delta}(\mathcal{M})). \tag{5.13}$$

Therefore,

$$\{u_{\epsilon}\} \text{ is bounded in } H^{1}(I, H^{-n/2-1-\delta}(\mathcal{M})). \tag{5.14}$$

Interpolating (5.12) and (5.14),

$$\{u_{\epsilon}\} \text{ is bounded in } H^{s}(I, H^{1-s(n/2+1+\delta)}), \tag{5.15}$$

for each $0 \leq s \leq 1$. Choosing $s > 0$ sufficiently small, Rellich's theorem implies

$$\{u_{\epsilon} : 0 < \epsilon \leq 1\} \text{ is compact in } L^{2}(I, H^{1-\gamma}(\mathcal{M})), \tag{5.16}$$

for any $\gamma > 0$.

For any $T < \infty$, we can choose a sequence $u_k = u_{\epsilon_k}$, $\epsilon_k \searrow 0$, such that

$$u_k \to u \text{ in } L^{2}([0, T], H^{1-\gamma}), \text{ in norm.} \tag{5.17}$$

Making a diagonal argument, it is possible to arrange that (5.17) holds for all $T < \infty$. We can also assume that u_k is weakly convergent in each space specified by (5.10), (5.12), and that $\frac{\partial u_k}{\partial t}$ is weakly convergent in the space (5.13). Furthermore, from (5.17),

$$F_j(P_{\epsilon_k} u_{\epsilon_k}) \to F_j(u), \text{ in } L^{1}([0, T], L^{1}(\mathcal{M})), \text{ in norm,} \tag{5.18}$$

as $k \to \infty$. Therefore,

$$\partial_j F_j(P_{\epsilon} u_{\epsilon}) \to \partial_j F_j(u) \text{ in } L^{1}([0, T], H^{-1,1}(\mathcal{M})). \tag{5.19}$$

Since $H^{-1,1}(\mathcal{M}) \subset H^{-n/2-1-\delta}(\mathcal{M})$, each term in (5.5) converges as $\epsilon_k \searrow 0$. Therefore, we have proved

Proposition 22. *If $|F_j(u)| \leq C\langle u\rangle^{2}$ and $|\nabla F_j(u)| \leq C\langle u\rangle$, then for each $f \in L^{2}(\mathcal{M})$, a $K \times K$ system of the form* (5.1) *satisfying the symmetry hypothesis* (5.3) *possesses a global weak solution,*

$$u \in L^{\infty}(\mathbb{R}^{+}, L^{2}(\mathcal{M})) \cap L^{2}_{loc}(\mathbb{R}^{+}, H^{1}(\mathcal{M})) \cap Lip_{loc}(\mathbb{R}^{+}, H^{-2}(\mathcal{M}) + H^{-n/2-1-\delta}(\mathcal{M})). \tag{5.20}$$

This argument can be generalized to the case when U is a bounded domain. In this case, we need smooth functions $w_k(x)$,

$$\{w_k\}_{k=1}^{\infty} \text{ is an orthonormal basis of } H_0^{1}(U), \tag{5.21}$$

and

$$\{w_k\}_{k=1}^{\infty} \text{ is an orthonormal basis of } L^{2}(U). \tag{5.22}$$

For example, we can take $\{w_k\}_{k=1}^{\infty}$ to be the complete set of appropriately normalized eigenfunctions for $L = -\Delta$ in $H_0^{1}(U)$.

Now we prove an important L^1–contractive property for a scalar equation.

Proposition 23. *Let u_j be solutions to the equation* (5.1) *with initial data $u_j(0) = f_j \in L^\infty(\mathcal{M})$. Then for each $t > 0$,*

$$\|u_1(t) - u_2(t)\|_{L^1(\mathcal{M})} \leq \|f_1 - f_2\|_{L^1(\mathcal{M})}. \tag{5.23}$$

Proof. Set $v = u_1 - u_2$. Then v solves

$$\frac{\partial v}{\partial t} = \nu \Delta v + \sum \partial_j [\Phi_j(u_1, u_2) v], \tag{5.24}$$

where

$$\Phi_j(u_1, u_2) = \int_0^1 F_j'(s u_1 + (1-s) u_2) ds. \tag{5.25}$$

Now set $G_j(t,x) = \Phi_j(u_1, u_2)$. Given $T > 0$, let w solve the backward heat equation

$$\frac{\partial w}{\partial t} = -\nu \Delta w + \sum G_j(t,x) \partial_j w, \qquad w(T) = w_0 \in C^\infty(\mathcal{M}). \tag{5.26}$$

Now then, $w(t)$ is well–defined for $t \leq T$, and the maximum principle implies

$$\|w(t)\|_{L^\infty} \leq \|w_0\|_{L^\infty}, \qquad \text{for} \qquad t \leq T. \tag{5.27}$$

