A COURSE ON ELLIPTIC PDE

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Contents

These notes are based on [Tay96], [T$^+$a], [T$^+$b], [Eva98], and [HL11].

1. The Dirichlet problem on the ball in $\mathbb{R}^n$

The study of elliptic partial differential equations begins with harmonic functions on a disc,

$$\begin{cases} \Delta u = 0 & : \text{ on } \mathbb{D}, \\ u = f & : \text{ on } S^1. \end{cases} \tag{1.1}$$

Suppose for now that $f \in C(\mathbb{S}^1)$. Then define the Fourier transform for any $k \in \mathbb{Z}$,

$$\hat{f}(k) = \mathcal{F}f(k) = \frac{1}{2\pi} \int_0^{2\pi} e^{-ik\theta} f(\theta) d\theta. \tag{1.2}$$

For $f \in L^1(\mathbb{S}^1)$, $\mathcal{F} : L^1(\mathbb{S}^1) \to l^\infty(\mathbb{Z})$. Then for any $r \in [0, 1)$, the sum

$$J_r f(\theta) = \sum_{k=-\infty}^{\infty} \hat{f}(k) r^{|k|} e^{ik\theta}, \tag{1.3}$$

converges absolutely.

Furthermore, taking $z = re^{i\theta}$,

$$\sum_{k=0}^{\infty} \hat{f}(k) r^k e^{ik\theta} = \sum_{k=0}^{\infty} \hat{f}(k) z^k, \tag{1.4}$$

is holomorphic on $\mathbb{D} = \{z : |z| < 1\}$, while the sum

$$\sum_{k=-\infty}^{-1} \hat{f}(k) r^{-k} e^{ik\theta} = \sum_{k=1}^{\infty} \hat{f}(-k) \bar{z}^k, \tag{1.5}$$

is anti-holomorphic on $\mathbb{D} = \{z : |z| < 1\}$. Therefore,

$$u = (PIf)(z) = \sum_{k=0}^{\infty} \hat{f}(k) z^k + \sum_{k=1}^{\infty} \hat{f}(-k) \bar{z}^k = (PI_+ f)(z) + (PI_- f)(z), \tag{1.6}$$

is the sum of a holomorphic function and an anti-holomorphic function on $\mathbb{D}$. A holomorphic function v satisfies

$$\frac{\partial v}{\partial \bar{z}} = \frac{1}{2}(\frac{\partial}{\partial x} + i \frac{\partial}{\partial y}) v = 0, \tag{1.7}$$

while an anti-holomorphic function satisfies

$$\frac{\partial v}{\partial z} = \frac{1}{2}(\frac{\partial}{\partial x} - i \frac{\partial}{\partial y}) v = 0. \tag{1.8}$$

Therefore, u defined by (1.6) is a harmonic function.

Next, we show that

Proposition 1. *If $f \in C(\mathbb{S}^1)$, $u \in C(\bar{\mathbb{D}})$ and $u|_{S^1} = f|_{S^1}$.*

Proof. Rewriting $J_r f$ in (1.3), for any $r \in [0, 1)$,

$$J_r f(\theta) = \sum_k \hat{f}(k) r^{|k|} e^{ik\theta} = \frac{1}{2\pi} \int_{S^1} f(\theta') \sum_k r^{|k|} e^{ik(\theta - \theta')} d\theta' = \frac{1}{2\pi} \int_{S^1} f(\theta') p(r, \theta - \theta') d\theta', \tag{1.9}$$

where

$$p(r,\theta) = \sum_{k=-\infty}^{\infty} r^{|k|} e^{ik\theta} = 1 + \sum_{k=1}^{\infty} (r^k e^{ik\theta} + r^k e^{-ik\theta}) = \frac{1-r^2}{1-2r\cos\theta + r^2}. \tag{1.10}$$

It is straightforward to verify that $p(r,\theta) > 0$ for any $r \in [0,1)$ and $\theta \in [0,2\pi)$. Furthermore, for any $r \in [0,1)$,

$$\frac{1}{2\pi}\int p(r,\theta)d\theta = 1, \tag{1.11}$$

and $p(r,\theta) \to 0$ as $r \nearrow 1$ uniformly for θ in a compact subset of S^1 that does not contain $\theta = 0$. Therefore, $\frac{1}{2\pi}p(r,\theta)$ is a kernel.

Now then, if f is continuous on S^1, then f is uniformly continuous on S^1. Fix $\theta \in S^1$. Then,

$$f(\theta) - \frac{1}{2\pi}\int_{S^1} p(r,\theta-\theta')f(\theta')d\theta' = \frac{1}{2\pi}\int_{S^1} p(r,\theta-\theta')[f(\theta)-f(\theta')]d\theta' \to 0, \tag{1.12}$$

as $r \nearrow 1$. This proves the proposition. □

Now we generalize this computation to n dimensions.

Proposition 2. *If $f \in C(S^{n-1})$, then the solution to* (1.1) *is given by*

$$u(x) = \frac{1-|x|^2}{A_{n-1}}\int_{S^{n-1}} \frac{f(x')}{|x-x'|^n} dS(x'), \tag{1.13}$$

where A_{n-1} is the area of the unit sphere $S^{n-1} \subset \mathbb{R}^n$.

Proof. We first prove that u given by (1.13) is harmonic on $\mathbb{D}$.

Lemma 1. *For a given $x' \in S^{n-1}$, set*

$$v(x) = (1-|x|^2)|x-x'|^{-n}. \tag{1.14}$$

Then v is harmonic on $\mathbb{R}^n \setminus \{x'\}$.

It is straightforward to see that Lemma 1 implies that (1.13) is harmonic on $\mathbb{D}$.

Proof of Lemma 1. Shifting $x \mapsto x + x'$, we show that

$$v(x) = (1-|x+x'|^2)|x|^{-n}, \tag{1.15}$$

is harmonic on $\mathbb{R}^n \setminus \{0\}$. Expanding,

$$(1-|x+x'|^2) = 1 - (1+|x|^2 + 2x\cdot x') = -2x\cdot x' - |x|^2. \tag{1.16}$$

When $n \geq 3$, $r > 0$,

$$\Delta|x|^{2-n} = (\partial_r^2 + \frac{n-1}{r}\partial_r)r^{2-n} = (2-n)(1-n)r^{-n} + \frac{(n-1)}{r}(2-n)r^{1-n} = 0. \tag{1.17}$$

Meanwhile, for $n > 2$,

$$-2x'\cdot x|x|^{-n} = -\frac{2x'}{2-n}\cdot\nabla(|x|^{2-n}), \tag{1.18}$$

so using the fact that $\Delta\nabla f = \nabla\Delta f$, $\Delta(-2x'\cdot x|x|^{-n}) = 0$ for $n > 2$.

Remark 1. *When $n = 1$, the harmonic functions and linear functions are identical. Plugging f into* (1.13),

$$u(x) = (1 + x)\frac{f(1)}{2} + (1 - x)\frac{f(-1)}{2}. \tag{1.19}$$

□

Now show that $u \in C(\bar{\mathbb{D}})$ and $u|_{S^{n-1}} = f|_{S^{n-1}}$. Let $x = r\omega$, where $\omega \in S^{n-1}$. Then,

$$u(r\omega) = \int_{S^{n-1}} p(r, \omega, \omega')f(\omega')dS(\omega'), \tag{1.20}$$

where

$$p(r, \omega, \omega') = \frac{1 - r^2}{A_{n-1}}|r\omega - \omega'|^{-n}. \tag{1.21}$$

It is clear that $p(r, \omega, \omega') \to 0$ as $r \nearrow 1$ if $\omega \neq \omega'$. Now then,

$$p(r, y, \omega') = \frac{1}{A_{n-1}}(1 - r^2|y|^2)|ry - \omega'|^{-n}, \tag{1.22}$$

is harmonic in y for $|y| < \frac{1}{r}$. Therefore, by the mean value property for harmonic functions,

$$\int_{S^{n-1}} p(r, \omega, \omega')dS(\omega) = p(0, e_n, \omega') = 1. \tag{1.23}$$

Since

$$\int_{S^{n-1}} p(r, \omega, \omega')dS(\omega'), \tag{1.24}$$

is independent of ω,

$$\int_{S^{n-1}} p(r, \omega, \omega')dS(\omega') = \frac{1}{A_{n-1}} \int_{S^{n-1}} \int_{S^{n-1}} p(r, \omega, \omega')dS(\omega)dS(\omega') = 1. \tag{1.25}$$

Therefore, $u|_{S^{n-1}} = f|_{S^{n-1}}$ and $u \in C(\bar{\mathbb{D}})$. □

It remains to prove the mean value property for harmonic functions.

Definition 1 (Mean value property)**.** *Suppose $\Omega \subset \mathbb{R}^n$ is a connected domain. For $u \in C(\Omega)$, u satisfies the mean value property if*

$$u(x) = \frac{1}{A_{n-1}r^{n-1}} \int_{\partial B_r(x)} u(y)dS(y), \qquad \textit{for any} \qquad B_r(x) \subset \Omega. \tag{1.26}$$

This definition is equivalent to the definition,

$$u(x) = \frac{n}{A_{n-1}r^n} \int_{B_r(x)} u(y)dy, \qquad \text{for any} \qquad B_r(x) \subset \Omega. \tag{1.27}$$

Indeed, if (1.26) holds,

$$\frac{n}{\omega_n r^n} \int_{B_r(x)} u(y)dy = \frac{n}{\omega_n r^n} \int_0^r u(x)\omega_n s^{n-1} ds = u(x). \tag{1.28}$$

Meanwhile, rewriting (1.27) and differentiating with respect to r,

$$u(x)r^n = \frac{n}{\omega_n} \int_{B_r(x)} u(y)dy, \qquad nr^{n-1}u(x) = \frac{n}{\omega_n} \int_{\partial B_r(x)} u(y)dS(y). \tag{1.29}$$

Theorem 1. *Let $u \in C^2(\Omega)$ be harmonic in Ω. Then u satisfies the mean value property in Ω.*

Proof. Take a ball $B_r(x) \subset \Omega$. For $\rho \in (0, r)$, apply the divergence theorem in $B_\rho(x)$. Then

$$0 = \int_{B_\rho(x)} \Delta u(y)dy = \int_{\partial B_\rho} \frac{\partial u}{\partial \nu} dS(y) = \rho^{n-1} \int_{|w|=1} \frac{\partial u}{\partial \rho}(x+\rho w) dS_w = \rho^{n-1} \frac{\partial}{\partial \rho} \int_{|w|=1} u(x+\rho w) dS_w. \tag{1.30}$$

For any continuous function,

$$\lim_{\rho \searrow 0} \int_{|w|=1} u(x+\rho w) dS_w = u(x), \tag{1.31}$$

which completes the proof. □

If u is a C^2 function, then the mean value property is equivalent to being a harmonic function.

Theorem 2. *If $u \in C(\Omega)$ has mean value property in Ω, then u is smooth and harmonic in Ω.*

Proof. Choose $\varphi \in C_0^\infty(B_1(0))$ to be a radial function that satisfies $\int_{B_1(0)} \varphi(y)dy = 1$. Then,

$$\omega_n \int_0^1 r^{n-1} \psi(r) dr = 1. \tag{1.32}$$

Now for $x \in \Omega$, $\epsilon < dist(x, \partial\Omega)$, then for $\varphi_\epsilon(z) = \frac{1}{\epsilon^n} \varphi(\frac{z}{\epsilon})$,

$$\int_\Omega u(y) \varphi_\epsilon(y - x) dy = u(x). \tag{1.33}$$

Therefore, $u(x) = (\varphi_\epsilon * u)(x)$ for any $x \in \Omega_\epsilon = \{y \in \Omega : d(y, \partial\Omega) > \epsilon\}$, so u is smooth.

Next,

$$\int_{B_r(x)} \Delta u(y)dy = r^{n-1} \frac{\partial}{\partial r} \int_{|w|=1} u(x+rw) dS_w = r^{n-1} \frac{\partial}{\partial r}(\omega_n u(x)) = 0, \qquad \text{for any} \qquad B_r(x) \subset \Omega. \tag{1.34}$$

Thus, $\Delta u = 0$ on Ω. □

The mean value property implies uniqueness for solutions to (1.1).

Proposition 3. *If $u \in C(\bar{\Omega})$ satisfies the mean value property in Ω, then u assumes its maximum and minimum only on $\partial\Omega$, unless u is constant.*

Proof. The proof is the same as for holomorphic functions. □

Since harmonic functions satisfy the mean value property, they satisfy the maximum and minimum principle. Thus, a solution to the Dirichlet problem,

$$\Delta u = f \qquad \text{in} \qquad \Omega, \qquad u = \varphi \qquad \text{on} \qquad \partial\Omega, \tag{1.35}$$

is unique. Indeed, suppose u and v are two solutions to (1.35). Then, $u - v$ solves (1.35) with $f = 0$ on $\partial\Omega$. Thus, the maximum and minimum of $u - v$ is 0, so $u - v = 0$ everywhere.

Remark 2. *In general, uniqueness does not hold for a solution on an unbounded domain.*

Harmonic functions display a Harnack inequality.

Lemma 2 (Harnack's inequality)**.** *Suppose u is harmonic in $B_R(x_0)$ and $u \geq 0$. Then,*

$$(\frac{R}{R+r})^{n-2}\frac{R-r}{R+r}u(x_0) \leq u(x) \leq (\frac{R}{R-r})^{n-2}\frac{R+r}{R-r}u(x_0), \tag{1.36}$$

where $r = |x - x_0| < R$.

Proof. Suppose without loss of generality that $x_0 = 0$. Then by the Poisson integral formula,

$$u(x) = \frac{1}{\omega_n R}\int_{\partial B_R}\frac{R^2 - |x|^2}{|x-y|^n}u(y)dS_y. \tag{1.37}$$

Since $R - |x| \leq |y - x| \leq R + |x|$, if $|y| = R$,

$$\frac{1}{\omega_n R}\cdot\frac{R-|x|}{R+|x|}(\frac{1}{R+|x|})^{n-2}\int_{\partial B_R}u(y)dS_y \leq u(x) \leq \frac{1}{\omega_n R}\cdot\frac{R+|x|}{R-|x|}(\frac{1}{R-|x|})^{n-2}\int_{\partial B_R}u(y)dS_y. \tag{1.38}$$

Then (1.36) follows from the mean value formula. □

Corollary 1. *If u is a harmonic function in $\mathbb{R}^n$ is bounded above or below, then u is a constant.*

We can also prove a result concerning the removable singularity.

Theorem 3. *Suppose u is harmonic on $B_R \setminus \{0\}$ and harmonic in B_R and satisfies $u(x) = o(\log|x|)$, $n = 2$, $o(|x|^{2-n})$, $n \geq 3$ as $|x| \to 0$. Then u can be defined so that it is C^2 and harmonic in B_R.*

Proof. Suppose u is continuous in $0 < |x| \leq R$. Let v solve $\Delta v = 0$ on B_R, $v = u$ on ∂B_R. Then set $w = v - u$ in $B_R \setminus \{0\}$ and let $M_r = \max_{\partial B_R}|w|$. Clearly, for $n \geq 3$,

$$|w(x)| \leq M_r \cdot \frac{r^{n-2}}{|x|^{n-2}}, \qquad \text{on} \qquad \partial B_r. \tag{1.39}$$

Then by the maximum principle, since $w = 0$ on ∂B_R,

$$|w(x)| \leq M_r \cdot \frac{r^{n-2}}{|x|^{n-2}}, \qquad \text{for any} \qquad x \in B_R \setminus B_r. \tag{1.40}$$

However, $M_r r^{n-2} \to 0$ as $r \searrow 0$, so $w = 0$ and $v = u$. □

2. The Dirichlet problem on a smooth domain

Having obtained the solution to the Dirichlet problem on a ball in $\mathbb{R}^n$, (1.13), the next goal is to obtain a solution on a general smooth domain Ω. Notice that, as in complex analysis, Proposition 3 can be generalized to smooth domains, which gives uniqueness.

Let $v(r)$ be a radial function that solves

$$v'' + \frac{n-1}{r}v' = 0. \tag{2.1}$$

Then let $w = v'$, $w = r^{-(n-1)}$, and

$$v(r) = c_1 + c_2 \log r, \qquad n = 2, \qquad v(r) = c_3 + c_4 r^{2-n}, \qquad n \geq 3. \tag{2.2}$$

Now choose c_2, c_4 such that

$$\int_{\partial B_r}\frac{\partial v}{\partial r}dS = 1, \qquad \text{for any} \qquad r > 0, \qquad c_2 = \frac{1}{2\pi}, \qquad c_4 = \frac{1}{(2-n)A_{n-1}}. \tag{2.3}$$

Then for any $a \in \mathbb{R}^n$, set

$$\Gamma(a,x) = \frac{1}{2\pi}\log|a-x|, \qquad n=2, \qquad \Gamma(a,x) = \frac{1}{(2-n)A_{n-1}}|a-x|^{2-n}, \qquad n \geq 3. \tag{2.4}$$

Thus, for a fixed $a \in \mathbb{R}^n$, $\Gamma(x,a)$ is harmonic at $x \neq a$, $\Delta_x \Gamma(a,x) = 0$ for any $x \neq a$, and

$$\int_{\partial B_r(a)} \frac{\partial \Gamma}{\partial n_x}(a,x)dS_x = 1, \qquad \text{for any} \qquad r > 0. \tag{2.5}$$

Theorem 4. *Suppose Ω is a bounded domain in $\mathbb{R}^n$ and that $u \in C^1(\bar{\Omega}) \cap C^2(\Omega)$. Then for any $a \in \Omega$ there holds*

$$u(a) = \int_\Omega \Gamma(a,x)\Delta u(x)dx - \int_{\partial\Omega}(\Gamma(a,x)\frac{\partial u}{\partial n_x}(x) - u(x)\frac{\partial \Gamma}{\partial n_x}(a,x))dS_x. \tag{2.6}$$

Proof. Apply Green's formula to u and $\Gamma(a,\cdot)$ in the domain $\Omega \setminus B_r(a)$ for $r > 0$ small. Then,

$$\int_{\Omega\setminus B_r(a)} (\Gamma\Delta u - u\Delta\Gamma)dx = \int_{\partial\Omega}(\Gamma\frac{\partial u}{\partial n} - u\frac{\partial\Gamma}{\partial n})dS_x - \int_{B_r(a)}(\Gamma\frac{\partial u}{\partial n} - u\frac{\partial\Gamma}{\partial n})dS_x. \tag{2.7}$$

Now then, $\Delta\Gamma = 0$ in $\Omega \setminus B_r(a)$, so

$$\int_{\Omega\setminus B_r(a)} \Gamma\Delta u dx = \int_{\partial\Omega}(\Gamma\frac{\partial u}{\partial n} - u\frac{\partial\Gamma}{\partial n})dS_x - \lim_{r\searrow 0}\int_{B_r(a)}(\Gamma\frac{\partial u}{\partial n} - u\frac{\partial\Gamma}{\partial n})dS_x. \tag{2.8}$$

Now then, by direct computation, since $u \in C^2(\Omega) \cap C^1(\bar{\Omega})$,

$$|\int_{\partial B_r(a)} \Gamma\frac{\partial u}{\partial n}dS| \to 0, \qquad \text{as} \qquad r \searrow 0, \tag{2.9}$$

and by (2.3),

$$\int_{\partial B_r(a)} u\frac{\partial\Gamma}{\partial n}dS = \frac{1}{A_{n-1}r^{n-1}}\int_{\partial B_r(a)} udS \to u(a), \qquad \text{as} \qquad r \searrow 0. \tag{2.10}$$

□

Now suppose Ω is a bounded domain in $\mathbb{R}^n$ and let $u \in C^2(\Omega) \cap C^1(\bar{\Omega})$. Then for $x \in \Omega$,

$$u(x) = \int_\Omega \Gamma(x,y)\Delta u(y)dy - \int_{\partial\Omega}(\Gamma(x,y)\frac{\partial u}{\partial n_y}(y) - u(y)\frac{\partial\Gamma}{\partial n_y}(x,y))dS_y. \tag{2.11}$$

A problem where both u and $\frac{\partial u}{\partial n}$ is known on the boundary is overdetermined, so suppose u solves the Dirichlet boundary problem,

$$\Delta u = f \qquad \text{in} \qquad \Omega, \qquad u = \varphi, \qquad \text{on} \qquad \partial\Omega. \tag{2.12}$$

Then,

$$u(x) = \int_\Omega \Gamma(x,y)f(y)dy - \int_{\partial\Omega}(\Gamma(x,y)\frac{\partial u}{\partial n_y}(y) - \varphi(y)\frac{\partial\Gamma}{\partial n_y}(x,y))dS_y. \tag{2.13}$$

Now then, for fixed $x \in \Omega$, consider

$$\gamma(x,y) = \Gamma(x,y) + \Phi(x,y), \tag{2.14}$$

for some $\Phi(x,\cdot) \in C^2(\bar{\Omega})$ and $\Delta_y\Phi(x,y) = 0$. Then redoing the proof of Theorem 4, for any $x \in \Omega$,

$$u(x) = \int_\Omega \gamma(x,y)\Delta u(y)dy - \int_{\partial\Omega}(\gamma(x,y)\frac{\partial u}{\partial n_y}(y) - u(y)\frac{\partial\gamma}{\partial n_y}(x,y))dS_y. \tag{2.15}$$

For a fixed $x \in \Omega$, choose $\Phi(x, \cdot) \in C^1(\bar{\Omega}) \cap C^2(\Omega)$ such that

$$\Delta_y \Phi(x,y) = 0, \qquad \text{for} \qquad y \in \Omega, \qquad \Phi(x,y) = -\Gamma(x,y), \qquad \text{for} \qquad y \in \partial\Omega. \tag{2.16}$$

Plugging this $\gamma(x,y)$, call it $G(x,y)$, into (2.15),

$$u(x) = \int_{\Omega} G(x,y) f(y) dy + \int_{\partial\Omega} \varphi(y) \frac{\partial G}{\partial n_y}(x,y) dS_y. \tag{2.17}$$

Remark 3. *The function $G(x,y)$ is called the Green's function.*

Remark 4. *The proof of the existence of $\Phi(x,y)$ satisfying* (2.16) *will be postponed to a later section. For now, we can observe that Proposition* 3 *implies uniqueness.*

Proposition 4. *Green's function $G(x,y)$ is symmetric in $\Omega \times \Omega$; that is, $G(x,y) = G(y,x)$ for $x \neq y \in \Omega$.*

Proof. Choose $x_1, x_2 \in \Omega$ with $x_1 \neq x_2$, and choose $r > 0$ suffficiently small such that $B_r(x_1) \cap B_r(x_2) = \emptyset$. Then set $G_1(y) = G(x_1, y)$ and $G_2(y) = G(x_2, y)$. Then,

$$\begin{aligned} \int_{\Omega \setminus B_r(x_1) \cup B_r(x_2)} (G_1 \Delta G_2 - G_2 \Delta G_1) &= \int_{\partial\Omega} (G_1 \frac{\partial G_2}{\partial n} - G_2 \frac{\partial G_1}{\partial n}) dS \\ - \int_{\partial B_r(x_1)} (G_1 \frac{\partial G_2}{\partial n} - G_2 \frac{\partial G_1}{\partial n}) dS &- \int_{\partial B_r(x_2)} (G_1 \frac{\partial G_2}{\partial n} - G_2 \frac{\partial G_1}{\partial n}) dS \\ = - \int_{\partial B_r(x_1)} (G_1 \frac{\partial G_2}{\partial n} - G_2 \frac{\partial G_1}{\partial n}) dS &- \int_{\partial B_r(x_2)} (G_1 \frac{\partial G_2}{\partial n} - G_2 \frac{\partial G_1}{\partial n}) dS. \end{aligned} \tag{2.18}$$

Since $x_1, x_2 \notin \partial\Omega$, $G(x_1, y) = G(x_2, y) = 0$ for $y \in \partial\Omega$.

Since G_i is harmonic for $y \neq x_i$, $\int_{\Omega \setminus B_r(x_1) \cup B_r(x_2)} (G_1 \Delta G_2 - G_2 \Delta G_1) = 0$, and therefore,

$$\int_{\partial B_r(x_1)} (G_1 \frac{\partial G_2}{\partial n} - G_2 \frac{\partial G_1}{\partial n}) dS + \int_{\partial B_r(x_2)} (G_1 \frac{\partial G_2}{\partial n} - G_2 \frac{\partial G_1}{\partial n}) dS = 0. \tag{2.19}$$

Now since $\Phi \in C^2(\bar{\Omega})$,

$$\int_{\partial B_r(x_1)} (\Gamma \frac{\partial G_2}{\partial n} - G_2 \frac{\partial \Gamma}{\partial n}) dS + \int_{\partial B_r(x_2)} (G_1 \frac{\partial \Gamma}{\partial n} - \Gamma \frac{\partial G_1}{\partial n}) dS \to 0, \qquad \text{as} \qquad r \searrow 0. \tag{2.20}$$

Taking the limit of (2.20) as $r \searrow 0$,

$$-G_2(x_1) + G_1(x_2) = 0, \qquad \text{which implies} \qquad G(x_2, x_1) = G(x_1, x_2). \tag{2.21}$$

□

Before dealing with the question of existence, it will be useful to use the Green's function to prove the Riemann mapping theorem.

Theorem 5. *Let Ω be a bounded domain in $\mathbb{C}$ with smooth boundary. Suppose Ω is connected and simply connected, which means that $\partial\Omega$ is diffeomorphic to the circle S^1. Let p be a point in Ω. There exists a holomorphic function Φ on Ω such that $\Phi(p) = 0$ and $\Phi : \bar{\Omega} \to \bar{\mathcal{D}}$ is a diffeomorphism, where $\mathcal{D} = \{z : |z| < 1\}$.*

Proof. Let $G(x,y)$ denote the Green's function in two dimensions for domain Ω and let $G_0(x,y)$ refer to $\Phi(x + iy, p)$ in (2.16),

$$G(x,y) = \log|x + iy - p| + G_0(x,y) = \log|z - p| + G_0(x,y). \tag{2.22}$$

Now let $H_0 \in C^\infty(\bar{\Omega})$ denote the harmonic conjugate of G_0,

$$H_0(z) = \int_p^z [-\frac{\partial G_0}{\partial y}dx + \frac{\partial G_0}{\partial x}dy]. \tag{2.23}$$

Since G_0 is harmonic, Green's theorem implies that the integral is independent of path. Indeed, plug in $u = 1$ and $\Gamma = G_0$ to (2.7). Furthermore,

$$\frac{\partial H_0}{\partial x} = -\frac{\partial G_0}{\partial y}, \qquad \frac{\partial H_0}{\partial y} = \frac{\partial G_0}{\partial x}, \tag{2.24}$$

so $G_0 + iH_0$ is holomorphic. Let

$$H(x,y) = Im \log(z-p) + H_0(x,y). \tag{2.25}$$

Let

$$\Phi(z) = e^{G+iH} = (z-p)e^{G_0+iH_0}, \tag{2.26}$$

which is a single–valued function on Ω with $\Phi(p) = 0$. Since a Green's function vanishes on the boundary,

$$\Phi : \partial\Omega \to S^1, \tag{2.27}$$

and therefore by the maximum modulus principle,

$$\Phi : \Omega \to \mathcal{D}. \tag{2.28}$$

In fact,

Theorem 6. *Φ is a holomorphic diffeomorphism of $\bar{\Omega}$ onto $\bar{\mathcal{D}}$.*

To show this, we must show that

$$\Phi : \bar{\Omega} \to \bar{\mathcal{D}}, \tag{2.29}$$

is one-to-one and onto, with nowhere vanishing derivative.

First, observe that the tangential derivative of H is nowhere vanishing on the boundary. This is equivalent to saying that

$$\frac{\partial G}{\partial \nu}(z) \neq 0, \qquad \text{for all} \qquad z \in \partial\Omega. \tag{2.30}$$

This follows from the fact that $G(z) \to -\infty$ as $z \to p$, $G(z)$ is maximal on $\partial\Omega$, G is harmonic on $\Omega \setminus \{p\}$, and Zaremba's principle. Therefore, (2.28) is a local diffeomorphism, and thus a covering map.

Remark 5. *By regularity theory, $G \in C^1(\bar{\Omega})$.*

Next, utilize the argument principle to show that (2.29) is one-to-one and onto.

Proposition 5 (Argument principle)**.** *Let $\Phi \in C^1(\bar{\Omega})$ be holomorphic inside Ω, where Ω is a bounded region in $\mathbb{C}$ with smooth boundary $\partial\Omega = \gamma$. Take $q \in \mathbb{C}$, not in the image of γ under Φ. The number of points $p_j \in \Omega$, counting multiplicity, for which $\Phi(p_j) = q$, is equal to the winding number of the curve $\Phi(\gamma)$ about q.*

Now then, for $q = 0$, it is clear from (2.26) that p is the unique, simple zero of Φ. Therefore, the map Φ is a simple diffeomorphism, and for any $q \in \mathcal{D}$, there is exactly one $w \in \Omega$ for which $\Phi(w) = q$. Therefore, $\Phi'(w) \neq 0$ for all $w \in \Omega$. □

Because of this fact, properties of harmonic functions imply a number of important results for holomorphic functions.

Lemma 3. *Suppose $u \in C(\bar{B}_R)$ is a nonnegative harmonic function in $B_R = B_R(x_0)$. Then*

$$|Du(x_0)| \leq \frac{n}{R} u(x_0). \tag{2.31}$$

Proof. Since $D_{x_i} u$ is a harmonic function, integrating by parts,

$$D_{x_i} u(x_0) = \frac{n}{\omega_n R^n} \int_{B_R(x_0)} D_{x_i} u(y) dy = \frac{n}{\omega_n R^n} \int_{\partial B_R(x_0)} u(y) \nu_i dS(y) \leq \frac{n}{R} u(x_0). \tag{2.32}$$

The last inequality uses the mean value formula and the fact that u is positive. □

Corollary 2. *A harmonic function bounded from above or below is constant.*

Proof. Suppose without loss of generality that u is bounded from below and that $u \geq 0$. Then by (2.32), $Du(x_0) = 0$. □

Proposition 6. *Suppose $u \in C(\bar{B}_R)$ is harmonic in $B_R = B_R(x_0)$. Then for any multiindex $|\alpha| = m$,*

$$|D^{\alpha} u(x_0)| \leq \frac{n^m e^{m-1} m!}{R^m} \max_{\bar{B}_R} |u|. \tag{2.33}$$

Proof. Prove by induction. When $m = 1$, the proposition follows from Lemma 3. Now suppose that the proposition holds for m, and prove for $m + 1$. For $0 < \theta < 1$, take $r = (1 - \theta)R$. Then by Lemma 3 and the induction assumption,

$$|D^{m+1} u(x_0)| \leq \frac{n}{r} \max_{\bar{B}_r} |D^m u| \leq \frac{n}{r} \frac{n^m e^{m-1} \cdot m!}{(R - r)^m} \max_{\bar{B}_R} |u| = \frac{n^{m+1} e^{m-1} m!}{R^{m+1} \theta^m (1 - \theta)} \max_{\bar{B}_R} |u|. \tag{2.34}$$

Now then, for $\theta = \frac{m}{m+1}$,

$$\frac{1}{\theta^m (1 - \theta)} = (1 + \frac{1}{m})^m (m + 1) < e(m + 1). \tag{2.35}$$

□

Theorem 7. *A harmonic function is analytic.*

Proof. Suppose u is a harmonic function in Ω. Fix $x \in \Omega$ and take $B_{2R}(x) \subset \Omega$ and $h \in \mathbb{R}^n$ with $|h| \leq R$. Then by Taylor expansion,

$$u(x + h) = u(x) + \sum_{i=1}^{m-1} \frac{1}{i!} [(h_1 \frac{\partial}{\partial x_1} + ... + h_n \frac{\partial}{\partial x_n})^i u](x) + R_m(h), \tag{2.36}$$

where

$$R_m(h) = \frac{1}{m!} [(h_1 \frac{\partial}{\partial x_1} + ... + h_n \frac{\partial}{\partial x_n})^m u](x_1 + \theta h_1, ..., x_n + \theta h_n), \qquad \text{for some} \qquad 0 < \theta < 1. \tag{2.37}$$

Since $x + h \in B_R(x)$,

$$|R_m(h)| \leq \frac{1}{m!} |h|^m n^m \cdot \frac{n^m e^{m-1} m!}{R^m} \max_{\bar{B}_{2R}} |u| \leq (\frac{|h| n^2 e}{R})^m \max_{\bar{B}_{2R}} |u|. \tag{2.38}$$

Then for $|h| n^2 e < \frac{R}{2}$, $R_m(h) \to 0$ as $m \to \infty$. □

We conclude with some properties of the Green's function.

Proposition 7. *For $x, y \in \Omega$ with $x \neq y$,*

(2.39)

$$0 > G(x,y) > \Gamma(x,y), \qquad for \qquad n \geq 3, \qquad 0 > G(x,y) > \Gamma(x,y) - \frac{1}{2\pi}\log diam(\Omega), \qquad for \qquad n = 2.$$

Proof. Fix $x \in \Omega$ and let $G(y) = G(x,y)$. Since $\lim_{y\to x} G(y) = -\infty$, there exists $r > 0$ such that $G(y) < 0$ in $B_r(x)$. Since G is harmonic in $\Omega \setminus B_r(x)$, $G = 0$ on $\partial\Omega$, and $G < 0$ on $\partial B_r(x)$, the maximum principle implies $G < 0$ in $\Omega \setminus B_r(x)$.

Now then, recall that $G(x,y) = \Gamma(x,y) + \Phi(x,y)$, where $\Delta\Phi = 0$ in Ω and $\Phi = -\Gamma$ on $\partial\Omega$. When $n \geq 3$, $\Gamma(x,y) = \frac{1}{(2-n)\omega_n}|x-y|^{2-n} < 0$ for $y \in \partial\Omega$, so by the maximum principle, $\Phi(x,\cdot) > 0$ on $\partial\Omega$. By the maximum principle, $\Phi > 0$ in Ω. When $n = 2$,

$$\Gamma(x,y) = \frac{1}{2\pi}\log|x-y| \leq \frac{1}{2\pi}\log diam(\Omega), \qquad \text{for} \qquad y \in \partial\Omega. \tag{2.40}$$

Therefore, the maximum principle implies $\Phi > -\frac{1}{2\pi}\log diam(\Omega)$ in Ω. □

3. Sobolev spaces

To prove existence and uniqueness of solutions, it is necessary to first identify the appropriate function space in which to work. Sobolev spaces are frequently the space that is useful.

When $k \geq 0$ is an integer, the Sobolev space $H^k(\mathbb{R}^n)$ is defined as follows,

Definition 2 (Sobolev space)**.**

$$H^k(\mathbb{R}^n) = \{u \in L^2(\mathbb{R}^n) : D^\alpha u \in L^2(\mathbb{R}^n), \qquad for \qquad |\alpha| \leq k\}, \tag{3.1}$$

where $D^\alpha = i^{-|\alpha|}\partial_{x_1}^{\alpha_1}\partial_{x_2}^{\alpha_2}\cdots\partial_{x_n}^{\alpha_n}, \qquad \alpha = (\alpha_1, ..., \alpha_n).$

Integrating by parts,

$$(2\pi)^{-n/2}\int e^{-ix\cdot\xi}\frac{1}{i}\partial_{x_1}f(x)dx = i(2\pi)^{-n/2}\int \partial_{x_1}(e^{-ix\cdot\xi})f(x)dx = \xi_1\hat{f}(\xi). \tag{3.2}$$

By Plancherel's theorem,

$$\|f\|_{L^2(\mathbb{R}^n)} = \|\hat{f}\|_{L^2(\mathbb{R}^n)}, \tag{3.3}$$

Therefore,

$$u \in H^k(\mathbb{R}^n) \Leftrightarrow \langle\xi\rangle^k\hat{u} \in L^2(\mathbb{R}^n), \qquad \langle\xi\rangle = (1+|\xi|^2)^{1/2}. \tag{3.4}$$

This definition can be extended to s when s need not be an integer,

$$H^s(\mathbb{R}^n) = \{u \in \mathcal{S}'(\mathbb{R}^n) : \langle\xi\rangle^s\hat{u} \in L^2(\mathbb{R}^n)\}. \tag{3.5}$$

Then,

$$u \in H^{s+1}(\mathbb{R}^n) \Leftrightarrow D_j u \in H^s(\mathbb{R}^n), \qquad \forall j. \tag{3.6}$$

For any $y \in \mathbb{R}^n$ let

$$\tau_y u(x) = u(x+y). \tag{3.7}$$

Proposition 8. *Let $(e_1, ..., e_n)$ be the standard basis of $\mathbb{R}^n$ and let $u \in H^s(\mathbb{R}^n)$. Then,*

$$\sigma^{-1}(\tau_{\sigma e_j}u - u) \tag{3.8}$$

is bounded in $H^s(\mathbb{R}^n)$ for $0 < \sigma \leq 1$ if and only if $D_j u \in H^s(\mathbb{R}^n)$.