Now then, for $0 < t < T$,

$$\frac{d}{dt}(v, w) = (\nu \Delta v, w) + \sum (\partial_j (G_j v), w) - (v, \nu \Delta w) + \sum (v, G_j \partial_j w) = 0. \tag{5.28}$$

Since $(v(0), w(0)) \leq \|v(0)\|_{L^1} \|w(0)\|_{L^\infty}$, the proof of (5.23) is complete. □

6. Navier–Stokes equation

Consider the Navier–Stokes equation for the viscous incompressible flow of a fluid. Now the Euler equation has the form

$$\frac{\partial u}{\partial t} + P \nabla_u u = 0, \qquad u(0) = u_0, \tag{6.1}$$

where P is the orthogonal projection of $L^2(\mathcal{M}, T\mathcal{M})$ onto the space of divergence–free vector fields, and the divergence of u_0 is equal to zero. On $\mathbb{R}^n$, the Leray projection P is defined by

$$P(u) = u - \nabla \Delta^{-1} (\nabla \cdot u). \tag{6.2}$$

Then the Navier–Stokes equation has the form

$$\frac{\partial u}{\partial t} + P \nabla_u u = \nu \Delta u, \qquad u(0) = u_0. \tag{6.3}$$

Define the Friedrichs mollifier,

$$j_\epsilon(x) = \epsilon^{-n} j(\epsilon^{-1} x), \qquad \int j(x) dx = 1, \qquad j \in \mathcal{S}(\mathbb{R}^n), \tag{6.4}$$

and let

$$J_\epsilon u(x) = j_\epsilon * u(x). \tag{6.5}$$

Now define the approximating equation

$$\frac{\partial u_\epsilon}{\partial t} + P J_\epsilon \nabla_{u_\epsilon} J_\epsilon u_\epsilon = \nu J_\epsilon \Delta J_\epsilon u_\epsilon, \qquad u_\epsilon(0) = u_0. \tag{6.6}$$

Then by direct computation,

$$\frac{d}{dt}\|u_\epsilon\|_{L^2}^2 = -2\nu\|\nabla J_\epsilon u_\epsilon\|_{L^2}^2, \tag{6.7}$$

and therefore,

$$\|u_\epsilon(t)\|_{L^2} \leq \|u_0\|_{L^2}. \tag{6.8}$$

Therefore, (6.6) is solvable for all $t \in \mathbb{R}$ whenever $\nu \geq 0$ and $\epsilon > 0$.

Now let $\mathcal{M} = \mathbb{T}^n$ and compute

$$\frac{d}{dt}\|u_\epsilon(t)\|_{H^k}^2 \leq C\|u_\epsilon(t)\|_{C^1}\|u_\epsilon(t)\|_{H^k}^2 - 4\nu\|\nabla J_\epsilon u_\epsilon\|_{L^2}^2. \tag{6.9}$$

Observe that the constant C in (6.9) is independent of $\nu \geq 0$. The estimate (6.9) is sufficient to establish a local existence theorem for a limit point of u_ϵ as $\epsilon \searrow 0$, which we denote u_ν.

Theorem 1. *Given $u_0 \in H^k(\mathcal{M})$, $k > \frac{n}{2}+1$, with $div(u_0) = 0$, there is a solution u_ν on an interval $I = [0, A)$ to* (6.3) *satisfying*

$$u_\nu \in L^\infty(I, H^k(\mathcal{M})) \cap Lip(I, H^{k-2}(\mathcal{M})). \tag{6.10}$$

The interval I and the estimate of u_ν in $L^\infty(I, H^k(\mathcal{M}))$ can be taken independent of $\nu \geq 0$.

We can establish the uniqueness and treat the stability and rate of convergence of u_ϵ to $u = u_\nu$ as before. For $\epsilon \in [0,1]$, compare a solution $u = u_\nu$ to a solution $u_{\nu\epsilon} = w$ to

$$\frac{\partial w}{\partial t} + PJ_\epsilon \nabla_w J_\epsilon w = \nu J_\epsilon \Delta J_\epsilon w, \qquad w(0) = w_0. \tag{6.11}$$

Setting $v = u_\nu - u_{\nu\epsilon}$, we have an estimate.

Proposition 24. *Given $k > \frac{n}{2}+1$, solutions to* (6.3) *satisfying* (6.10) *are unique. They are limits of solutions $u_{\nu\epsilon}$ to* (6.3)*, and for $t \in I$,*

$$\frac{d}{dt}\|v\|_{L^2}^2 = -2\nu\|\nabla v\|_{L^2}^2 + K_1(t)\|I - J_\epsilon\|_{\mathcal{L}(H^{k-1}, L^2)}. \tag{6.12}$$

Next, we can deduce

$$\frac{d}{dt}\|D^\alpha J_\epsilon u_\nu(t)\|_{L^2}^2 = -2(D^\alpha J_\epsilon L(u_\nu, D)u_\nu, D^\alpha J_\epsilon u_\nu) - 2\nu\|\nabla J_\epsilon u_\nu(t)\|_{L^2}^2. \tag{6.13}$$

Therefore,

$$\frac{d}{dt}\|u_\nu(t)\|_{H^k}^2 \leq C\|u_\nu(t)\|_{C^1}\|u_\nu(t)\|_{H^k}^2. \tag{6.14}$$

Thus, u_ν is continuous in t with values in $H^k(\mathcal{M})$ at $t = 0$. At other points $t \in I$, u_ν is right continuous. u_ν is not left continuous, since the evolution equation is not well–posed backward in time.