Proof. If $u \in H^s(\mathbb{R}^n)$ then $\sigma^{-1}(\tau_{\sigma e_j} u - u)$ converges to $iD_j u$ in $H^{s-1}(\mathbb{R}^n)$. Since (3.8) is bounded for any compact subset of $(0,1]$, $D_j u \in H^s$ implies that (3.8) is bounded in H^s.

Meanwhile, if $\sigma^{-1}(\tau_{\sigma e_i} u - u)$ is bounded in H^s, then there exists a sequence $\sigma_\nu \searrow 0$ such that

$$\sigma_\nu^{-1}(\tau_{\sigma_\nu e_j} u - u), \tag{3.9}$$

converges weakly to an element $w \in H^s(\mathbb{R}^n)$. Therefore, $w = iD_j u$. Since $w \in H^s(\mathbb{R}^n)$, the proof is complete. □

Corollary 3. *Given $u \in H^s(\mathbb{R}^n)$, $u \in H^{s+1}(\mathbb{R}^n)$ if and only if $\tau_y u$ is a Lipschitz continuous function of y with values in $H^s(\mathbb{R}^n)$.*

Proof. $\tau_y u$ is Lipschitz continuous with values in $H^s(\mathbb{R}^n)$ if and only if (3.8) holds. □

Proposition 9. *If $s > n/2$ then each $u \in H^s(\mathbb{R}^n)$ is bounded and continuous. If $u \in H^s$ for some $s > n/2 + k$, then $u \in C^k(\mathbb{R}^n)$.*

Proof. Using Cauchy's inequality, if $s > n/2$,

$$\int |\hat{u}(\xi)| d\xi \leq (\int |\hat{u}(\xi)|^2 \langle \xi \rangle^{2s} d\xi)^{1/2} \cdot (\int \langle \xi \rangle^{-2s} d\xi)^{1/2} \leq C(s) \|u\|_{H^s}. \tag{3.10}$$

Equation (3.6) implies $u \in C^k$ if $s > n/2 + k$. □

If $s = n/2 + \alpha$ for some $0 < \alpha < 1$, we can establish Hölder continuity.

Proposition 10. *If $s = n/2 + \alpha$, $0 < \alpha < 1$, then $H^s(\mathbb{R}^n) \subset C^\alpha(\mathbb{R}^n)$.*

Proof. Using Cauchy's inequality,

$$\begin{aligned} |u(x+y) - u(x)| &= (2\pi)^{-n/2} | \int \hat{u}(\xi) e^{ix \cdot \xi} (e^{iy \cdot \xi} - 1) d\xi | \\ &\leq C(\int |\hat{u}(\xi)|^2 \langle \xi \rangle^{n+2\alpha} d\xi)^{1/2} (\int |e^{iy \cdot \xi} - 1|^2 \langle \xi \rangle^{-n-2\alpha} d\xi)^{1/2}. \end{aligned} \tag{3.11}$$

For $|y| \leq 1/2$,

$$\int |e^{iy \cdot \xi} - 1|^2 \langle \xi \rangle^{-n-2\alpha} d\xi \leq C \int_{|\xi| \leq \frac{1}{|y|}} |y|^2 |\xi|^2 \langle \xi \rangle^{-n-2\alpha} d\xi + 4 \int_{|\xi| \geq \frac{1}{|y|}} \langle \xi \rangle^{-n-2\alpha} d\xi \leq C(\alpha) |y|^{2\alpha}. \tag{3.12}$$

□

Now consider the trace map $\tau : \mathcal{S}(\mathbb{R}^n) \to \mathcal{S}(\mathbb{R}^{n-1})$ given by $\tau u = f$, where $f(x') = u(0, x')$ if $x = (x_1, ..., x_n)$ and $x' = (x_2, ..., x_n)$.

Proposition 11. *The map τ extends uniquely to a continuous linear map,*

$$\tau : H^s(\mathbb{R}^n) \to H^{s-1/2}(\mathbb{R}^{n-1}), \qquad \text{for} \qquad s > \frac{1}{2}. \tag{3.13}$$

Proof. If $f = \tau u$, where $u \in \mathcal{S}(\mathbb{R}^{n-1})$, then

$$\hat{f}(\xi') = (2\pi)^{-1/2} \int \hat{u}(\xi) d\xi_1. \tag{3.14}$$

Therefore, if $s > 1/2$,

$$|\hat{f}(\xi')|^2 \leq (\int |\hat{u}(\xi)|^2 \langle \xi \rangle^{2s} d\xi_1) \cdot (\int \langle \xi \rangle^{-2s} d\xi_1). \tag{3.15}$$

Then,

$$\int \langle\xi\rangle^{-2s} d\xi_1 = \int (1+|\xi'|^2+\xi_1^2)^{-s} d\xi_1 = C(1+|\xi'|^2)^{-s+1/2} = C\langle\xi'\rangle^{-2(s-1/2)}. \tag{3.16}$$

Therefore,

$$\langle\xi'\rangle^{2(s-1/2)}|\hat{f}(\xi')|^2 \leq C\int |\hat{u}(\xi)|^2\langle\xi\rangle^{2s} d\xi_1. \tag{3.17}$$

Therefore,

$$\|f\|^2_{H^{s-1/2}(\mathbb{R}^{n-1})} \leq C\|u\|^2_{H^s(\mathbb{R}^n)}. \tag{3.18}$$

□

Proposition 12. *The map* (3.13) *is surjective for each* $s > 1/2$.

Proof. If $g \in H^{s-1/2}(\mathbb{R}^{n-1})$, let

$$\hat{u}(\xi) = \hat{g}(\xi')\frac{\langle\xi'\rangle^{2(s-1/2)}}{\langle\xi\rangle^{2s}}. \tag{3.19}$$

It is clear that $u \in H^s(\mathbb{R}^n)$ and that $u(0,x') = cg(x')$. □

Now let Ω be a bounded domain, start with $\mathbb{R}^n_+ = \{x \in \mathbb{R}^n : x_1 > 0\}$. For any $k \geq 0$, let

$$H^k(\mathbb{R}^n_+) = \{u \in L^2(\mathbb{R}^n_+) : D^\alpha u \in L^2(\mathbb{R}^n_+) \quad \text{for} \quad |\alpha| \leq k\}. \tag{3.20}$$

Next, define the extension operator,

$$Eu(x) = u(x), \quad \text{for} \quad x_1 \geq 0, \quad \sum_{j=1}^N a_j u(-jx_1, x'), \quad \text{for} \quad x_1 < 0. \tag{3.21}$$

Lemma 4. *It is possible to choose* $\{a_1, ..., a_N\}$ *such that the map* E *has a unique continuous extension to*

$$E : H^k(\mathbb{R}^n_+) \to H^k(\mathbb{R}^n), \quad \textit{for} \quad k \leq N-1. \tag{3.22}$$

Proof. This was proved in Lemma 4.1 in chapter four of [Tay96]. □

Now then, suppose $\partial\Omega$ is a smooth, compact manifold on which Sobolev spaces have been defined. By using local coordinate systems, flattening out $\partial\Omega$, combined with the extension map and the trace theorem, we have the following result on the trace map,

Proposition 13. *For* $s > 1/2$, τ *extends uniquely to a continuous map*

$$\tau : H^s(\Omega) \to H^{s-1/2}(\partial\Omega). \tag{3.23}$$

Now let $H^k_0(\Omega)$ be the closure of $C^\infty_0(\Omega)$ in $H^s(\Omega)$. Then,

$$H^k_0(\Omega) = \{u \in H^k : supp(u) \subset \bar{\Omega}\}. \tag{3.24}$$

Proposition 14. *For* Ω *open in* $\mathcal{M}$ *with smooth boundary,* $k \geq 0$ *an integer, we have a natural isomorphism*

$$H^k_0(\Omega)^* \approx H^{-k}(\Omega). \tag{3.25}$$

Proof. Let P be a differential operator of order $2k$ with smooth coefficients on $\bar{\Omega}$. Suppose

$$P = \sum_{j=1}^{L} A_j B_j, \tag{3.26}$$

where A_j and B_j are differential operators of order k with smooth coefficients on $\bar{\Omega}$. Now take the inner product for $u, v \in H_0^k(\Omega)$,

$$\langle u, Pv \rangle = \sum_{j=1}^{L} \langle A_j^* u, B_j v \rangle_{L^2(\Omega)}. \tag{3.27}$$

This dual pairing gives

$$P : H^s(\Omega) \to H^{s-2k}(\Omega), \tag{3.28}$$

for any $s \in \mathbb{R}$. To show that any $\psi \in H^{-k}$ can be written in the form Pu for some $u \in H_0^k(\Omega)$ is the topic of the next section. □

4. Existence and regularity of solutions to the Dirichlet problem

Now turn to the question of whether or not equation (2.16) even has a solution. More generally, does the Dirichlet problem,

$$\Delta u = 0, \qquad u|_{\partial\Omega} = f, \tag{4.1}$$

have a solution for Ω compact, smoothly bounded, and $f \in C^\infty(\partial\Omega)$. For $u \in C_0^\infty(\Omega)$, integrating by parts,

$$(-\Delta u, u) = \|\nabla u\|_{L^2(\Omega)}^2, \tag{4.2}$$

where $(\cdot, \cdot)$ is the usual inner product, $(f, g) = \int_\Omega f(x) g(x) dx$. Furthermore, if Ω has a nonempty boundary,

$$\|u\|_{L^2(\Omega)}^2 \leq C(\Omega) \|\nabla u\|_{L^2(\Omega)}^2, \qquad u \in C_0^\infty(\Omega). \tag{4.3}$$

Now define the Sobolev space,

Definition 3.

$$\|\nabla u\|_{H^1(\Omega)}^2 = \|u\|_{L^2(\Omega)}^2 + \|\nabla u\|_{L^2(\Omega)}^2, \tag{4.4}$$

and let $H_0^1(\Omega)$ *denote the closure of* $C_0^\infty(\Omega)$ *in* $H^1(\Omega)$. *Let* $H^{-1}(\Omega)$ *denote the dual of* $H_0^1(\Omega)$.

Then (4.3) implies

$$\|\nabla u\|_{L^2(\Omega)}^2 \approx \|u\|_{H^1(\Omega)}^2, \qquad \text{for} \qquad u \in H_0^1(\Omega). \tag{4.5}$$

Furthermore, note that

$$\Delta : H_0^1(\Omega) \to H^{-1}(\Omega), \tag{4.6}$$

is well-defined, by the Riesz representation theorem.

Proposition 15. *The map in* (4.6) *is one-to-one and onto.*

Proof. First observe that by (4.2) and (4.5),

$$\|\Delta u\|_{H^{-1}(\Omega)} \geq C\|u\|_{H^1(\Omega)}, \tag{4.7}$$

for some $C > 0$, which implies that (4.6) is one-to-one. If (4.6) is not surjective, then there must be an element of $(H^{-1}(\Omega))^* = H_0^1(\Omega)$ that is orthogonal to the range. Then there exists $u_0 \in H_0^1(\Omega)$ such that

$$(-\Delta u, u_0) = 0, \qquad \text{for all} \qquad u \in H_0^1(\Omega). \tag{4.8}$$

Taking $u = u_0$ in (4.8) implies that by (4.5), $u = 0$. □

Thus there is a uniquely determined inverse

$$T : H^{-1}(\Omega) \to H_0^1(\Omega). \tag{4.9}$$

Proposition 16. *The inverse T to Δ in (4.6) is a compact negative self adjoint operator on $L^2(\Omega)$.*

Proof. If $\varphi = \Delta u$, $\psi = \Delta v$, with $u, v \in H_0^1(\Omega)$, then

$$(T\varphi, \psi) = (T\Delta u, \Delta v) = (u, \Delta v) = -(\nabla u, \nabla v) = (\Delta u, v) = (\varphi, T\psi). \tag{4.10}$$

Indeed, (4.2) extends to

$$(-\Delta u, v) = (\nabla u, \nabla v), \qquad \text{for} \qquad u, v \in H_0^1(\Omega). \tag{4.11}$$

Therefore, restricting T to $L^2(\Omega)$,

$$T = T^*, \tag{4.12}$$

so T is self-adjoint. Since $T : L^2(\Omega) \to H_0^1(\Omega)$, then by Rellich's theorem, T is compact on $L^2(\Omega)$. □

Since T is a compact operator, the spectral theorem implies that there exists an orthonormal basis $\{u_j\}$ of $L^2(\Omega)$ consisting of eigenfunctions of T,

$$Tu_j = -\mu_j u_j, \qquad \mu_j \searrow 0. \tag{4.13}$$

Then by (4.9),

$$u_j \in H_0^1(\Omega), \qquad \text{for each} \qquad j. \tag{4.14}$$

Moreover,

$$\Delta u_j = -\lambda_j u_j, \qquad \lambda_j = \frac{1}{\mu_j} \nearrow \infty. \tag{4.15}$$

Now consider a more general operator of the form

$$Lu = -\Delta u + Xu, \tag{4.16}$$

where X is a first-order differential operator with smooth coefficients on $\bar{\Omega}$.

Theorem 8. *Given $f \in H^{k-1}(\Omega)$ for $k = 0, 1, 2, ...,$ a solution $u \in H_0^1(\Omega)$ to*

$$Lu = f, \qquad u \in H_0^1(\Omega), \tag{4.17}$$

belongs to $H^{k+1}(\Omega)$ and we have the estimate

$$\|u\|^2_{H^{k+1}} \leq C\|Lu\|^2_{H^{k-1}} + C\|u\|^2_{H^k}, \tag{4.18}$$

for all $u \in H^{k+1}(\Omega) \cap H_0^1(\Omega)$. Here,

$$H^k(\Omega) = \{u : D^\alpha u \in L^2(\Omega), \qquad \text{for all} \qquad |\alpha| \leq k\}. \tag{4.19}$$

A corollary of Theorem 8 implies

Corollary 4. *The eigenfunctions u_j of Δ belong to $C^{\infty}(\bar{\Omega})$.*

Apply Theorem 8 to the boundary value problem

$$\Delta u = 0, \qquad \text{on} \qquad \Omega, \qquad u|_{\partial\Omega} = f, \tag{4.20}$$

where

$$f \in C^{\infty}(\partial\Omega), \tag{4.21}$$

is given. Then construct $F \in C^{\infty}(\bar{\Omega})$ so that $F|_{\partial\Omega} = f$. Then (4.20) is equivalent to

$$u = F + v, \tag{4.22}$$

where

$$\Delta v = g = -\Delta F, \qquad v|_{\partial\Omega} = 0. \tag{4.23}$$

Since $g \in C^{\infty}(\bar{\Omega})$,

$$v = Tg \in H_0^1(\Omega), \tag{4.24}$$

satisfies (4.23) and then by Theorem 8, $v \in C^{\infty}(\bar{\Omega})$. Thus, for any $f \in C^{\infty}(\partial\Omega)$ we have $u \in C^{\infty}(\bar{\Omega})$ solving (4.20), assuming that each connected component of $\partial\Omega$ has nonempty boundary.

Proof of Theorem 8. First prove (4.18) with $k = 0$. By Hölder's inequality, for any $\epsilon > 0$,

$$|(Xu, u)| \leq C\|u\|_{H^1}\|u\|_{L^2} \leq \frac{C}{2}[\epsilon\|u\|_{H^1}^2 + \frac{1}{\epsilon}\|u\|_{L^2}^2]. \tag{4.25}$$

By (4.2), (4.5), and (4.25), for $u \in H_0^1(\Omega)$,

$$Re(Lu, u) \geq C\|u\|_{H^1}^2 - C'\|u\|_{L^2}^2. \tag{4.26}$$

Therefore,

$$\|u\|_{H^1}^2 \leq CRe(Lu, u) + C'\|u\|_{L^2}^2 \leq C\|Lu\|_{H^{-1}}\|u\|_{H^1} + C'\|u\|_{L^2}^2 \leq C\epsilon\|u\|_{H^1}^2 + \frac{C}{\epsilon}\|Lu\|_{H^{-1}}^2 + C'\|u\|_{L^2}^2. \tag{4.27}$$

Taking $\epsilon > 0$ sufficiently small,

$$\|u\|_{H^1}^2 \leq C\|Lu\|_{H^{-1}}^2 + C\|u\|_{L^2}^2, \qquad u \in H_0^1(\Omega). \tag{4.28}$$

Now prove Theorem 8 by induction on k. Suppose it is true that

$$u \in H_0^1(\Omega), Lu = f \in H^{k-1}(\Omega) \Rightarrow u \in H^{k+1}(\Omega), \tag{4.29}$$

and that (4.18) is true. Also suppose that

$$u \in H_0^1(\Omega), \qquad Lu \in H^k(\Omega). \tag{4.30}$$

Since we know that $u \in H^{k+1}(\Omega)$, we want to show $u \in H^{k+2}(\Omega)$ and that (4.18) holds with k replaced by $k + 1$.

Now for any $\chi \in C^{\infty}(\bar{\Omega})$,

$$L(\chi u) = \chi(Lu) + [L, \chi]u. \tag{4.31}$$

The commutator $[L, \chi]$ is a first order differential operator, by (4.29), along with $u \in H^{k+1}(\Omega)$, implies $L(\chi u) \in H^k(\Omega)$. Therefore, the analysis of u on $\bar{\Omega}$ can be localized. Now suppose $u \in$

$H^{k+1}(\Omega)$ satisfying (4.30) is supported on a coordinate neighborhood $\mathcal{O}$, either with no boundary, or a boundary given by $\{x_n = 0\}$. Applying (4.18) to

$$D_{j,h}u(x) = \frac{1}{h}[u(x+he_j) - u(x)], \tag{4.32}$$

where $e_1, ..., e_n$ are the standard coordinate vectors in $\mathbb{R}^n$. If $\mathcal{O}$ has no boundary, we can take $1 \le j \le n$, otherwise $1 \le j \le n-1$. Then by (4.18),

(4.33)
$$\|D_{j,h}u\|^2_{H^{k+1}} \le C\|LD_{j,h}u\|^2_{H^{k-1}} + C\|u\|^2_{H^{k+1}} \le C\|D_{j,h}Lu\|^2_{H^{k-1}} + C\|[L, D_{j,h}]u\|^2_{H^{k-1}} + C\|u\|^2_{H^{k+1}}.$$

Lemma 5. *As $h \searrow 0$, $[L, D_{j,h}]$ is a bounded family of operators of order two,*

$$\|[L, D_{j,h}]u\|_{H^{k-1}} \le C\|u\|_{H^{k+1}}, \qquad k \ge 0, \tag{4.34}$$

given $u \in H^1_0(\Omega) \cap H^{k+1}(\Omega)$ supported in $\mathcal{O}$.

Then

$$\|D_{j,h}u\|^2_{H^{k+1}} \le C\|Lu\|^2_{H^k} + C\|u\|^2_{H^{k+1}}, \tag{4.35}$$

and passing to the limit $h \searrow 0$ gives

$$D_j u \in H^{k+1}(\Omega). \tag{4.36}$$

If $\mathcal{O}$ has no boundary then we are done. If $\mathcal{O}$ has a boundary then we have (4.36) for $1 \le j \le n-1$. Therefore it remains to establish

$$D_n u \in H^{k+1}(\Omega). \tag{4.37}$$

Since $k \ge 0$, we need to show

$$D_j D_n u \in H^k(\Omega), \qquad 1 \le j \le n. \tag{4.38}$$

If $1 \le j \le n-1$, $D_j D_n u = D_n D_j u$ which gives (4.36). Finally,

$$D_n^2 u = -\sum_{j=1}^{n-1} D_j^2 u. \tag{4.39}$$

All the terms on the right hand side have been shown to be in $H^k(\Omega)$, so the proof is complete for the case when the boundary is given by $x_n > 0$.

For a general boundary $x_n > \psi(x_1, ..., x_{n-1})$, make the transformation

$$\begin{aligned} \Phi &: (x_1, ..., x_n) \mapsto (x_1, ..., x_{n-1}, x_n + \psi(x_1, ..., x_{n-1})), \\ \Psi &: (x_1, ..., x_n) \mapsto (x_1, ..., x_{n-1}, x_n - \psi(x_1, ..., x_{n-1})). \end{aligned} \tag{4.40}$$

If $v(y) = u(\Psi(y))$,

$$\tilde{L}v = f(\Psi(y)), \tag{4.41}$$

where

$$\tilde{L} = \Delta - 2\sum_{i=1}^{n-1} \psi_{x_i}\partial_i\partial_n - (\sum_{i=1}^{n} \psi^2_{x_i})\partial_{nn} + X, \tag{4.42}$$

where X is an operator of first order. In a local coordinate patch, $|\nabla\psi| \lesssim \epsilon$, so the computations in (4.1)–(4.12) also hold for $\tilde{L}$. $\square$

Remark 6. *It is possible to generalize the above argument proving Proposition* 16.

Let L denote the operator

$$Lu = -\partial_j(a_{ij}(x)\partial_i u) + b_i(x)\partial_i u + c(x)u, \tag{4.43}$$

and let a be the bilinear form associated with the operator L,

$$a(u,v) = \int_\Omega (a_{ij}D_i u D_j v + b_i D_i u v + cuv)dx, \qquad u, v \in H_0^1(\Omega). \tag{4.44}$$

If $a_{ij} = a_{ji}$ and $b_i = 0$, then a is symmetric,

$$a(u,v) = a(v,u), \qquad \forall u, v \in H_0^1(\Omega). \tag{4.45}$$

Compare (4.45) to the $H_0^1(\Omega)$ inner product defined by

$$(u,v)_{H_0^1(\Omega)} = \int_\Omega \nabla u \cdot \nabla v dx. \tag{4.46}$$

Theorem 9 (Lax-Milgram theorem)**.** *Let a_{ij}, b_i, and c be bounded functions in Ω and $f \in L^2(\Omega)$. Suppose that the bilinear form a is coercive,*

$$a(u,u) \geq c_0\|u\|^2_{H_0^1(\Omega)}, \qquad \forall u \in H_0^1(\Omega). \tag{4.47}$$

Then there exists a unique weak solution $u \in H_0^1(\Omega)$ of $Lu = f$.

Proof. Define a linear functional F on $H_0^1(\Omega)$ by

$$F(\varphi) = \int_\Omega f\varphi dx, \qquad \forall \varphi \in H_0^1(\Omega). \tag{4.48}$$

Then,

$$|F(\varphi)| \leq \|f\|_{L^2}\|\varphi\|_{L^2} \leq C\|f\|_{L^2}\|\varphi\|_{H_0^1}, \tag{4.49}$$

so F is a bounded linear functional on H_0^1. Now suppose $a_{ij} = a_{ji}$, $b_i = 0$, and $c \geq 0$. Then $a(u,v)$ is an inner product on $H_0^1(\Omega)$ that is equivalent to the standard $H_0^1(\Omega)$ inner product. Then by the Riesz representation theorem, for any $f \in L^2(\Omega)$ there is a unique u such that $a(u,\varphi) = F(\varphi)$ for all $\varphi \in H_0^1(\Omega)$. □

Theorem 10 (Riesz representation theorem)**.** *Let H be a Hilbert space whose inner product is $\langle x, y\rangle$. For every linear functional $\varphi \in H^*$, there exists a unique vector $f_\varphi \in H$ such that $\varphi(x) = \langle x, f_\varphi\rangle$ for all $x \in H$.*

It is possible to say the same when a_{ij} is not symmetric.

Theorem 11. *Let a_{ij}, b_i, and c be bounded functions on Ω and $f \in L^2(\Omega)$. Then there exists a $\mu_0(a_{ij}, b_i, c)$ such that, for $\mu \geq \mu_0$, there exists a unique weak solution $u \in H_0^1(\Omega)$ of $(L+\mu)u = f$. Here,*

$$\sum_{i,j=1}^n a_{ij}(x)\xi_i\xi_j \geq c_0|\xi|^2, \tag{4.50}$$

for some $c > 0$.

Proof. In this case, a need not be symmetric.

$$|a(u,v)| \leq C\|u\|_{H_0^1(\Omega)}\|v\|_{H_0^1(\Omega)}, \qquad \forall u, v \in H_0^1(\Omega). \tag{4.51}$$

Then for any fixed $u \in H_0^1(\Omega)$, the mapping $v \mapsto a(u,v)$ is a bounded linear functional on $H_0^1(\Omega)$. Then by the Riesz representation theorem, there exists a unique $w \in H_0^1(\Omega)$ such that

$$a(u,v) = (w,v)_{H_0^1(\Omega)}, \qquad \forall v \in H_0^1(\Omega). \tag{4.52}$$

Then set $w = Au$,

$$a(u,v) = (Au,v)_{H_0^1(\Omega)}, \qquad \forall u, v \in H_0^1(\Omega). \tag{4.53}$$

Then A is a bounded linear operator on $H_0^1(\Omega)$. Indeed,

$$\begin{gathered} \|Au\|_{H_0^1}^2 = (Au, Au)_{H_0^1} = a(u, Au) \le C\|u\|_{H_0^1}\|Au\|_{H_0^1}, \\ \text{which implies} \qquad \|Au\|_{H_0^1} \le C\|u\|_{H_0^1}, \qquad \forall u \in H_0^1(\Omega). \end{gathered} \tag{4.54}$$

By coerciveness,

$$c_0\|u\|_{H_0^1(\Omega)}^2 \le a(u,u) = (Au,u)_{H_0^1(\Omega)} \le \|u\|_{H_0^1(\Omega)}\|Au\|_{H_0^1(\Omega)}, \tag{4.55}$$

and therefore,

$$c_0\|u\|_{H_0^1(\Omega)} \le \|Au\|_{H_0^1(\Omega)}, \qquad \forall u \in H_0^1(\Omega). \tag{4.56}$$

Therefore, A is one-to-one and the range of A is closed in $H_0^1(\Omega)$.

Next, for $w \in R(A)^\perp$,

$$c_0\|w\|_{H_0^1}^2 \le a(w,w) = (Aw,w)_{H_0^1} = 0. \tag{4.57}$$

Therefore, $w = 0$, so $R(A)^\perp = \{0\}$. Therefore, $R(A) = H_0^1(\Omega)$, so A is onto.

For the bounded linear functional F in $H_0^1(\Omega)$, by the Riesz representation theorem, there exists $w \in H_0^1(\Omega)$ such that

$$(w,v)_{H_0^1(\Omega)} = F(v), \qquad \forall v \in H_0^1(\Omega). \tag{4.58}$$

Since A is onto, there exists $u \in H_0^1(\Omega)$ such that $Au = w$. Therefore, for any $v \in H_0^1(\Omega)$,

$$a(u,v) = (Au,v)_{H_0^1(\Omega)} = (w,v)_{H_0^1(\Omega)} = F(v). \tag{4.59}$$

This proves existence.

Now assume $\tilde{u} \in H_0^1(\Omega)$ also satisfies $a(\tilde{u},v) = F(v)$, for any $v \in H_0^1(\Omega)$. Then, $a(u-\tilde{u},v) = 0$ for any $v \in H_0^1(\Omega)$. So for $v = u - \tilde{u}$, $a(u-\tilde{u}, u-\tilde{u}) = 0$, so by coerciveness, $u = \tilde{u}$.

For the general operator

$$L = a_{ij}\partial_i\partial_j + b_i\partial_i + c, \tag{4.60}$$

for $\mu > 0$ sufficiently large,

$$a_\mu(u,v) = a(u,v) + \mu(u,v)_{L^2}. \tag{4.61}$$

Then, a_μ is coercive for μ sufficiently large. Applying the Lax–Milgram theorem to $L + \mu$ proves the theorem. □

5. The weak and strong maximum principles

Having dealt with the question of existence and uniqueness, we now prove some weak and strong maximum principles. The weak and strong maximum principle was already utilized, namely Zaremba's principle in (2.30). Let

$$L = \Delta + X, \tag{5.1}$$

where X is a real vector field. If L is a Laplacian on a manifold $\mathcal{M}$, L has the form

$$L = g^{jk}(x)\partial_j \partial_k + b^j(x)\partial_j, \tag{5.2}$$

where $(g^{jk}(x))$ is a positive–definite matrix and $b^j(x)$ is smooth and real–valued.

Theorem 12. *Suppose Ω is an open bounded domain in $\mathbb{R}^n$ and L is given by (5.2) with coefficients smooth on a neighborhood of $\bar{\Omega}$. If $u \in C(\bar{\Omega}) \cap C^2(\Omega)$ and*

$$Lu \geq 0, \qquad on \qquad \Omega, \tag{5.3}$$

then

$$\sup_{x \in \Omega} u(x) = \sup_{y \in \partial\Omega} u(y). \tag{5.4}$$

Furthermore, if

$$Lu = 0, \qquad on \qquad \Omega, \tag{5.5}$$

then also

$$\sup_{x \in \Omega} |u(x)| = \sup_{y \in \partial\Omega} |u(y)|. \tag{5.6}$$

Proof. First note that if $Lu > 0$ then u cannot attain an interior maximum at $x_0 \in \Omega$, since in that case, $\nabla u(x_0) = 0$ and $g^{jk}(x_0)\partial_j \partial_k u(x_0) \leq 0$.

For $Lu \geq 0$, observe that for any $\Omega \subset\subset \mathbb{R}^n$,

$$L(e^{\gamma x_1}) = (\gamma^2 g^{11}(x) + \gamma b^1(x))e^{\gamma x_1} > 0, \tag{5.7}$$

for $\gamma > 0$ sufficiently large. Then, for any $\epsilon > 0$, let $u_\epsilon(x) = u(x) + \epsilon e^{\gamma x_1}$, so

$$\sup_{x \in \Omega} u(x) + \epsilon e^{\gamma x_1} = \sup_{y \in \partial\Omega} u(y) + \epsilon e^{\gamma y_1}. \tag{5.8}$$

Taking $\epsilon \searrow 0$ proves (5.4). If $Lu = 0$ then replace u by $-u$ and do the same calculation, giving (5.6). □

Now we can prove Zaremba's principle. This is also sometimes called the Hopf lemma.

Proposition 17 (Zaremba's principle)**.** *Suppose that, in addition the hypotheses above, $\partial\Omega$ is smooth and $u \in C^1(\bar{\Omega}) \cap C^2(\Omega)$. If $Lu \geq 0$ and $y \in \partial\Omega$ is a point such that*

$$u(y) > u(x), \qquad for\ all \qquad x \in \Omega, \tag{5.9}$$

then if ν is the inward pointing normal to $\partial\Omega$,

$$\frac{\partial u}{\partial \nu}(y) > 0. \tag{5.10}$$

Proof. Let $\mathcal{O}$ be a small ball in Ω whose boundary is tangent to $\partial\Omega$ at y. Let p be the center of $\mathcal{O}$ and let R denote the radius of $\mathcal{O}$. Then for $x \in \mathcal{O}$, let

$$r(x) = |x - p|^2. \tag{5.11}$$

If $h(x) = e^{-\alpha r(x)^2} - e^{-\alpha R^2}$, then by direct calculation,

$$Lh = e^{-\alpha r(x)^2}\{4\alpha^2 \sum_{i,j=1}^{n} g^{ij}(x)(x_i - p_i)(x_j - p_j) - 2\alpha \sum_{i=1}^{n} g^{ii}(x) - 2\alpha \sum_{i=1}^{n} b^i(x)(x_i - p_i)\}. \tag{5.12}$$

Fixing $\rho \in (0, R)$, for $\alpha > 0$ sufficiently large, if

$$w = e^{-\alpha r^2} - e^{-\alpha R^2}, \tag{5.13}$$

then $Lw > 0$ on the shell $\mathcal{A}$, where

$$\mathcal{A} = \{x \in \mathcal{O} : r(x) > \rho\}. \tag{5.14}$$

Therefore, for any $\epsilon > 0$, if w is given by (5.13) then $Lu \geq 0$ implies

$$L(u + \epsilon w) > 0, \qquad \text{on} \qquad \mathcal{A}. \tag{5.15}$$

Therefore, by Theorem 12,

$$\sup_{\mathcal{A}}(u + \epsilon w) = \sup_{\partial\mathcal{A}}(u + \epsilon w). \tag{5.16}$$

Observe that $w = 0$ on $\partial\mathcal{O} = \{r(x) = R\}$. By (5.9),

$$\sup_{\{r(x)=\rho\}} u(x) < u(y), \tag{5.17}$$

so for $\epsilon > 0$ sufficiently small,

$$u(x) + \epsilon w(x) \leq u(y), \qquad \text{for all} \qquad x \in \mathcal{A}. \tag{5.18}$$

Doing some algebra, since $w(y) = 0$,

$$\frac{u(y) - u(x)}{|y - x|} \geq \epsilon \frac{w(x)}{|y - x|} = \epsilon \frac{w(x) - w(y)}{|y - x|}. \tag{5.19}$$

Therefore,

$$\liminf_{t \searrow 0} \frac{1}{t}[u(y) - u(y - t\nu)] \geq -\epsilon \frac{\partial w}{\partial \nu}(y). \tag{5.20}$$

By definition of w, $\frac{\partial w}{\partial \nu}(y) < 0$, which proves the Proposition. □

Proposition 18 (Strong maximum principle)**.** *Let $u \in C^2(\Omega) \cap C(\bar{\Omega})$ satisfy $Lu \geq 0$, then either u is constant or*

$$u(x) < \sup_{z \in \partial\Omega} u(z), \qquad \textit{for all} \qquad x \in \Omega. \tag{5.21}$$

Proof. Let M be the nonnegative maximum of u and let $\Sigma = \{x \in \Omega : u(x) = M\}$. This set is relatively closed in Ω. We show that $\Sigma = \Omega$.

If Σ is a proper subset of Ω then there exists an open ball $B \subset \Omega \backslash \Sigma$ with a point on its boundary belonging to Σ. Suppose $x_0 \in \partial B \cap \Sigma$. Then $Lu \geq 0$ in B and $u(x) < u(x_0)$ for all $x \in B$, so $\frac{\partial u}{\partial n}(x_0) > 0$. However, this contradicts the fact that $x_0 \in \Omega$ is an interior maximum, and thus, $\nabla u(x_0) = 0$. □

Recall from the previous section that we have a map

$$PI : C^{\infty}(\partial\Omega) \to C^{\infty}(\bar{\Omega}), \tag{5.22}$$

where PI is the solution operator to the Dirichlet problem. By Proposition 18, this map has a unique continuous extension to

$$PI : C(\partial\Omega) \to C(\bar{\Omega}). \tag{5.23}$$

Indeed, approximate a continuous function uniformly by a sequence of smooth functions.

We can apply these calculations to the study of eigenvalues. Let λ_0 be the smallest eigenvalue of $-\Delta$. By (4.2) and (4.5), $\lambda_0 > 0$. Suppose $\bar{\Omega}$ is a connected, compact manifold with nonempty smooth boundary.

Proposition 19. *If $u_0 \in H_0^1(\Omega)$ is an eigenvalue for $-\Delta$ corresponding to λ_0,*

$$\Delta u_0 = -\lambda_0 u_0, \tag{5.24}$$

then u_0 is nowhere vanishing on the interior of Ω.