Now we prove a local well–posedness result for (6.3).

Proposition 25. *If $div(u_0) = 0$ and $u_0 \in L^p(\mathcal{M})$, with $p > n = dim(\mathcal{M})$, and if $\nu > 0$, then* (6.3) *has a unique short–time solution on an interval $I = [0, T]$,*

$$u = u_\nu = C(I, L^p(\mathcal{M})) \cap C^\infty((0, T) \times \mathcal{M}). \tag{6.15}$$

Proof. It is useful to rewrite ((6.3) as

$$\frac{\partial u}{\partial t} + Pdiv(u \otimes u) = \nu \Delta u, \qquad u(0) = u_0. \tag{6.16}$$

Indeed, since u is divergence free,

$$\nabla_u u = u_j \nabla_j u_i = \nabla_j (u_j u_i) = div(u \otimes u). \tag{6.17}$$

Now then, rewrite (6.16) as an integral equation,

$$u(t) = e^{t\nu\Delta} u_0 - \int_0^t e^{(t-s)\nu\Delta} Pdiv(u(s) \otimes u(s)) ds = \Psi u(t). \tag{6.18}$$

Then we look for a fixed point,

$$\Psi : C(I, X) \to C(I, X), \qquad X = L^p(\mathcal{M}) \cap kerdiv. \tag{6.19}$$

Then by Proposition 7, fix $\alpha > 0$ and set

$$X = \{u \in C([0, T], X) : u(0) = u_0, \qquad \|u(t) - u_0\|_X \leq \alpha\}, \tag{6.20}$$

and show that if $T > 0$ is sufficiently small, then $\Psi : Z \to Z$ is a contraction map.

Then we need a Banach space such that

$$\Phi : X \to Y, \qquad \text{is Lipschitz, uniformly on bounded sets,} \qquad e^{t\nu\Delta} : Y \to X, \qquad \text{for} \qquad t > 0, \tag{6.21}$$

and for some $\gamma < 1$,

$$\|e^{t\nu\Delta}\|_{\mathcal{L}(Y,X)} \leq Ct^{-\gamma}, \qquad \text{for} \qquad t \in (0, 1]. \tag{6.22}$$

The map Φ in (6.21) is

$$\Phi(u) = Pdiv(u \otimes u), \tag{6.23}$$

and then set

$$Y = H^{-1,p/2}(\mathcal{M}) \cap kerdiv. \tag{6.24}$$

These conditions hold if $p > n$. Thus, we have the solution u_ν to (6.16) belonging to

$$u_\nu \in C([0, T], L^p(\mathcal{M})). \tag{6.25}$$

The proof of smoothness from Proposition 8) applies essentially verbatim. □

Thus, we can get global well–posedness if we can bound $\|u(t)\|_{L^p(\mathcal{M})}$ for some $p > n$.

Proposition 26. *Given $\nu > 0$, $p > n$, if $u \in C([0, T), L^p(\mathcal{M}))$ solves* (6.3), *and if the vorticity ω satisfies*

$$\sup_{t \in [0,T)} \|\omega(t)\|_{L^q} \leq K < \infty, \qquad q = \frac{np}{n + p}, \tag{6.26}$$

then the solution u continues to an interval $[0, T')$, for some $T' > T$,

$$u \in C([0, T'), L^p(\mathcal{M})) \cap C^\infty((0, T') \times \mathcal{M}), \tag{6.27}$$

solving (6.3).

Proof. We have

$$u = Aw + P_0 u, \tag{6.28}$$

where $A \in OPS^{-1}(\mathcal{M})$ and P_0 is a projection onto a finite–dimensional space of smooth fields. Then by the Sobolev embedding theorem,

$$A : L^q(\mathcal{M}) \to L^p(\mathcal{M}). \tag{6.29}$$

□

Now then, when $dim\mathcal{M} = 2$, the vorticity $\omega = \partial_1 u_2 - \partial_2 u_1$ is a scalar and satisfies the PDE

$$\frac{\partial \omega}{\partial t} + \nabla_u \omega = \nu(\Delta + c_0)\omega, \tag{6.30}$$

Then by the maximum principle,

$$\|\omega(t)\|_{L^\infty} \leq e^{\nu c_0 t}\|\omega(0)\|_{L^\infty}. \tag{6.31}$$

When $\mathcal{M} = \mathbb{R}^2$, $c_0 = 0$. When $dim\mathcal{M} = 3$, $\omega = curl(u)$ is a vector field, and then

$$\frac{\partial \omega}{\partial t} + \nabla_u \omega - \nabla_\omega u = \nu \Delta \omega. \tag{6.32}$$

In this case we cannot use the maximum principle to control ω, and the Navier–Stokes equation remains an open problem. We can prove a global result for small data.