Proof. We have $u_0 \in C^{\infty}(\bar{\Omega})$. Let

$$\begin{aligned} u_0^+(x) &= \max\{u_0(x), 0\}, \\ u_0^-(x) &= \min\{u_0(x), 0\}. \end{aligned} \tag{5.25}$$

We have u_0^+, $u_0^- \in H_0^1(\Omega)$ and

$$\|\nabla u_0^{\pm}\|_{L^2(\Omega)}^2 = \int_{\Omega^{\pm}} |\nabla u|^2, \tag{5.26}$$

where

$$\Omega^{\pm} = \{x \in \Omega : \pm u(x) > 0\}. \tag{5.27}$$

Since

$$\lambda_0 = \inf\{\|\nabla u\|_{L^2(\Omega)}^2 : u \in H_0^1(\Omega), \qquad \|u\|_{L^2} = 1\}, \tag{5.28}$$

either u_0^+ or u_0^- must be a λ_0 eigenfunction of $-\Delta$. Therefore, to prove Proposition 19, it suffices to prove it under the additional hypothesis that $u_0(x) \geq 0$ on Ω. In that case,

$$\Delta(-u_0) = \lambda_0 u_0 \geq 0, \qquad \text{on} \qquad \Omega, \tag{5.29}$$

so then applying Proposition 18 to $-u_0$, since $u_0|_{\Omega} = 0$,

$$-u_0(x) < 0, \qquad \text{for all} \qquad x \in \Omega. \tag{5.30}$$

□

Corollary 5. *If λ_0 is the smallest eigenvalue of $-\Delta$ for Ω, with Dirichlet boundary conditions, then the corresponding λ_0-eigenspace is one-dimensional.*

Proof. If there were u_1 orthogonal to u_0 then u_1 must change sign, contradicting Proposition 19. □

Now let L have a zeroth order term,

$$L = g^{ij}(x)\partial_i\partial_j + b^i(x)\partial_i + c(x), \tag{5.31}$$

and let

$$\mathcal{L} = L - c(x). \tag{5.32}$$

Suppose $c \in C(\bar{\Omega})$ with $\Omega \subset \mathbb{R}^n$ bounded.

Proposition 20. *Suppose* $c(x) \geq 0$ *in* (5.32). *For* $u, v \in C^2(\Omega) \cap C(\bar{\Omega})$,

$$\mathcal{L}u \leq \mathcal{L}v \quad \text{on} \quad \Omega, \quad u \geq v \quad \text{on} \quad \partial\Omega \Rightarrow u \geq v \quad \text{on} \quad \Omega. \tag{5.33}$$

Proof. By linearity, it suffices to show that for $\mathcal{L}v \geq 0$ on Ω and $v \leq 0$ on $\partial\Omega$, $v \leq 0$ on Ω. Let $\mathcal{O} = \{x \in \Omega : v(x) > 0\}$. Then $Lv = cv \geq 0$ on $\mathcal{O}$ and $v = 0$ on $\partial\mathcal{O}$. Then by Proposition **??**, $\sup_{\mathcal{O}} v = \sup_{\partial\mathcal{O}} v$, which is impossible for $\mathcal{O} \neq \emptyset$. □

Corollary 6. *If* $c(x) \geq 0$ *in* $\mathcal{L}u = 0$ *then if* $\alpha = \sup_{\partial\Omega} u$,

$$\alpha \geq 0 \Rightarrow \sup_{\Omega} u = \alpha, \qquad \text{and} \qquad \alpha < 0 \Rightarrow \sup_{\Omega} u < 0. \tag{5.34}$$

Proof. The first implication follows from (5.33), since $\alpha \geq 0$ implies $\mathcal{L}\alpha \leq 0$. For the second implication let $\mathcal{O} = \{x \in \Omega : u(x) > 0\}$. If $\mathcal{O} \neq \emptyset$, $\bar{\mathcal{O}} \subset \Omega$ and $u = 0$ on $\partial\mathcal{O}$. But then the first implication of (5.34) applies to $u|_{\partial\mathcal{O}}$, which gives a contradiction. □

When $L = \Delta$, it is possible to strengthen Proposition 20.

Proposition 21. *Assume* $c \in C(\bar{\Omega})$ *and that* $\mathcal{L} = \Delta - c$ *is negative-definite with Dirichlet boundary condition,*

$$-\|\nabla u\|_{L^2}^2 - (cu, u) < 0, \qquad \text{for nonzero} \qquad u \in H_0^1(\Omega). \tag{5.35}$$

Then for $v \in H^1(\Omega)$,

$$(\Delta - c)v \geq 0, \quad \text{on} \quad \Omega, \quad v \leq 0 \quad \text{on} \quad \partial\Omega \Rightarrow v \leq 0, \quad \text{on} \quad \Omega. \tag{5.36}$$

Proof. Let $v_+ = \max\{v, 0\}$. Then by (5.9), $v_+ \in H_0^1(\Omega)$,

$$-(\nabla v, \nabla v_+) - (cv, v_+) \geq 0. \tag{5.37}$$

Since $(\nabla v, \nabla v_+) = (\nabla v_+, \nabla v_+)$, $-(\nabla v_+, \nabla v_+) - (cv_+, v_+) \geq 0$. Then by (5.35), $v_+ = 0$, which proves the proposition. □

6. Single and double layer potential methods

Before moving on to the Dirichlet problem with a non-smooth domain, we study the method of layer potentials. Begin with the Dirichlet problem for a half-space:

$$\Delta u = 0, \quad \text{in} \quad \mathbb{R}_+^{n+1} = \{x \in \mathbb{R}^{n+1} : x_{n+1} > 0\}, \quad u = f, \quad \text{on} \quad \partial\mathbb{R}_+^{n+1} = \mathbb{R}^n \times \{0\}. \tag{6.1}$$

Using the Poisson integral formula,

$$u(x,y) = P_y * f(x), \qquad (x,y) \in \mathbb{R}_+^{n+1} = \mathbb{R}^n \times \mathbb{R}_+, \qquad P_y(x) = \frac{\Gamma(\frac{n+1}{2})}{\pi^{\frac{n+1}{2}}} \frac{y}{(|x|^2 + y^2)^{\frac{n+1}{2}}}, \qquad f \in C_0(\mathbb{R}^{n+1}). \tag{6.2}$$

Indeed, by direct computation,

$$\begin{aligned} (\Delta_x + \partial_{yy}) \frac{y}{(|x|^2+y^2)^{\frac{n+1}{2}}} &= \frac{-n(n+1)y}{(|x|^2+y^2)^{\frac{n+3}{2}}} + (n+1)(n+3)\frac{|x|^2 y}{(|x|^2+y^2)^{\frac{n+5}{2}}} \\ -(n+1)\frac{y}{(|x|^2+y^2)^{\frac{n+3}{2}}} &- \frac{2(n+1)y}{(|x|^2+y^2)^{\frac{n+3}{2}}} + (n+1)(n+3)\frac{y^3}{(|x|^2+y^2)^{\frac{n+5}{2}}} = 0. \end{aligned} \tag{6.3}$$

The estimates on convolutions imply that

$$\sup_{y>0} \|u(\cdot, y)\|_{L^p(\mathbb{R}^n)} \leq \|f\|_{L^p(\mathbb{R}^n)}, \qquad \text{for all} \qquad 1 \leq p \leq \infty. \tag{6.4}$$

Indeed, this follows from the fact that

$$\frac{\Gamma(\frac{n+1}{2})}{\pi^{\frac{n+1}{2}}} \int_{\mathbb{R}^n} \frac{y}{(|x|^2 + y^2)^{\frac{n+1}{2}}} dx = 1. \tag{6.5}$$

Indeed, by a change of variables,

$$\frac{\Gamma(\frac{n+1}{2})}{\pi^{\frac{n+1}{2}}} A_{n-1} \int_0^\infty \frac{1}{(1+r^2)^{\frac{n+1}{2}}} r^{n-1} dr = 2\frac{A_{n-1}}{A_n} \int_0^{\pi/2} (\sin\theta)^{n-1} d\theta = \frac{A_{n-1}}{A_n} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (\cos\theta)^{n-1} d\theta = 1. \tag{6.6}$$

The last equality uses the geometric implication that $A_n = A_{n-1} \int_{-\pi/2}^{\pi/2} (\cos\theta)^{n-1} d\theta$. Since $P_y(x) = \frac{1}{y^n} P_1(\frac{x}{y})$, (6.4) holds, and furthermore, for any $f \in C_0(\mathbb{R}^n)$,

$$\lim_{y \searrow 0} P_y * f \to f, \tag{6.7}$$

uniformly as $y \searrow 0$.

Now we can reverse the implication.

Lemma 6. *For $1 < p \leq \infty$, if a harmonic function u in $\mathbb{R}^{n+1}_+$ satisfies (6.4), then u has a non-tangential limit a.e. on $\partial\mathbb{R}^{n+1}_+$, and the limit function $u_0 = u(\cdot, 0) \in L^p(\mathbb{R}^n)$ satisfies $u(x,y) = P_y * u_0$. If $p = 1$ then u_0 is a Radon measure.*

Proof. Suppose u is harmonic in $\mathbb{R}^{n+1}_+$ with

$$\sup_{y>0} \|u(\cdot, y)\|_{L^p(\mathbb{R}^n)} < \infty. \tag{6.8}$$

Now then,

$$u(x, y + \rho) = P_y * u_\rho(x), \qquad u_\rho(x) = u(x + \rho), \qquad y > 0, \qquad \rho > 0. \tag{6.9}$$

Then for some sequence $\rho_n \searrow 0$, $u_{\rho_n} \rightharpoonup v$ in $L^p(\mathbb{R}^n)$. Then, $P_y * u_{\rho_n}(x) \to P_y * v(x)$ for all $y > 0$ as $\rho_n \searrow 0$. Meanwhile, $P_y * u_{\rho_n}(x) = u(x, y + \rho_n)$, so $P_y * v(x) = u(x,y)$, where $v \in L^p(\mathbb{R}^n)$. □

Remark 7. *The proof uses uniqueness of solutions to the Dirichlet problem on the half space.*

Lemma 7. *If u is a bounded, continuous solution to (6.1) and $u = 0$ on $y = 0$, then $u \equiv 0$.*

Proof. Suppose $u(x,y)$ is a harmonic function on $\mathbb{R}^{n+1}_+$ such that $u(x,0) = 0$. Then for $y > 0$, let

$$v(x, -y) = -u(x,y), \qquad v(x,y) = u(x,y). \tag{6.10}$$

Then v solves the mean value property. Therefore, by Theorem 2, v is harmonic.

Now apply an argument similar to the proof of Liouville's theorem in complex analysis.

Lemma 8. *Suppose $u \in C(\bar{B}_R)$ is harmonic in $B_R(x_0) = B_R$. Then there holds*

$$|\nabla u(x_0)| \leq \frac{n}{R} \max_{\bar{B}_R} |u|. \tag{6.11}$$

Proof. Suppose $x \in \Omega$ is such that $B_r(x) \subset \Omega$. Then,

$$u(x) - u(y) = \frac{n}{\omega_n r^n}[\int_{B_r(x)} u(z)dz - \int_{B_r(y)} u(z)dz] \leq \frac{n}{\omega_n r^n} \int_{B_r(x)\Delta B_r(y)} |u(z)|dz. \tag{6.12}$$

Since $B_r(x)\Delta B_r(y) \subset B_{r+|x-y|}(x)$,

$$|B_r(x)\Delta B_r(y)| \leq \omega_n r^{n-1}|x-y| + o(|x-y|), \tag{6.13}$$

so taking $|x-y| \searrow 0$ proves the lemma. □

Since v is harmonic and bounded on $\mathbb{R}^{n+1}$, $\nabla v = 0$. □

The proof of Lemma 7 can be modified to the case when $\|v(\cdot, y)\|_{L^p(\mathbb{R}^n)}$ is uniformly bounded for $y \in \mathbb{R}$. Then,

$$\frac{1}{R^{n+1}} \int_{B(x,R)} |v(z', y)|dz'dy \leq CR^{-n/p}. \tag{6.14}$$

Plugging (6.14) into (6.12),

$$|u(x', y') - u(x, y)| \leq \frac{R^{\frac{n-1}{p'}}|x-y|^{\frac{1}{p'}}}{R^n} \sup_y \|u(\cdot, y)\|_{L^p}. \tag{6.15}$$

Taking $R \to \infty$ implies that u is constant.

Now assume that Ω is a bounded, connected domain in $\mathbb{R}^n$, $n \geq 3$, with C^2 boundary. Consider the Dirichlet problem,

$$\Delta u = 0, \qquad \text{in} \qquad \Omega, \qquad u|_{\partial\Omega} = f \in C(\partial\Omega). \tag{6.16}$$

Then let $\gamma(x) = \frac{C_n}{|x|^{n-2}}$ be the fundamental solution of the Laplace operator on $\mathbb{R}^n$, $C_n = -\frac{1}{n-2}\frac{\Gamma(\frac{n}{2})}{2\pi^{n/2}}$.

Now let $R(x, y) = -\gamma(x-y)$, and for $f \in C(\partial\Omega)$, define the double-layer potential

$$\mathcal{D}f(p) = \int_{\partial\Omega} \frac{\partial}{\partial n_Q} R(P, Q) f(Q) d\mathcal{H}^{n-1}(Q), \qquad P \notin \partial\Omega, \tag{6.17}$$

and the single layer potential

$$\mathcal{S}(f)(P) = \int_{\partial\Omega} R(P, Q) f(Q) d\mathcal{H}^{n-1}(Q), \qquad P \notin \partial\Omega. \tag{6.18}$$

Now then, since $R(P, Q)$ is harmonic for $P \notin \partial\Omega$,

$$\Delta\mathcal{D}f(P) = 0, \qquad \text{for} \qquad P \in \mathbb{R}^n \setminus \partial\Omega. \tag{6.19}$$

Now study the boundary behavior of $\mathcal{D}f(P)$ on $\partial\Omega$.

Lemma 9. *If $f \in C(\partial\Omega)$, then*
(1) $\mathcal{D}f \in C(\bar{\Omega})$,
(2) $\mathcal{D}f \in C(\bar{\Omega}^c)$.

Thus, $\mathcal{D}f$ can be extended continuously from inside Ω to $\bar{\Omega}$ and from outside Ω to $\bar{\Omega}^c$. Let $\mathcal{D}_+ f$ and $\mathcal{D}_- f$ be restrictions of these two functions to $\partial\Omega$. Set

$$K(P, Q) = \frac{\partial}{\partial n_Q} R(P, Q) = \frac{1}{\omega_n} \frac{\langle P - Q, n_Q \rangle}{|P - Q|^n}. \tag{6.20}$$

Therefore, $K \in C(\partial\Omega \times \partial\Omega \setminus \{(P,P) : P \in \partial\Omega\})$ and $|K(P,Q)| \leq \frac{C}{|P-Q|^{n-2}}$ for $P, Q \in \partial\Omega$ and some $C < \infty$. Then for $f \in C(\partial\Omega)$, define the operator

$$Tf(P) = \int_{\partial\Omega} K(P,Q) f(Q) d\mathcal{H}^{n-1}(Q), \qquad P \in \partial\Omega. \tag{6.21}$$

Lemma 10 (Jump relations for $\mathcal{D}$)**.** (1) $\mathcal{D}_+ = \frac{1}{2}I + T$, *and*
(2) $\mathcal{D}_- = -\frac{1}{2}I + T$.
Moreover, $T : C(\partial\Omega) \to C(\partial\Omega)$ *is compact.*

Now solve the Dirichlet problem (6.1). Take $g \in C(\partial\Omega)$ and let $u(x) = \mathcal{D}g(x)$ for $x \in \Omega$. Then $\Delta u = 0$ in Ω and $u \in C(\bar{\Omega})$. Moreover, $u|_{\partial\Omega} = (\frac{1}{2}I + T)g$. This map is one-to-one. Indeed, if $(\frac{1}{2}I + T)g = 0$ then

$$u|_{\partial\Omega} = 0, \qquad \text{which implies} \qquad u = 0, \qquad \text{in} \qquad \Omega. \tag{6.22}$$

Now then, by Lemma 8, the jump of $\mathcal{D}$ across $\partial\Omega$ is g. Therefore, for $v = \mathcal{D}_- g$,

$$v|_{\partial\Omega} = -g, \qquad \Delta v = 0, \qquad \text{on} \qquad \mathbb{R}^n \setminus \Omega. \tag{6.23}$$

It is also clear from the formula (6.17) that v vanishes at infinity. Accepting for a moment that there is no jump across $\partial\Omega$ of $\frac{\partial}{\partial\nu}\mathcal{D}f$,

$$\frac{\partial}{\partial\nu} v = 0, \qquad \text{on} \qquad \partial\Omega. \tag{6.24}$$

However, this contradicts Zaremba's principle at the maximum of g on $\partial\Omega$.

Since $\frac{1}{2}I + T$ is a Fredholm operator, $\frac{1}{2}I + T$ is injective if and only if $\frac{1}{2}I + T$ is surjective. Therefore, $\frac{1}{2}I + T$ is an isomorphism.

Proof of Lemmas 8 *and* 10. Define

$$K_N(P,Q) = sign K(P,Q) \cdot \min\{N, |K(P,Q)|\}, \qquad N \in \mathbb{Z}_+. \tag{6.25}$$

Therefore, K_N is continuous on $\partial\Omega \times \partial\Omega$, and the Arzela–Ascoli theorem implies that $T_N f = \int_{\partial\Omega} K_N(P,Q) f(Q) d\mathcal{H}^{n-1}(Q)$ is compact on $C(\partial\Omega)$. Next,

$$\|T_N\| \leq \sup_{P \in \partial\Omega} \|K_N(P,Q)\|_{L^1(\partial\Omega)} \leq C < \infty, \tag{6.26}$$

where C is independent of N. Also,

$$\|T_N - T_{N+1}\| \leq C((\frac{1}{N})^{\frac{1}{n-2}} - (\frac{1}{N+1})^{\frac{1}{n-2}}) \leq CN^{-1-\frac{1}{n-2}}. \tag{6.27}$$

Therefore, $T = \lim_{N \to \infty} T_N$ is a compact operator on $C(\partial\Omega)$.