Proposition 27. *Let $k > \frac{n}{2} + 1$, $\nu > 0$. If $\|u_0\|_{H^k}$ is sufficiently small, then* (6.3) *has a global solution in $C([0,\infty), H^k) \cap C^\infty((0,\infty) \times \mathcal{M})$.*

Proof. If $\mathcal{M} = \mathbb{R}^n$, we can choose constants A and B such that

$$\|\nabla u\|_{H^k}^2 \geq A\|u\|_{H^k}^2 - B\|u\|_{L^2}^2. \tag{6.33}$$

Therefore, (6.13) yields

$$\frac{d}{dt}\|u(t)\|_{H^k}^2 \leq \{C\|u(t)\|_{C^1} - 2\nu A\}\|u\|_{H^k}^2 + 2\nu B\|u(t)\|_{L^2}^2. \tag{6.34}$$

Now suppose

$$\|u_0\|_{L^2}^2 \leq \delta, \qquad \text{and} \qquad \|u_0\|_{H^k}^2 \leq L\delta, \tag{6.35}$$

where L is specified below. For $L\delta$ sufficiently small,

$$\|v\|_{H^k}^2 \leq 2L\delta \qquad \text{implies} \qquad \|v\|_{C^1} \leq \frac{\nu A}{C}. \tag{6.36}$$

Recall that $\|u(t)\|_{L^2} \leq \|u_0\|_{L^2}$. Therefore, if $\|u(t)\|_{H^k}^2 \leq 2L\delta$,

$$\frac{dy}{dt} \leq -\nu A y + 2\nu B\delta, \qquad y(t) = \|u(t)\|_{H^k}^2. \tag{6.37}$$

Therefore, (6.37) implies

$$y(t) \leq \max\{y(t_0), 2BA^{-1}\delta\}, \qquad \text{for} \qquad t \geq t_0. \tag{6.38}$$

Therefore, if we take $L = \frac{2B}{A}$ and $\delta > 0$ sufficiently small so that (6.36) holds, we have a global bound $\|u(t)\|_{H^k}^2 \leq L\delta$, which gives global existence. □

Now we prove the Hopf theorem proving global weak solutions exist for $\nu > 0$. Suppose $c_0 = 0$, which corresponds to $Ric = 0$.

Proposition 28. *Given $u_0 \in L^2(\mathcal{M})$, $div(u_0) = 0$, $\nu > 0$,* (6.3) *has a weak solution for $t \in (0, \infty)$,*

$$u \in L^\infty(\mathbb{R}^+, L^2(\mathcal{M})) \cap L^2_{loc}(\mathbb{R}^+, H^1(\mathcal{M})) \cap Lip_{loc}(\mathbb{R}^+, H^{-2}(\mathcal{M}) + H^{-1,1}(\mathcal{M})). \tag{6.39}$$

Proof. We produce u as a limit point of solutions u_ϵ to a slight modification of (6.6), namely we require J_ϵ to be a projection, $J_\epsilon = \chi(\epsilon\Delta)$, where $\chi(\lambda)$ is the characteristic function of $[-1, 1]$. Then J_ϵ commutes with Δ and with P. We also require $u_\epsilon(0) = J_\epsilon u_0$ and then $u_\epsilon(t) = J_\epsilon u_\epsilon(t)$. Now, by (6.7),

$$\{u_\epsilon : \epsilon \in (0, 1]\} \qquad \text{is bounded in} \qquad L^\infty(\mathbb{R}^+, L^2). \tag{6.40}$$

Furthermore, for $\mathcal{M} = \mathbb{R}^n$,

$$2\nu \int_0^T \|\nabla u_\epsilon(t)\|^2_{L^2} dt = \|J_\epsilon u_0\|^2_{L^2} - \|u_\epsilon(T)\|^2_{L^2}. \tag{6.41}$$

Therefore, for each bounded interval $I = [0, T]$,

$$\{u_\epsilon\} \qquad \text{is bounded in} \qquad L^2(I, H^1(\mathcal{M})). \tag{6.42}$$

Now then, since $J_\epsilon \Delta J_\epsilon u_\epsilon = \Delta u_\epsilon$.

$$\frac{\partial u_\epsilon}{\partial t} + PJ_\epsilon div(u_\epsilon \otimes u_\epsilon) = \nu \Delta u_\epsilon. \tag{6.43}$$

Now by (6.40),

$$\{u_\epsilon \otimes u_\epsilon : \epsilon \in (0, 1]\} \qquad \text{is bounded in} \qquad L^\infty(\mathbb{R}^+, L^1(\mathcal{M})). \tag{6.44}$$

Since $L^1(\mathcal{M}) \subset H^{-n/2-\delta}(\mathcal{M})$, for each $\delta > 0$,

$$\{\partial_t u_\epsilon\} \qquad \text{is bounded in} \qquad L^2(I, H^{-n/2-1-\delta}(\mathcal{M})), \tag{6.45}$$

so

$$\{u_\epsilon\} \qquad \text{is bounded in} \qquad H^1(I, H^{-n/2-1-\delta}(\mathcal{M})). \tag{6.46}$$

Interpolating between (6.46) and (6.42),

$$\{u_\epsilon\} \qquad \text{is bounded in} \qquad H^s(I, H^{1-s-s(\frac{n}{2}+1+\delta)}(\mathcal{M})), \tag{6.47}$$

and therefore,

$$\{u_\epsilon\} \qquad \text{is compact in} \qquad L^2(I, H^{1-\gamma}(\mathcal{M})), \tag{6.48}$$

for all $\gamma > 0$. Therefore, we can pick a sequence $u_k = u_{\epsilon_k}$ such that

$$u_k \to u, \qquad \text{in} \qquad L^2([0, T], H^{1-\gamma}(\mathcal{M})), \qquad \text{in norm.} \tag{6.49}$$

Therefore, u is the desired weak solution of (6.3). □

Solutions of (6.3) that are obtained as limits of u_ϵ are called Leray–Hopf solutions to the Navier Stokes equations. Uniqueness and smoothness of a Leray–Hopf solution remain open problems if $dim(\mathcal{M}) \geq 3$.

Proposition 29. *If $dim(\mathcal{M}) = 3$ and u is a Leray–Hopf solution of* (6.3)*, then there is an open dense subset $\mathcal{J}$ of $(0, \infty)$ such that $\mathbb{R}^+ \setminus \mathcal{J}$ has Lebesgue measure zero and*

$$u \in C^\infty(\mathcal{J} \times \mathcal{M}). \tag{6.50}$$

Proof. Fix $T > 0$ arbitrary, $I = [0, T]$. Passing to a subsequence, $u_k = u_{\epsilon_k}$, with

$$\|u_{k+1} - u_k\|_E \leq 2^{-k}, \qquad E = L^2(I, H^{1-\gamma}(\mathcal{M})). \tag{6.51}$$

Now set $\Gamma(t) = \sup_k \|u_k(t)\|_{H^{1-\gamma}}$. Then

$$\Gamma(t) \leq \|u_1(t)\|_{H^{1-\gamma}} + \sum_{k=1}^{\infty} \|u_{k+1}(t) - u_k(t)\|_{H^{1-\gamma}}. \tag{6.52}$$

Therefore, $\Gamma \in L^2(I)$. Therefore, $\Gamma(t)$ is finite almost everywhere. Let

$$S = \{t \in I : \Gamma(t) < \infty\}. \tag{6.53}$$

For $\gamma > 0$ small, $H^{1-\gamma}(\mathcal{M}) \subset L^p(\mathcal{M})$, with p close to 6 when $dim(\mathcal{M}) = 3$. Therefore, the product of two elements in $H^{1-\gamma}$ belongs to $H^{1/2-\gamma'}$ for $\gamma' > 0$. Applying the local well–posedness result, for each $t_0 \in S$, there exists $T(\Gamma(t_0)) > 0$ such that, for $\gamma' > 0$,

$$\{u_k\} \qquad \text{bounded in} \qquad C([t_0, t_0 + T(t_0)], H^{1-\gamma}) \cap C^{\infty}((t_0, t_0 + T(t_0)) \times \mathcal{M}). \tag{6.54}$$

Therefore, in the set

$$\mathcal{J}_T = \cup_{t_0 \in S}(t_0, t_0 + T(t_0)), \tag{6.55}$$

and the weak limit u has the property $u \in C^{\infty}(\mathcal{J}_T \times \mathcal{M})$.