Now apply the divergence theorem on $\Omega \setminus B_\delta(P)$. Then,

$$\int_{\partial\Omega} \frac{\partial}{\partial n_Q} R(P,Q) d\mathcal{H}^{n-1}(Q) = 1, \qquad \text{if} \qquad P \in \Omega, \tag{6.28}$$

$$\int_{\partial\Omega} K(P,Q) d\mathcal{H}^{n-1}(Q) = \frac{1}{2}, \qquad \text{if} \qquad P \in \partial\Omega. \tag{6.29}$$

Let $P_0 \in \partial\Omega$ and $P \in \Omega$ such that $P \to P_0$. Then,

$$\mathcal{D}f(P) \to \frac{1}{2} f(P_0) + Tf(P_0). \tag{6.30}$$

This finishes the proof of Lemmas 8 and 10 in the case of Ω.

To see why (6.30) is true, first observe that for all $P \in \partial\Omega$,

$$\int_{\partial\Omega} |\frac{\partial}{\partial n_Q} R(P,Q)| d\mathcal{H}^{n-1}(Q) \leq C < \infty. \tag{6.31}$$

If $P_0 \notin supp(f)$, then

$$\int_{\partial\Omega} \frac{\partial}{\partial n_Q} R(P,Q) f(Q) d\mathcal{H}^{n-1}(Q) \to \int_{\partial\Omega} K(P_0,Q) f(Q) d\mathcal{H}^{n-1}(Q) = Tf(P_0). \tag{6.32}$$

If $P_0 \in supp(f)$ and $f(P_0) = 0$, then take $\{f_k\} \subset C(\partial\Omega)$ such that

$$\|f - f_k\|_{L^\infty(\partial\Omega)} \to 0, \tag{6.33}$$

and $P_0 \notin supp(f_k)$, for each k. Then,

$$\begin{aligned} |\mathcal{D}f(P) - Tf(P)| &\leq |\mathcal{D}(f - f_k)(P)| + |T(f - f_k)(P)| + |\mathcal{D}f_k(P) - Tf_k(P)| \\ &\leq C\|f - f_k\|_{L^\infty(\partial\Omega)} + \|T\| \|f - f_k\|_{L^\infty(\partial\Omega)} + |\mathcal{D}f_k(P) - Tf_k(P)|. \end{aligned} \tag{6.34}$$

Then for k sufficiently large such that the first two terms are small, and then take $P \to P_0$, so the last term goes to 0.

Finally, for the general case when $f(P_0) \neq 0$, observe that by (6.29), when $f \equiv 1$, $\mathcal{D}f = \frac{1}{2}$. □

To show that there is no jump of $\frac{\partial}{\partial \nu}\mathcal{D}f$, we use the single–layer and double–layer potentials.

Lemma 11. *If $f \in C(\partial\Omega)$ then*
(1) $\mathcal{D}_+\mathcal{S}(f) = \frac{\partial}{\partial \nu}\mathcal{S}(f) \in C(\bar{\Omega_{\delta_0}})$,
(2) $\mathcal{D}_-\mathcal{S}(f) = \frac{\partial}{\partial \nu}\mathcal{S}(f) \in C(\bar{\Omega^c_{\delta_0}})$.
Here, $\bar{\Omega_{\delta_0}} = \{x \in \bar{\Omega} : dist(x, \partial\Omega) \leq \delta_0\}$ for some $\delta_0 > 0$ small.

Now let

$$K^*(P,Q) = K(Q,P), \tag{6.35}$$

and define

$$T^*f(P) = \int_{\partial\Omega} K^*(P,Q) f(Q) d\mathcal{H}^{n-1}(Q), \qquad P \in \partial\Omega. \tag{6.36}$$

Lemma 12 (Jump relations for $\mathcal{D}S(f)$)**.** (1) $\mathcal{D}_+S(f) = -\frac{1}{2}I + T^*$, *and*
(2) $\mathcal{D}_-S(f) = \frac{1}{2}I + T^*$.

Define the Neumann operator

$$\mathcal{N} : C^\infty(\partial\Omega) \to C^\infty(\partial\Omega), \tag{6.37}$$

where $u \in C^\infty(\bar{\Omega})$ solves

$$\Delta u = 0, \qquad \text{on} \qquad \Omega, \qquad u = f \qquad \text{on} \qquad \partial\Omega, \tag{6.38}$$

and then let

$$\mathcal{N}f = \frac{\partial u}{\partial \nu}|_{\partial\Omega}. \tag{6.39}$$

Now then, by Green's formula, (2.6),

$$\int_{\partial\Omega} [f(Q)\frac{\partial}{\partial n_Q} R(P,Q) - \mathcal{N}f(Q) R(P,Q)] d\mathcal{H}^{n-1}(Q) = u(P), \qquad P \in \Omega, \qquad 0, \qquad P \notin \overline{\Omega}. \tag{6.40}$$

Also, by (6.17) and (6.18),

$$(6.40) = \mathcal{D}f - \mathcal{S}\mathcal{N}f. \tag{6.41}$$

Taking the limit from each side,

$$\frac{\partial}{\partial \nu}\mathcal{D}_{+}f - \frac{\partial}{\partial \nu}\mathcal{S}\mathcal{N}f = \mathcal{N}f, \qquad \frac{\partial}{\partial \nu}\mathcal{D}_{-}f - \frac{\partial}{\partial \nu}\mathcal{S}\mathcal{N}f_{-} = 0. \tag{6.42}$$

Plugging in Lemmas 10, 11, and 12,

$$\frac{\partial}{\partial \nu}\mathcal{D}_{+}f = -\frac{1}{2}\mathcal{N}f + T^{*}\mathcal{N}f + \mathcal{N}f, \qquad \frac{\partial}{\partial \nu}\mathcal{D}_{-}f = \frac{1}{2}\mathcal{N}f + T^{*}\mathcal{N}f. \tag{6.43}$$

This proves that there is no jump across the boundary.

7. The Neumann boundary problem

In the study of layer potentials, the Dirichlet and Neumann problems,

$$\Delta u = f, \qquad \text{on} \qquad \Omega, \qquad \frac{\partial u}{\partial \nu} = 0, \qquad \text{on} \qquad \partial\Omega, \tag{7.1}$$

are inextricably linked. Here $\bar{\Omega}$ is connected and compact with nonempty boundary.

By Green's formula, for u and v smooth on $\bar{\Omega}$,

$$(-\Delta u, v)_{L^2} = (\nabla u, \nabla v)_{L^2} - \int_{\partial\Omega} v \frac{\partial u}{\partial \nu} dS. \tag{7.2}$$

If $\frac{\partial u}{\partial \nu} = 0$ on $\partial\Omega$, we are motivated to consider the operator

$$\mathcal{L}_N : H^1(\Omega) \rightarrow H^1(\Omega)^{*}, \tag{7.3}$$

defined by

$$(\mathcal{L}_N u, v)_{L^2} = (\nabla u, \nabla v)_{L^2}, \qquad u, v \in H^1(\Omega). \tag{7.4}$$

The operator $\mathcal{L}_N$ annihilates constants, so $\mathcal{L}_N$ is not injective. However,

$$((\mathcal{L}_N + 1)u, u) = \|\nabla u\|_{L^2}^2 + \|u\|_{L^2}^2, \tag{7.5}$$

so

Proposition 22. *The map*

$$\mathcal{L}_N + 1 : H^1(\Omega) \rightarrow H^1(\Omega)^{*}, \tag{7.6}$$

is one-to-one and onto.

As in section three, the inverse map

$$T_N : H^1(\Omega)^{*} \rightarrow H^1(\Omega), \tag{7.7}$$

restricts to a compact, self–adjoint operator on $L^2(\Omega)$. Therefore, there is an orthonormal basis u_j of $L^2(\Omega)$ consisting of eigenfunctions of T_N:

$$T_N u_j = \mu_j u_j, \qquad \mu_j \searrow 0, \qquad u_j \in H^1(\Omega). \tag{7.8}$$

Now then,

$$\mathcal{L}_N u_j = \lambda_j u_j, \qquad \lambda_j = \frac{1}{\mu_j} - 1 \nearrow \infty. \tag{7.9}$$

Now let $i : H^1(\Omega)^* \to \mathcal{D}'(\Omega)$ be the adjoint of the inclusion $C_0^\infty(\Omega) \hookrightarrow H^1(\Omega)$. Then,

$$i(\mathcal{L}_N u) = -\Delta u, \qquad \text{in} \qquad \mathcal{D}'(\Omega), \qquad \text{for} \qquad u \in H^1(\Omega). \tag{7.10}$$

Therefore, in the distributional sense,

$$\Delta u_j = -\lambda_j u_j, \qquad \text{on} \qquad \Omega. \tag{7.11}$$

Regularity theorems imply that each u_j belongs to $C^\infty(\bar{\Omega})$.

Proposition 23. *Given $f \in L^2(\Omega)$, $u = T_N f$ satisfies*

$$u \in H^2(\Omega), \qquad \frac{\partial u}{\partial \nu}|_{\partial\Omega} = 0, \tag{7.12}$$

and

$$(-\Delta + 1)u = f. \tag{7.13}$$

Furthermore,

$$\|u\|_{H^2}^2 \le C\|\Delta u\|_{L^2}^2 + C\|u\|_{H^1}^2, \tag{7.14}$$

for all u satisfying (7.12)*.*

Proof. By (7.5) and Cauchy's inequality,

$$\|u\|_{H^1}^2 = (\mathcal{L}_N u, u)_{L^2} + \|u\|_{L^2}^2 \le \|\mathcal{L}_N u\|_{H^{1*}} \|u\|_{H^1} + \|u\|_{L^2}^2 \le \frac{1}{2}\|u\|_{H^1}^2 + \frac{1}{2}\|\mathcal{L}_N u\|_{H^{1*}}^2 + \|u\|_{L^2}^2, \tag{7.15}$$

which implies

$$\|u\|_{H^1}^2 \le C\|\mathcal{L}_N u\|_{H^{1*}}^2 + C\|u\|_{L^2}^2. \tag{7.16}$$

Now take $\chi \in C^\infty(\bar{\Omega})$ and either $\chi \in C_0^\infty(\Omega)$ or $\frac{\partial \chi}{\partial \nu} = 0$ on $\partial\Omega$. Let $M_\chi u = \chi u$. Then,

$$(\mathcal{L}_N M_\chi u, v)_{L^2} = (\nabla(\chi u), \nabla v)_{L^2} = ((\nabla \chi)u, \nabla v) + (\chi \nabla u, \nabla v), \tag{7.17}$$

$$(M_\chi \mathcal{L}_N u, v)_{L^2} = (\mathcal{L}_N u, \chi v) = (\nabla u, (\nabla \chi) v) + (\nabla u, \chi \nabla v). \tag{7.18}$$

Therefore,

$$([\mathcal{L}_N, M_\chi]u, v) = ((\nabla \chi)u, \nabla v) - (\nabla \chi \cdot \nabla u, v). \tag{7.19}$$

Integrating by parts, since $\frac{\partial \chi}{\partial \nu} = 0$ on $\partial\Omega$,

$$[\mathcal{L}_N, M_\chi]u = -(\Delta \chi)u - 2\nabla \chi \cdot \nabla u. \tag{7.20}$$

Therefore,

$$[\mathcal{L}_N, M_\chi] : H^1(\Omega) \to L^2(\Omega). \tag{7.21}$$

Now then, by (7.14),

$$\|D_{j,h}u\|_{L^2}^2 \le C\|\mathcal{L}_N D_{j,h} u\|_{H^{1*}}^2 + C\|D_{j,h}u\|_{L^2}^2 \le C\|D_{j,h}\mathcal{L}_N u\|_{H^{1*}}^2 + C\|[\mathcal{L}_N, D_{j,h}]u\|_{H^{1*}}^2 + C\|D_{j,h}u\|_{L^2}^2. \tag{7.22}$$

Then,

$$(\mathcal{L}_N D_{j,h} u, v) = (dD_{j,h}u, dv) = (du, D_{j,h}^{(1)} dv), \tag{7.23}$$

where

$$D_{j,h}^{(1)} \varphi = h^{-1}(\tau_{j,h}^* \varphi - \varphi). \tag{7.24}$$

The adjoint of $D_{j,h}$ is $D_{j,-h}$, so

$$(D_{j,h}\mathcal{L}_N u, v) = (\mathcal{L}_N u, D_{j,-h} v) = (du, D^{(1)}_{j,-h} dv). \tag{7.25}$$

Therefore,

$$([\mathcal{L}_N, D_{j,h}]u, v) = ([D^{(1)*}_{j,-h} - D^{(1)}_{j,h}]du, dv). \tag{7.26}$$

Lemma 13. *If β is a one–form on Ω,*

$$\|[D^{(1)*}_{j,-h} - D^{(1)}_{j,h}]\beta\|_{L^2} \leq C\|\beta\|_{L^2}. \tag{7.27}$$

Proof. The proof is similar to the proof of Lemma 5. □

Plugging in Lemma 13, we have

$$\|[\mathcal{L}_N, D_{j,h}]u\|_{H^{1*}} \leq C\|du\|_{L^2}. \tag{7.28}$$

Therefore,

$$\|D_{j,h}u\|_{H^1}^2 \leq C\|D_{j,h}\mathcal{L}_N u\|_{H^{1*}}^2 + C\|u\|_{H^1}^2. \tag{7.29}$$

Now then, $(D_{j,h} f_1, v) = (f_1, D_{j,-h} v)$ for $v \in H^1$, so

$$\|D_{j,h} f_1\|_{H^{1*}}^2 \leq C\|f_1\|_{L^2}^2, \tag{7.30}$$

which implies

$$\|D_{j,h}u\|_{H^1}^2 \leq C\|f_1\|_{L^2}^2 + C\|u\|_{H^1}^2. \tag{7.31}$$

Therefore, $D_j u \in H^1$ for $j = 1, ..., n-1$. As in section three, it remains to estimate D_{nn}, which uses the fact that the operator is elliptic.

It remains to show that u satisfies the Neumann boundary condition. If $u = T_N f \in H^2(\Omega)$, $v \in H^1(\Omega)$,

$$(f, v) = (\mathcal{L}_N u, v) + (u, v) = (\nabla u, \nabla v) + (u, v) = ((-\Delta + 1)u, v) + \int_{\partial\Omega} \bar{v}\frac{\partial u}{\partial \nu} dS = (f, v) + \int_{\partial\Omega} \bar{v}\frac{\partial u}{\partial \nu} dS. \tag{7.32}$$

Since this holds for all $v \in H^1(\Omega)$, $\frac{\partial u}{\partial \nu} = 0$ on $\partial\Omega$. □

Arguing by induction,

Proposition 24. *For $k = 1, 2, 3, ...$, given $f_1 \in H^k(\Omega)$, a function $u \in H^{k+1}(\Omega)$ satisfying*

$$\Delta u = f_1, \quad on \quad \Omega, \quad \frac{\partial u}{\partial \nu} = 0, \quad on \quad \partial\Omega, \tag{7.33}$$

belongs to $H^{k+2}(\Omega)$, and we have the estimate

$$\|u\|_{H^{k+2}}^2 \leq C\|\Delta u\|_{H^k}^2 + C\|u\|_{H^{k+1}}^2, \tag{7.34}$$

for all $u \in H^{k+2}(\Omega)$ such that $\frac{\partial u}{\partial \nu} = 0$ on $\partial\Omega$.

We can also analyze nonhomogeneous boundary value problems such as

$$(-\Delta + 1)u = f, \quad \text{in} \quad \Omega, \quad \frac{\partial u}{\partial \nu} = g, \quad \text{on} \quad \partial\Omega. \tag{7.35}$$

For $g \in H^{k+1/2}(\partial\Omega)$, we can choose $h \in H^{k+2}(\Omega)$ such that $\frac{\partial h}{\partial \nu} = g$ on $\partial\Omega$, and then write $u = v+h$, where

$$(7.36) \qquad (-\Delta + 1)v = f + (\Delta - 1)h, \qquad \text{in} \qquad \Omega, \qquad \frac{\partial v}{\partial \nu} = 0, \qquad \text{on} \qquad \partial\Omega.$$

The fact that 0 is an eigenvalue in (7.9) with eigenspace consisting of constants, implies

Proposition 25. *Given $f \in L^2(\Omega)$, the boundary value problem* (7.1) *has a solution $u \in H^2(\Omega)$ if and only if*

$$(7.37) \qquad \int_\Omega f(x)dx = 0.$$

Provided this condition holds, the solution u is unique up to an additive constant and belongs to $H^{k+2}(\Omega)$ if $f \in H^k(\Omega)$, $k \geq 0$.

Proof. Integrating by parts, if $u \in H^2(\Omega)$ solves (7.1),

$$(7.38) \qquad \int_\Omega f(x)dx = \int_\Omega \Delta u dx = \int_{\partial\Omega} \frac{\partial u}{\partial \nu} dS(x) = 0.$$

On the other hand, if (7.37) holds, then f is orthogonal to the constant function, and we can use the formula (7.9) to compute the inverse. □

We have an extension for the nonhomogeneous boundary problem

$$(7.39) \qquad \Delta u = f \qquad \text{on} \qquad \Omega, \qquad \frac{\partial u}{\partial \nu} = g, \qquad \text{on} \qquad \partial\Omega.$$

By Green's formula (7.2),

$$(7.40) \qquad \int_\Omega \Delta u(x)dx = \int_{\partial\Omega} \frac{\partial u}{\partial \nu} dS(x).$$

Thus, a necessary condition for (7.39) to have a solution is

$$(7.41) \qquad \int_\Omega f(x)dx = \int_{\partial\Omega} g(x)dS(x).$$

This condition is also sufficient.

Proposition 26. *If $k \geq 0$, $f \in H^k(\Omega)$, and $g \in H^{k+\frac{1}{2}}(\partial\Omega)$, then* (7.39) *has a solution $H^{k+2}(\Omega)$ if and only if* (7.41) *holds.*

Proof. Define the linear operator

$$(7.42) \qquad \mathcal{T} : H^{k+2}(\Omega) \to H^k(\Omega) \oplus H^{k+1/2}(\partial\Omega), \qquad \mathcal{T}u = (\Delta u, \frac{\partial u}{\partial \nu}).$$

Now use a proposition from functional analysis.