It remains to show that $I \setminus \mathcal{J}_T$ has Lebesgue measure zero. Fix $\delta_1 > 0$. Since $meas(I \setminus S) = 0$, there exists $\delta_2 > 0$ such that if $S_{\delta_2} = \{t \in S : T(t) \geq \delta_2\}$, then $meas(I \setminus S_{\delta_2}) < \delta_1$. Now then, if $T(t_0) \geq \delta_2$ then $t_0 + \frac{\delta_2}{2} \in \mathcal{J}_T$. Therefore, $meas(I \setminus \mathcal{J}_T) \leq \delta_1 + \frac{\delta_2}{2}$. This completes the proof. □

7. Harmonic maps

Let $\mathcal{M}$ and $\mathcal{N}$ be compact Riemannian manifolds. Using the Nash embedding result, we can take $\mathcal{N} \subset \mathbb{R}^k$. A harmonic map is a critical point for the energy functional

$$E(u) = \frac{1}{2} \int_{\mathcal{M}} |\nabla u(x)|^2 dV(x). \tag{7.1}$$

Remark 3. *Recall that an isometric embedding f is an embedding that preserves the metric. That is, for $v, w \in T_x\mathcal{M}$, if g and h are the metrics,*

$$g(v, w) = h(df(v), df(w)). \tag{7.2}$$

Therefore, the quantity (7.1) *only depends on the metrics of $\mathcal{M}$ and $\mathcal{N}$, not on the embedding.*

Suppose u_s is a smooth family of maps from $\mathcal{M}$ to $\mathcal{N}$. Then,

$$\frac{d}{ds} E(u_s)|_{s=0} = -\int v(x) \Delta u(x) dV(x), \tag{7.3}$$

where $u = u_0$ and $v(x) = \frac{\partial}{\partial s} u_s(x) \in T_{u(x)}\mathcal{N}$. It is possible to vary u_0 so that v is any map $\mathcal{M} \to \mathbb{R}^k$ that satisfies $v(x) \in T_{u(x)}\mathcal{N}$. Therefore, the stationary condition is that

$$\Delta u(x) \perp T_{u(x)}\mathcal{N}, \qquad \text{for all} \qquad x \in \mathcal{M}. \tag{7.4}$$

It is possible to rewrite the stationary condition (7.4). Suppose that near a point $z \in \mathcal{N} \subset \mathbb{R}^k$, N is given by

$$f_l(y) = 0, \qquad 1 \leq l \leq L, \tag{7.5}$$

where $L = k - dim\mathcal{N}$, and with $\nabla f_l(y)$ linearly independent in $\mathbb{R}^k$ for each y near z. Now then, if $u : \mathcal{M} \to \mathcal{N}$ is smooth and $u(x)$ is close to z, then we have

$$\sum_\nu \frac{\partial f_l}{\partial u_\nu}\frac{\partial u_\nu}{\partial x_j} = 0, \qquad 1 \le l \le L, \qquad 1 \le j \le m, \tag{7.6}$$

where $(x_1, ..., x_m)$ is a local coordinate system on $\mathcal{M}$. Multiplying (7.6) by g^{jk} and differentiating with respect to x_k,

$$\sum_\nu \frac{\partial f_l}{\partial u_\nu}\Delta u_\nu = -\sum_{\mu,\nu,j,k} g^{jk}\frac{\partial^2 f_l}{\partial u_\mu \partial u_\nu}\frac{\partial u_\mu}{\partial x_k}\frac{\partial u_\nu}{\partial x_j}. \tag{7.7}$$

Since $\{\nabla_y f_l(y) : 1 \le l \le L\}$ is a basis for the orthogonal complement in $\mathbb{R}^k$ of $T_y\mathcal{N}$, the normal component of Δu depends only on the first order derivatives of u and is quadratic in ∇u. That is,

$$(\Delta u)^N = \Gamma(u)(\nabla u, \nabla u). \tag{7.8}$$

Thus, the stationary solution for (7.4) is equivalent to

$$\Delta u - \Gamma(u)(\nabla u, \nabla u) = 0. \tag{7.9}$$

Let $\tau(u)$ denote the left hand side of (7.9). Then by (7.8), given $u \in C^2(\mathcal{M}, \mathcal{N})$, $\tau(u)$ is tangent to $\mathcal{N}$ at $u(x)$. There is a result of Eells and Sampson.

Theorem 2. *Suppose $\mathcal{N}$ has negative sectional curvature everywhere. Then, given $v \in C^\infty(\mathcal{M}, \mathcal{N})$, there exists a harmonic map $w \in C^\infty(\mathcal{M}, \mathcal{N})$ that is homotopic to v.*

The existence of w is established by solving the PDE,

$$\frac{\partial u}{\partial t} = \Delta u - \Gamma(u)(\nabla u, \nabla u), \qquad u(0) = v. \tag{7.10}$$

Under the hypothesis of negative sectional curvature on $\mathcal{N}$, there is a smooth solution to (7.10) for all $t \ge 0$, and that, for a sequence $t_k \to \infty$, $u(t_k)$ tends to the desired w. By Proposition 8, equation (7.10) is locally solvable on some interval $[0, T)$. Since $\tau(u)$ is tangent to $\mathcal{N}$ for $u \in C^\infty(\mathcal{M}, \mathcal{N})$, it follows that $u(t) : \mathcal{M} \to \mathcal{N}$ for each $t \in [0, T)$. To show that $T = \infty$, it suffices to estimate $\|u(t)\|_{C^1}$.