Proposition 27. *If K is compact operator from V to Y, T is a linear operator from V to W, and*

$$(7.43) \qquad \|u\|_V \leq \|Tu\|_W + \|Ku\|_Y,$$

then T has closed range.

Then, since

$$\|u\|_{H^{k+2}(\Omega)}^2 \leq C\|\Delta u\|_{H^k(\Omega)}^2 + C\|\frac{\partial u}{\partial \nu}\|_{H^{k+1/2}(\partial\Omega)}^2 + C\|u\|_{H^{k+1}(\Omega)}^2, \tag{7.44}$$

$\mathcal{T}$ has closed range. We also know that the kernel of $\mathcal{T}$ consists of constants. By (7.41), $(-1,1) \in C^{\infty}(\bar{\Omega}) \oplus C^{\infty}(\partial\Omega)$ is orthogonal to the range of $\mathcal{T}$. If $\mathcal{T}$ is Fredholm of index zero, then this is all of the orthogonal complement of the range of $\mathcal{T}$. Now,

$$\mathcal{T}^{\sharp} : H^{k+2}(\Omega) \to H^k(\Omega) \oplus H^{k+1/2}(\partial\Omega), \qquad \mathcal{T}^{\sharp} u = ((\Delta - 1)u, \frac{\partial u}{\partial \nu}). \tag{7.45}$$

The operator $\mathcal{T}^{\sharp}$ differs from $\mathcal{T}$ by the operator $\mathcal{K}u = (-u, 0)$, which is compact. Since $\mathcal{T}^{\sharp}$ is an isomorphism, $\mathcal{T}$ is Fredholm of index zero. □

8. A class of semilinear equations

The Dirichlet and Neumann problems are linear problems. This means that two solutions to the problem $\Delta u = 0$ may be added together to obtain another solution to the problem, a phenomenon that has proved to be very useful to solving such equations. Here, we consider a class of semilinear equations,

$$\Delta u = f(x, u), \qquad \text{on} \qquad \mathcal{M}, \tag{8.1}$$

where $\mathcal{M}$ is a Riemannian manifold, or the interior of a compact manifold $\bar{\mathcal{M}}$ with smooth boundary. Consider the Dirichlet boundary condition,

$$u|_{\partial\mathcal{M}} = g, \tag{8.2}$$

where $\bar{\mathcal{M}}$ is connected and has nonempty boundary. Suppose $f \in C^{\infty}(\bar{\mathcal{M}} \times \mathbb{R})$, and also suppose that

$$\frac{\partial f}{\partial u} \geq 0. \tag{8.3}$$

Now suppose $F(x, u) = \int_0^u f(x, s) ds$, so $f(x, u) = \partial_u F(x, u)$. Then by (8.3), $F(x, u)$ is a convex function of u. Let

$$I(u) = \frac{1}{2}\|du\|_{L^2(\mathcal{M})}^2 + \int_{\mathcal{M}} F(x, u(x)) dV(x). \tag{8.4}$$

Then, a solution to (8.1) is a critical point of I on the space of functions u on $\mathcal{M}$ that are equal to g on ∂M. Indeed, let $\psi \in C_0^{\infty}(\mathcal{M})$ be a perturbation of u. Then,

(8.5)

$$I(u+\epsilon\psi) = I(u)+(\epsilon d\psi, du)+\epsilon\int_0^1\int_{\mathcal{M}} \frac{\partial F}{\partial u}(x, u+\tau\epsilon\psi) dV(x) d\tau + O(\epsilon^2) = I(u)+\epsilon(-\Delta u+f(x, u), \psi)+o(\epsilon).$$

Then, u is a minimizer of (8.5) if and only if u is a weak solution to $\Delta u - f(x, u) = 0$.

Remark 8. *Let L be the linear operator,*

$$Lu = -\partial_j(a_{ij}\partial_i u) + b_i D_i u + cu, \qquad \lambda|\xi|^2 \leq a_{ij}(x)\xi_i\xi_j \leq \Lambda|\xi|^2. \tag{8.6}$$

Define the weak solution

Definition 4 (Weak solution)**.** *Let $f \in L^2(\Omega)$ and $u \in H_{loc}^1(\Omega)$. Then u is a weak solution of $Lu = f$ in Ω if for any $\varphi \in H_0^1(\Omega)$,*

$$\int_{\Omega} (a_{ij}(x)\partial_i u \partial_j \varphi + b_i(x)\partial_i u\varphi + c(x)u\varphi) dx = \int_{\Omega} f\varphi dx. \tag{8.7}$$

As a warm–up, first consider a special case of (8.1). Let Ω be a bounded domain, $a_{ij}(x) = a_{ji}(x)$ and $c(x)$ be bounded functions on Ω. Define

$$J(u) = \frac{1}{2}\int_\Omega (a_{ij}(x)\partial_i u \partial_j u + c(x)u^2)dx - \int_\Omega u(x)f(x)dx. \tag{8.8}$$

The minimizer u is a weak solution of

$$-\partial_j(a_{ij}(x)\partial_i u) + c(x)u = f(x), \qquad \text{in} \qquad \Omega. \tag{8.9}$$

Since (8.8) is a convex function, (8.8) has a unique minimizer.

Theorem 13. *Let a_{ij} and c be bounded functions in Ω with $a_{ij} = a_{ji}$, $c \geq 0$, and $f \in L^2(\Omega)$. Then J admits a minimizer $u \in H_0^1(\Omega)$.*

Proof. First show that J has a lower bound in $H_0^1(\Omega)$. By the Poincare inequality,

$$\int_\Omega u^2 dx \leq C(\Omega)\int_\Omega |\nabla u|^2 dx. \tag{8.10}$$

Then for any $\lambda > 0$,

$$\int_\Omega |uf| dx \leq \frac{\lambda}{4}\int |\nabla u|^2 dx + \frac{C}{\lambda}\int f^2 dx. \tag{8.11}$$

Therefore,

$$J(u) \geq \frac{\lambda}{4}\int |\nabla u|^2 - \frac{C}{\lambda}\int f^2 dx. \tag{8.12}$$

Therefore, J has a lower bound in $H_0^1(\Omega)$. Set,

$$J_0 = \inf\{J(u) : u \in H_0^1(\Omega)\}. \tag{8.13}$$

Now consider a minimizing sequence $\{u_k\} \subset H_0^1(\Omega)$, with $J(u_k) \to J_0$. Then,

$$\int_\Omega |\nabla u_k|^2 dx \leq \frac{4}{\lambda}J(u_k) + \frac{4C}{\lambda^2}\int_\Omega f^2 dx. \tag{8.14}$$

Since $J(u_k)$ is bounded, $\|u_k\|_{H_0^1}$ is also bounded. Therefore, by Rellich's theorem, there exists a sequence $\{u_{k'}\}$ and $u \in H_0^1(\Omega)$ such that $u_{k'} \to u$ in the L^2 norm as $k' \to \infty$.

By the Hilbert space structure, $J(u) \leq \lim_{k'\to\infty} J(u_{k'}) = J_0$, so since J_0 is a minimizer, $J(u) = J_0$. Therefore, J_0 is attained in $H_0^1(\Omega)$. □

Returning to (8.1), make a temporary restriction on F. For $|u| \geq K$, let $\partial_u f(x, u) = 0$. Therefore, for some L, $|\partial_u F(x, u)| \leq L$ on $\bar{\mathcal{M}} \times \mathbb{R}$. Let,

$$V = \{u \in H^1(\mathcal{M}) : u = g, \qquad \text{on} \qquad \partial M\}. \tag{8.15}$$

Lemma 14. *We have the following results about the functional $I : V \to \mathbb{R}$.*

1. *I is strictly convex,*
2. *F is Lipschitz continuous,*
3. *I is bounded below,*
4. *$I(v) \to \infty$, as $\|v\|_{H^1} \to \infty$.*

Proof. Convexity follows from (8.3). The fact that I is Lipschitz continuous follows from $|F(x,u) - F(x,v)| \leq L|u-v|$, which follows from the bounds on $|\partial_u F(x,u)| \leq L$. Since F is convex, $F(x,u) \geq -C_0|u| - C_1$, so

$$I(u) \geq \frac{1}{2}\|du\|^2 - C_0\|u\|_{L^1} - C_1' \geq \frac{1}{4}\|du\|_{L^2}^2 + \frac{1}{2}B\|u\|_{L^2}^2 - C\|u\|_{L^2} - C', \tag{8.16}$$

since

$$\frac{1}{2}\|du\|_{L^2}^2 \geq B\|u\|_{L^2}^2 - C''. \tag{8.17}$$

□

Lemma 14 implies that I has a unique minimum on V.

Proposition 28. *Under the above hypotheses, $I(u)$ has a unique minimum on V.*

Proof. Let $\alpha_0 = \inf_V I(u)$. Since I is bounded below, $\alpha_0 > -\infty$. Choose R such that $K = V \cap B_R(0) \neq \emptyset$ and such that $\|u\|_{H^1} \geq R$ implies $I(u) \geq \alpha_0 + 1$. Then K is a closed, convex, bounded subset of $H^1(\mathcal{M})$. Let $u_k \in H^1(\mathcal{M})$ be a sequence in $B_R(0)$, where

$$I(u_k) \searrow \alpha_0. \tag{8.18}$$

By Rellich's theorem, after passing to a subsequence, u_k converges in L^2 to some $u_0 \in L^2$. Furthermore, as in the proof of Theorem 13, $u_k \to u_0$ in $H^1(\mathcal{M})$. Therefore, $\inf I(u)$ is assumed at u_0, and by the strict convexity of $I(u)$, u_0 is unique. The unique minimum of $I(u)$ is the solution to (8.1). □

We have the regularity result.

Proposition 29. *For $k = 1, 2, 3, ...$, if $g \in H^{k+1/2}(\partial\mathcal{M})$, then any solution $u \in V$ to* (8.1) *and* (8.2) *belongs to $H^{k+1}(\mathcal{M})$. Hence, if $g \in C^\infty(\partial\mathcal{M})$, then $u \in C^\infty(\mathcal{M})$.*

Proof. Start with $u \in H^1(\mathcal{M})$. Then the right hand side belongs to $L^2(\mathcal{M})$, which implies that $u \in H^2(\mathcal{M})$ if $g \in H^{3/2}(\mathcal{M})$. Arguing by induction proves the proposition. □

We have uniqueness of the element $u \in V$ minimimizing $I(u)$ under the hypotheses. In fact, under the hypothesis $\frac{\partial f}{\partial u} \geq 0$, there is uniqueness of solutions to (8.1) and (8.2) which are sufficiently smooth.

Proposition 30. *Let u and $v \in C^2(\mathcal{M}) \cap C(\bar{\mathcal{M}})$ satisfy* (8.1) *with $u = g$ and $v = h$ on $\partial\mathcal{M}$. Then,*

$$\sup_{\mathcal{M}}(u - v) \leq \sup_{\partial\mathcal{M}}(g - h) \vee 0, \tag{8.19}$$

and

$$\sup_{\mathcal{M}}|u - v| \leq \sup_{\partial\mathcal{M}}|g - h|. \tag{8.20}$$

Proof. Let $w = u - v$. Then,

$$\Delta w = \lambda(x) w, \qquad w|_{\partial\mathcal{M}} = g - h, \tag{8.21}$$

where

$$\lambda(x) = \frac{f(x,u) - f(x,v)}{u - v} \geq 0. \tag{8.22}$$

If $\mathcal{O} = \{x \in \mathcal{M} : w(x) \geq 0\}$, then $\Delta w \geq 0$ on $\mathcal{O}$, so applying the maximum principle, Theorem 12, on $\mathcal{O}$ gives (8.19). Replacing w by $-w$ gives (8.20). □

Corollary 7. *Let $f(x,0)=\varphi(x)\in C^\infty(\bar{\mathcal{M}})$. Take $g\in C^\infty(\partial\mathcal{M})$, and let $\Phi\in C^\infty(\bar{\mathcal{M}})$ be the solution to*

$$\Delta\Phi=\varphi,\qquad on\qquad \mathcal{M},\qquad \Phi=g,\qquad on\qquad \partial\mathcal{M}. \tag{8.23}$$

Then under the hypothesis $\frac{\partial f}{\partial u}\geq 0$, a solution u to (1.1) satisfies

$$\sup_{\mathcal{M}} u\leq \sup_{\mathcal{M}}\Phi+(\sup_{\mathcal{M}}(-\Phi)\vee 0), \tag{8.24}$$

and

$$\sup_{\mathcal{M}}|u|\leq \sup_{\mathcal{M}} 2|\Phi|. \tag{8.25}$$

Proof. We have

$$\Delta(u-\Phi)=f(x,u)-f(x,0)=\lambda(x)u, \tag{8.26}$$

with $\lambda(x)=\frac{f(x,u)-f(x,0)}{u}\geq 0$. Therefore, $\Delta(u-\Phi)\geq 0$ on $\mathcal{O}=\{x\in\mathcal{M}:u(x)>0\}$, so

$$\sup_{\mathcal{O}}(u-\Phi)=\sup_{\partial\mathcal{O}}(u-\Phi)\leq\sup_{\mathcal{M}}(-\Phi)\vee 0. \tag{8.27}$$

The last inequality follows from the fact that $u=\Phi$ on ∂M, and if $\partial\mathcal{O}$ has a point that is not contained in $\partial\mathcal{M}$, then $u=0$ on that point. Also, $\Delta(\Phi-u)\geq 0$ on $\mathcal{O}^-=\{x\in\mathcal{M}:u(x)<0\}$, so

$$\sup_{\mathcal{O}^-}(\Phi-u)=\sup_{\partial\mathcal{O}^-}(\Phi-u)\leq\sup_{\mathcal{M}}\Phi\vee 0. \tag{8.28}$$

□

Theorem 14. *Suppose $f(x,u)$ satisfies $\frac{\partial f}{\partial u}\geq 0$. Given $g\in C^\infty(\partial\mathcal{M})$, there is a unique solution $u\in C^\infty(\bar{\mathcal{M}})$ to (8.1) and (8.2).*

Proof. Let $f_j(x,u)$ be smooth and satisfying

$$f_j(x,u)=f(x,u),\qquad \text{for}\qquad |u|\leq j, \tag{8.29}$$

and suppose that (8.3) and (8.4) hold for $f_j(x,u)$, and furthermore $f_j(x,u)=\partial_u F_j(x,u)$, $\partial_u f(x,u)=0$ for $|u|\geq K_j$, and $|\partial_u F_j(x,u)|\leq L_j$ on $\bar{\mathcal{M}}\times\mathbb{R}$. Then by Proposition 28, we have a unique solution $u_j\in C^\infty(\bar{\mathcal{M}})$ to

$$\Delta u_j=f_j(x,u_j),\qquad u_j|_{\partial\mathcal{M}}=g. \tag{8.30}$$

If $f_j(x,0)=f(x,0)=\varphi(x)$, so then (8.25) holds for all u_j, so

$$\sup_{\mathcal{M}}|u_j|\leq\sup_{\mathcal{M}}2|\Phi|. \tag{8.31}$$

Thus, the sequence (u_j) stabilizes for large j, and the proof is complete. □

When $f(x,u)=f(u)$, and f is locally Lipschitz in $\mathbb{R}$, it is possible to prove some symmetry results for a solution to (8.1).

Lemma 15. *Suppose that Ω is a bounded domain that is convex in the x_1-direction and is symmetric with respect to the plane $\{x_1=0\}$. Suppose $u\in C(\bar{\Omega})\cap C^2(\Omega)$ is a positive solution of*

$$\Delta u+f(u)=0,\qquad in\qquad \Omega,\qquad u=0,\qquad on\qquad \partial\Omega, \tag{8.32}$$

where f is locally Lipschitz in $\mathbb{R}$. Then u is symmetric with respect to x_1 and $\partial_{x_1}u(x)<0$ for any $x\in\Omega$ when $x_1>0$.

Proof. Write $x = (x_1, y) \in \Omega$, $y \in \mathbb{R}^{n-1}$. We prove that

$$u(x_1, y) < u(x_1^*, y), \tag{8.33}$$

for any $x_1 > 0$ and $x_1^* < x_1$ with $x_1^* + x_1 > 0$. Letting $x_1^* \to -x_1$ gives $u(x_1, y) \leq u(-x_1, y)$ for any x_1. Changing the direction $x_1 \mapsto -x_1$ gives the symmetry.

Now let $a = \sup_{(x_1,y)\in\Omega} x_1$. For $0 < \lambda < a$ define

$$\begin{aligned} \Sigma_\lambda &= \{x \in \Omega : x_1 > \lambda\}, \\ T_\lambda &= \{x_1 = \lambda\}, \\ \Sigma'_\lambda &= \text{reflection of } \Sigma_\lambda \text{ with respect to } T_\lambda, \\ x_\lambda &= (2\lambda - x_1, x_2, ..., x_n), \qquad \text{for} \qquad x = (x_1, ..., x_n). \end{aligned} \tag{8.34}$$

and let

$$w_\lambda(x) = u(x) - u(x_\lambda), \qquad \text{for} \qquad x = (x_1, ..., x_n). \tag{8.35}$$

By the mean value theorem and (8.33), since f is locally Lipschitz,

$$\begin{aligned} \Delta w_\lambda + c(x, \lambda) w_\lambda = 0, \qquad &\text{in} \qquad \Sigma_\lambda, \\ w_\lambda \leq 0, \qquad \text{and} \qquad w_\lambda \not\equiv 0, \qquad &\text{on} \qquad \partial\Sigma_\lambda, \end{aligned} \tag{8.36}$$

where $c(x, \lambda)$ is a bounded function in Σ_λ.

Now we need to show that $w_\lambda < 0$ in Σ_λ for any $\lambda \in (0, a)$. In particular, this implies that w_λ assumes its maximum in Σ_λ along $\partial(\Sigma_\lambda \cap \Omega)$. Then by the Hopf lemma, for any such $\lambda \in (0, a)$,

$$\partial_{x_1} w_\lambda|_{x_1=\lambda} = 2\partial_{x_1} u|_{x_1=\lambda} < 0. \tag{8.37}$$

For λ close to a, $w_\lambda < 0$, using the maximum principle for small domains (see Theorem 2.32 of [HL11]) combined with the strong maximum principle.