Let $e(t, x)$ denote the energy density,

$$e(t, x) = \frac{1}{2}|\nabla_x u(t, x)|^2. \tag{7.11}$$

Now then, we have the identity

$$\frac{\partial e}{\partial t} - \Delta e = -|{}^N\nabla^2 u|^2 - \frac{1}{2}\langle du \cdot Ric^{\mathcal{M}}(e_j), du \cdot e_j\rangle + \frac{1}{2}\langle R^N(du \cdot e_j, du \cdot e_k)du \cdot e_k, du \cdot e_j\rangle. \tag{7.12}$$

Since $\mathcal{N}$ has negative sectional curvature, we have the identity

$$\frac{\partial e}{\partial t} - \Delta e \le ce. \tag{7.13}$$

Now then, if $f(t, x) = e^{-ct}e(t, x)$,

$$\frac{\partial f}{\partial t} - \Delta f \le 0, \tag{7.14}$$

which by the maximum principle, $f(t, x) \le \|f(0, \cdot)\|_{L^\infty}$. Therefore,

$$e(t, x) \le e^{ct}\|\nabla v\|^2_{L^\infty}. \tag{7.15}$$

This C^1 estimate implies global existence of a solution by Proposition 7.

Now then, for the total energy,

$$E(t) = \int_{\mathcal{M}} e(t,x)dV(x) = \frac{1}{2}\int_{\mathcal{M}} |\nabla u|^2 dV(x). \tag{7.16}$$

Then by (7.3),

$$E'(t) = -\int_{\mathcal{M}} |u_t|^2 dV(x). \tag{7.17}$$

Indeed, by (7.3),

$$E'(t) = \int_{\mathcal{M}} \langle u_t, \Delta u\rangle dV(x). \tag{7.18}$$

Since u_t is tangent to $\mathcal{N}$ and $\Gamma(u)(\nabla u, \nabla u)$ is normal to $\mathcal{N}$, (7.17) follows.

Lemma 3. *Let $e(t,x) \geq 0$ satisfy the differential inequality* (7.12)*. Assume that*

$$E(t) = \int e(t,x)dV(x) \leq E_0, \tag{7.19}$$

is bounded. Then there exists a uniform estimate

$$e(t,x) \leq e^c K E_0, \qquad t \geq 1, \tag{7.20}$$

where K depends on the geometry of $\mathcal{M}$.

Proof. Let $\frac{\partial e}{\partial t} - \Delta e = ce - g$, $g(t,x) \geq 0$. Then for $0 \leq s \leq 1$,

$$e(t+s,x) = e^{s(\Delta + c)} e(t,x) - \int_0^s e^{(s-\tau)(\Delta+c)} g(\tau, x) d\tau \leq e^{s(\Delta+c)} e(t,x). \tag{7.21}$$

Since $e^{s(\Delta+c)}$ is uniformly bounded from $L^1(\mathcal{M})$ to $L^\infty(\mathcal{M})$ for $s \in [\frac{1}{2}, 1]$, the bound (7.20) for $t \in [\frac{1}{2}, \infty)$ follows from the hypothesized $L^1(\mathcal{M})$ bound on $e(t)$. □

Now then, Lemma 3 applies to $e(t,x) = |\nabla u|^2$ when u solves (7.10) satisfy

$$\|u(t)\|_{C^1} \leq K_1 \|v\|_{C^1}, \qquad \text{for all} \qquad t \geq 0. \tag{7.22}$$

Then by the regularity estimates in Proposition 11,

$$\|u(t)\|_{C^l} \leq K_l \|v\|_{C^1}, \qquad t \geq 1. \tag{7.23}$$

Now by (7.17), $E(t)$ is positive and monotonically decreasing. Therefore, $\int_{\mathcal{M}} |u_t(t,x)|^2 dV(x)$ is an integrable function of t, so there exists a sequence $t_j \to \infty$ such that

$$\|u_t(t_j, \cdot)\|_{L^2} \to 0. \tag{7.24}$$

Also by (7.23) and the PDE (7.10), we have the bounds

$$\|u_t(t, \cdot)\|_{H^k} \leq C_k, \tag{7.25}$$

and interpolating with (7.24) gives

$$\|u_t(t_j, \cdot)\|_{L^2} \to 0. \tag{7.26}$$

Therefore, by (7.10), for $u_j(x) = u(t_j, x)$,

$$\Delta u_j - \Gamma(u_j)(\nabla u_j, \nabla u_j) \to 0, \qquad \text{in} \qquad H^l(\mathcal{M}). \tag{7.27}$$

Therefore, the subsequence converges in a strong norm to an element $w \in C^\infty(\mathcal{M},\mathcal{N})$ solving (7.9) and homotopic to v.

Theorem 3. *If we are given $v \in C^\infty(\mathcal{M},\mathcal{N})$ there exists a smooth map $w : \mathcal{M} \to \mathcal{N}$ that is harmonic and homotopic to v and such that $E(w) \le E(\tilde{v})$ for any $\tilde{v} \in C^\infty(\mathcal{M},\mathcal{N})$, homotopic to v.*

Proof. Let α be the infimum of the energy of smooth maps homotopic to v. Choose a sequence v_ν homotopic to v such that $E(v_\nu) \searrow \alpha$. Then solve (7.10) with $u_\nu(0) = v_\nu$. Then we have a sequence $u_\nu(t_{\nu j}) \to w_\nu \in C^\infty(\mathcal{M},\mathcal{N})$, harmonic, $E(w_\nu) \le E(v_\nu)$ so that

$$E(w_\nu) \searrow \alpha. \tag{7.28}$$

Also, we have uniform C^l bounds of w_ν for all l. Thus, the limit point has all the desired properties. □

Now let

$$F(x, D_x^1 u) = B(u)(\nabla u, \nabla u), \tag{7.29}$$

a quadratic form in ∇u. In this case, take

$$X = H^{1,p}, \qquad Y = L^q, \qquad q = \frac{p}{2}, \qquad p > n. \tag{7.30}$$

Then

$$H^{s,p} \subset L^{\frac{np}{n-sp}}, \qquad p < \frac{n}{s}. \tag{7.31}$$

Proposition 30. *If (7.29) is a quadratic form in ∇u, then the PDE*

$$\frac{\partial u}{\partial t} = \Delta u + B(u)(\nabla u, \nabla u), \qquad u(0) = f, \tag{7.32}$$

has a solution in $C([0,T], H^{1,p}) \cap C^\infty((0,T) \times \mathcal{M})$, provided $f \in H^{1,p}(\mathcal{M})$, $p > n$.

8. Reaction–diffusion equations

A reaction diffusion equation is an $l \times l$ system of the form,

$$\frac{\partial u}{\partial t} = Lu + X(u), \qquad u(0) = f, \tag{8.1}$$

where $u = u(t, x)$ takes values in $\mathbb{R}^l$, X is a real vector field on $\mathbb{R}^l$, and L is a second order differential operator that is a negative semi-definite, self-adjoint operator on $L^2(\mathcal{M})$. The manifold $\mathcal{M}$ is complete, either $\mathbb{R}^n$ or a compact manifold. The operator L need not be elliptic.

One example is the Fitzhugh–Nagumo system,

$$\begin{aligned} \frac{\partial v}{\partial t} &= D\frac{\partial^2 v}{\partial x^2} + f(v) - w, \\ \frac{\partial w}{\partial t} &= \epsilon(v - \gamma w), \end{aligned} \tag{8.2}$$

with

$$f(v) = v(a - v)(v - 1). \tag{8.3}$$

In this case,

$$L = \begin{pmatrix} D\partial_x^2 & 0 \\ 0 & 0 \end{pmatrix}, \qquad D > 0. \tag{8.4}$$

The operator L has the following generalization of the maximum principle.

Proposition 31 (Invariance property)**.** *There is a compact, convex neighborhood K of the origin in $\mathbb{R}^l$ such that if $f \in L^2(\mathcal{M})$, then for all $t \geq 0$,*

$$f(x) \in K, \qquad \text{for all } x \qquad \text{implies} \qquad e^{tL} f(x) \in K, \qquad \text{for all } x. \tag{8.5}$$

Therefore, if $f, g \in L^2(\mathcal{M})$ have compact support,

$$\|e^{tL} f\|_{L^\infty} \leq \kappa \|f\|_{L^\infty}, \tag{8.6}$$

where κ is independent of $t \geq 0$. If we define a norm on $\mathbb{R}^l$ so that $K \cap (-K)$ is the unit ball, then we have $\kappa = 1$. For such f and g, we have

$$|(e^{tL} f, g)| = |(f, e^{tL} g)| \leq \kappa \|f\|_{L^1} \|g\|_{L^\infty}, \tag{8.7}$$

so then $\|e^{tL} f\|_{L^1} \leq \kappa \|f\|_{L^1}$. Therefore, e^{tL} has a unique extension to the linear map

$$e^{tL} : L^p(\mathcal{M}) \to L^p(\mathcal{M}), \qquad \|e^{tL}\| \leq \kappa_p, \tag{8.8}$$

for $1 \leq p \leq 2$, by interpolation. Then by duality (8.8) holds for $2 \leq p \leq \infty$.

9. A nonlinear Trotter product formula

10. The Stefan problem

11. Quasilinear parabolic equations 1

12. Quasilinear parabolic equations 2, sharper estimates

13. Quasilinear parabolic equations 3, Nash–Moser estimates

References

[Taya] Michael E Taylor. *Partial differential equations*, volume 2. Springer.
[Tayb] Michael E Taylor. *Partial differential equations*, volume 3. Springer.