Let (λ_0, a) be the largest interval of values of λ such that $w_\lambda < 0$ in Σ_λ. If $\lambda_0 > 0$, then by continuity $w_{\lambda_0} \leq 0$ in Σ_{λ_0} and $w_{\lambda_0} \not\equiv 0$ on $\partial\Sigma_{\lambda_0}$. If $w_{\lambda_0} \equiv 0$ on $\partial\Sigma_{\lambda_0}$. Then by the strong maximum principle, $w_{\lambda_0} < 0$ in Σ_{λ_0}. It suffices to show that for any $\epsilon > 0$ small, $w_{\lambda_0 - \epsilon} < 0$ in $\Sigma_{\lambda_0 - \epsilon}$.

Now let K be a closed subset in Σ_{λ_0} such that $|\Sigma_{\lambda_0} \setminus K| < \frac{\delta}{2}$. Since $w_{\lambda_0} < 0$ in Σ_{λ_0}, $w_{\lambda_0 - \epsilon} < 0$ in K, when $\epsilon > 0$ is small. Also, $|\Sigma_{\lambda_0 - \epsilon} \setminus K| < \delta$. Then again by the maximum principle on small domains, $w_{\lambda_0 - \epsilon} \leq 0$ in $\Sigma_{\lambda_0 - \epsilon} \setminus K$. Therefore, $w_{\lambda_0 - \epsilon} < 0$ in $\Sigma_{\lambda_0 - \epsilon} \setminus K$. Therefore, $w_{\lambda_0 - \epsilon} < 0$ in $\Sigma_{\lambda_0 - \epsilon}$. □

Theorem 15. *Suppose $u \in C(\bar{B}_1) \cap C^2(B_1)$ is a positive solution of*

$$\Delta u + f(u) = 0, \qquad \text{in} \qquad B_1, \qquad u = 0, \qquad \text{on} \qquad \partial B_1, \tag{8.38}$$

where f is locally Lipschitz in $\mathbb{R}$. Then u is radially symmetric in B_1 and $\frac{\partial u}{\partial r}(x) < 0$ for $x \neq 0$.

9. Alexandroff maximum principle

The maximum principle for small domains is also called the Alexandroff maximum principle. Suppose Ω is a bounded domain and let

$$L = a_{ij}(x)\partial_i\partial_j + b_i(x)\partial_i + c(x), \tag{9.1}$$

where a_{ij}, b_i, and c are continuous. Also, suppose that the coefficient matrix $a_{ij}(x)$ is positive definite everywhere in Ω, and that

$$0 < \lambda(x)|\xi|^2 \leq a_{ij}(x)\xi_i\xi_j \leq \Lambda(x)|\xi|^2. \tag{9.2}$$

Remark 9. *There is no assumption of uniform ellipticity in* (9.2).

Let $D = det(A)$ and $D^* = D^{1/n}$. Then,

$$0 < \lambda(x) \leq D^*(x) \leq \Lambda(x), \tag{9.3}$$

where D^* is the geometric mean of the eigenvalues of $a_{ij}(x)$, $\lambda > 0$ is the minimum eigenvalue and $\Lambda < \infty$ is the maximum eigenvalue.

Definition 5 (Contact set)**.** *For $u \in C^2(\Omega)$, define*

$$\Gamma^+ = \{y \in \Omega : u(x) \leq u(y) + \nabla u(y) \cdot (x - y) \qquad \textit{for any} \qquad x \in \Omega\}. \tag{9.4}$$

The set Γ^+ is called the upper contact set of u. In this case The Hessian matrix $D^2u = \partial_i\partial_j u$ is nonpositive on Γ^+.

If u is continuous, it is possible to define the contact set

$$\Gamma^+ = \{y \in \Omega : u(x) \leq u(y) + p \cdot (x - y), \qquad \forall x \in \Omega, \qquad \textit{for some} \qquad p = p(y) \in \mathbb{R}^n\}. \tag{9.5}$$

Then u is concave if and only if $\Gamma^+ = \Omega$. If $u \in C^1(\Omega)$, then $p(y) = \nabla u(y)$, and any support plane is the tangent plane to the graph.

Now consider the equation,

$$Lu = f, \qquad \text{in} \qquad \Omega, \qquad f \in C(\Omega). \tag{9.6}$$

Theorem 16. *Suppose $u \in C(\bar{\Omega}) \cap C^2(\Omega)$ satisfies $Lu \geq f$ in Ω with the following conditions:*

$$\frac{|b|}{D^*}, \frac{f}{D^*} \in L^n(\Omega), \qquad \textit{and} \qquad c \leq 0, \qquad \textit{in} \qquad \Omega. \tag{9.7}$$

Then there holds

$$\sup_{\Omega} u \leq \sup_{\partial\Omega} u^+ + C\|\frac{f^-}{D^*}\|_{L^n(\Gamma^+)}, \tag{9.8}$$

where Γ^+ is the upper contact set of u and $C = C(n, diam(\Omega), \|\frac{b}{D^*}\|_{L^n(\Gamma^+)})$. In fact,

$$C = d \cdot \{\exp(\frac{2^{n-2}}{\omega_n n^n}(\|\frac{b}{D^*}\|^n_{L^n(\Gamma^+)} + 1) - 1\}, \tag{9.9}$$

where ω_n is the volume of the unit ball in $\mathbb{R}^n$ and $\overrightarrow{b} = (b_1, ..., b_n)$.

Remark 10. *The integral domain can be replaced by*

$$\Gamma^+ \cap \{x \in \Omega : u(x) > \sup_{\partial\Omega} u^+, \qquad u^+ = \max\{u, 0\}\}. \tag{9.10}$$

We begin with the lemma.

Lemma 16. *Suppose $g \in L^1_{loc}(\mathbb{R}^n)$ is nonnegative. Then for any $u \in C(\bar{\Omega}) \cap C^2(\Omega)$, there holds*

$$\int_{B_{\tilde{M}}(0)} g \leq \int_{\Gamma^+} g(Du)|detD^2u|. \tag{9.11}$$

Here Γ^+ is the upper contact set of u and $\tilde{M} = \frac{(\sup_\Omega u - \sup_{\partial\Omega} u^+)}{d}$ with $d = diam(\Omega)$.

Remark 11. *If A is any positive definite matrix,*

$$det(-D^2u) \leq \frac{1}{D}(\frac{-a_{ij}D_{ij}u}{n})^n, \qquad on \qquad \Gamma^+. \tag{9.12}$$

Therefore,

$$\int_{B_{\tilde{M}}(0)} g \leq \int_{\Gamma^+} g(Du)(\frac{-a_{ij}D_{ij}u}{nD^*})^n. \tag{9.13}$$

Remark 12. *A special case corresponds to $g = 1$. In this case,*

$$\sup_{\Omega} u \leq \sup_{\partial\Omega} u^+ + \frac{d}{\omega_n^{1/n}}(\int_{\Gamma^+} |detD^2u|)^{1/n} \leq \sup_{\partial\Omega} u^+ + \frac{d}{\omega_n^{1/n}}(\int_{\Gamma^+} (-\frac{a_{ij}D_{ij}u}{nD^*})^n)^{1/n}. \tag{9.14}$$

This is Theorem 16 *when $b_i \equiv 0$ and $c \equiv 0$.*

Proof of Lemma 16. Suppose without loss of generality that $u \leq 0$ on $\partial\Omega$. Then let $\Omega^+ = \{u > 0\}$. Then,

$$\int_{Du(\Gamma^+ \cap \Omega^+)} g \leq \int_{\Gamma^+ \cap \Omega^+} g(Du)|det(D^2u)|, \tag{9.15}$$

where $|det(D^2u)|$ is the Jacobian of the map $Du : \Omega \to \mathbb{R}^n$.

Now then, consider $\chi_\epsilon = Du - \epsilon Id : \Omega \to \mathbb{R}^n$, then $D\chi_\epsilon = D^2u - \epsilon I$, which is negative definite on Γ^+. Then by the change of variables formula,

$$\int_{\chi_\epsilon(\Gamma^+ \cap \Omega^+)} g = \int_{\Gamma^+ \cap \Omega^+} g(\chi_\epsilon)|det(D^2u - \epsilon I)|, \tag{9.16}$$

which implies (9.15) when $\epsilon \searrow 0$.

Now it remains to show that $B_{\tilde{M}}(0) \subset Du(\Gamma^+ \cap \Omega^+)$. That is, for any $a \in \mathbb{R}^n$, $|a| < \tilde{M}$, there exists $x \in \Gamma^+ \cap \Omega^+$ such that $a = Du(x)$. Suppose u attains its maximum $m > 0$ at $0 \in \Omega$, that is,

$$u(0) = m = \sup_{\Omega} u. \tag{9.17}$$

Now consider an affine function for $|a| < \frac{m}{d} \equiv \tilde{M}$,

$$L(x) = m + a \cdot x. \tag{9.18}$$

Then $L(x) > 0$ for any $x \in \Omega$, and $L(0) = m$. Since u assumes its minimum at 0, $Du(0) = 0$. There exists x_1 close to 0 such that $u(x_1) > L(x_1) > 0$. Note that $u \leq 0 < L$ on $\partial\Omega$. Therefore, there exists $\tilde{x} \in \Omega$ such that $Du(\tilde{x}) = DL(\tilde{x}) = a$. Then translate the plane $y = L(x)$ to the highest such point. In that case the entire surface $y = u(x)$ lies below the plane, and thus $x \in \Gamma^+$. Also, at this point, $u(x)$ is positive, so $x \in$ (9.10). □

Proof of Theorem 16. We want to choose g so that we can apply Lemma 16. By the Cauchy inequality, since $c \leq 0$ in Ω,

$$\begin{aligned} -a_{ij}D_{ij}u &\leq b_iD_iu + cu - f \leq b_iD_iu - f, \qquad \text{in} \qquad \Omega^+ = \{x : u(x) > 0\}, \\ &\leq |b||Du| + f^- \leq (|b|^n + \frac{(f^-)^n}{\mu^n})^{1/n}(|Du|^n + \mu^n)^{1/n} \cdot (1+1)^{\frac{n-2}{n}}. \end{aligned} \tag{9.19}$$

In particular,

$$(-a_{ij}D_{ij}u)^n \leq (|b|^n + (\frac{f^-}{\mu})^n)(|Du|^n + \mu^n) \cdot 2^{n-2}. \tag{9.20}$$

Now choose $g(p) = \frac{1}{|p|^n + \mu^n}$. By Lemma 16,

$$\int_{B_{\tilde{M}}(0)} g \le \frac{2^{n-2}}{n^n} \int_{\Gamma^+ \cap \Omega^+} \frac{|b|^n + \mu^{-n}(f^-)^n}{D}. \tag{9.21}$$

Now evaluate the left hand side.

$$\int_{B_{\tilde{M}}(0)} g = \omega_n \int_0^{\tilde{M}} \frac{r^{n-1}}{r^n + \mu^n} dr = \frac{\omega_n}{n} \log \frac{\tilde{M}^n + \mu^n}{\mu^n} = \frac{\omega_n}{n} \log(\frac{\tilde{M}^n}{\mu^n} + 1). \tag{9.22}$$

Therefore,

$$\tilde{M}^n \le \mu^n \{\exp\{\frac{2^{n-2}}{\omega_n n^n} [\| \frac{b}{D^*} \|^n_{L^n(\Gamma^+ \cap \Omega^+)} + \mu^{-n} \| \frac{f^-}{D^*} \|^n_{L^n(\Gamma^+ \cap \Omega^+)} \} - 1\}. \tag{9.23}$$

If f is not identically zero then choose $\mu = \| \frac{f^-}{D^*} \|_{L^n(\Gamma^+ \cap \Omega^+)}$. If $f \equiv 0$ then let $\mu \searrow 0$. □

Now consider

$$Lu \equiv a_{ij} D_{ij} u + b_i D_i u + cu, \qquad \text{in} \qquad \Omega, \tag{9.24}$$

where a_{ij} is positive definite, $det(a_{ij}) \ge \lambda$ and $|b_i| + |c| \le \Lambda$. Now we can prove the maximum principle for small domains.

Theorem 17. *Suppose $u \in C(\bar{\Omega}) \cap C^2(\Omega)$ satisfies $Lu \ge 0$ in Ω with $u \le 0$ on $\partial\Omega$. Suppose $diam(\Omega) \le d$. Then there exists $\delta(n, \lambda, \Lambda, d) > 0$ such that if $|\Omega| \le \delta$, then $u \le 0$ in Ω.*

Proof. Split $c = c^+ - c^-$. Then,

$$a_{ij} D_{ij} u + b_i D_i u - c^- u \ge -c^+ u \equiv f. \tag{9.25}$$

Now then, by Theorem 16,

$$\sup_\Omega u \le c(n, \lambda, \Lambda, d) \|c^+ u^+\|_{L^n(\Omega)} \le c(n, \lambda, \Lambda, d) \|c^+\|_{L^\infty} |\Omega|^{1/n} \cdot \sup_\Omega u \le \frac{1}{2} \sup_\Omega u, \tag{9.26}$$

when $|\Omega|$ is small. Therefore, $u \le 0$ in Ω. □

Now we derive some estimates for solutions to quasi-linear equations and fully nonlinear equations, which will be useful going forward.

Proposition 31. *Suppose $u \in C(\bar{\Omega}) \cap C^2(\Omega)$ satisfies*

$$Qu = a_{ij}(x, u, Du) D_{ij} u + b(x, u, Du) = 0, \qquad in \qquad \Omega, \tag{9.27}$$

where $a_{ij}(x, z, p)\xi_i \xi_j > 0$ for any $(x, z, p) \in \Omega \times \mathbb{R} \times \mathbb{R}^n$ and $\xi \in \mathbb{R}^n$. Suppose there exist nonnegative functions $g \in L^n_{loc}(\mathbb{R}^n)$ and $h \in L^n(\Omega)$ such that

$$\frac{|b(x, z, p)|}{nD^*} \le \frac{h(x)}{g(p)}, \qquad for\ any \qquad (x, z, p) \in \Omega \times \mathbb{R} \times \mathbb{R}^n, \tag{9.28}$$

$$\int_\Omega h^n(x) dx < \int g^n(p) dp \equiv g_\infty. \tag{9.29}$$

Then there holds $\sup_\Omega |u| \le \sup_{\partial\Omega} |u| + C diam(\Omega)$, where $C = C(g, h)$.

Proof of Proposition 31. Suppose $Qu \geq 0$ in Ω. Then we have

$$-a_{ij} D_{ij} u \leq b, \qquad \text{in} \qquad \Omega. \tag{9.30}$$

By definition of Γ^{+}, $\{D_{ij} u\}$ is non-positive in Γ^{+}. Thus, $-a_{ij} D_{ij} u \geq 0$, so $b(x, u, Du) \geq 0$ in Γ^{+}. Then in $\Gamma^{+} \cap \Omega^{+}$,

$$\frac{b(x, z, Du)}{nD^{*}} \leq \frac{h(x)}{g(Du)}. \tag{9.31}$$

Then by Lemma 16,

$$\int_{B_{\tilde{M}}(0)} g^{n} \leq \int_{\Gamma^{+} \cap \Omega^{+}} g^{n}(Du)(\frac{-a_{ij} D_{ij} u}{nD^{*}})^{n} \leq \int_{\Gamma^{+} \cap \Omega^{+}} g^{n}(Du)(\frac{b}{nD^{*}})^{n} \leq \int_{\Gamma^{+} \cap \Omega^{+}} h^{n} \leq \int_{\Omega} h^{n}. \tag{9.32}$$

Therefore, by (9.29) there exists $C(g, h)$ such that $\tilde{M} \leq C$. Thus,

$$\sup_{\Omega} u \leq \sup_{\partial \Omega} u^{+} + C diam(\Omega). \tag{9.33}$$

□

The mean curvature equation is a fully nonlinear equation. The prescribed mean curvature equation is given by

$$(1 + |Du|^{2}) \Delta u - D_{i} u D_{j} u D_{ij} u = nH(x)(1 + |Du|^{2})^{3/2}, \qquad \text{for some} \qquad H \in C(\Omega). \tag{9.34}$$

Then,

$$\begin{aligned} a_{ij}(x, z, p) = (1 + |p|^{2}) \delta_{ij} - p_{i} p_{j} \Rightarrow D = (1 + |p|^{2})^{n-1}, \\ b = -nH(x)(1 + |p|^{2})^{3/2}. \end{aligned} \tag{9.35}$$

This implies

$$\frac{|b(x, z, p)|}{nD^{*}} \leq \frac{|H(x)|(1 + |p|^{2})^{3/2}}{(1 + |p|^{2})^{\frac{n-1}{n}}} = |H(x)|(1 + |p|^{2})^{\frac{n+2}{2n}}, \tag{9.36}$$

and

$$g_{\infty} = \int_{\mathbb{R}^{n}} g^{n}(p) dp = \int_{\mathbb{R}^{n}} (1 + |p|^{2})^{-\frac{n+2}{2}} dp = \omega_{n}. \tag{9.37}$$

Corollary 8. *Suppose $u \in C(\bar{\Omega}) \cap C^{2}(\Omega)$ satisfies*

$$(1 + |Du|^{2}) \Delta u - D_{i} u D_{j} u D_{ij} u = nH(x)(1 + |Du|^{2})^{3/2}, \qquad in \qquad \Omega, \tag{9.38}$$

for some $H \in C(\Omega)$. Then if

$$H_{0} \equiv \int_{\Omega} |H(x)|^{n} dx < \omega_{n}, \tag{9.39}$$

we have

$$\sup_{\Omega} |u| \leq \sup_{\partial \Omega} |u| + C diam(\Omega), \tag{9.40}$$

where $C = C(n, H_{0})$.

Now discuss Monge–Ampere equations.

Corollary 9. *Suppose $u \in C(\bar{\Omega}) \cap C^2(\Omega)$ satisfies*

$$det(D^2u) = f(x, u, Du), \qquad in \qquad \Omega, \tag{9.41}$$

for some $f \in C(\Omega \times \mathbb{R} \times \mathbb{R}^n)$. Suppose there exist nonnegative functions $g \in L^1_{loc}(\mathbb{R}^n)$ and $h \in L^1(\Omega)$ such that

$$|f(x, z, p)| \leq \frac{h(x)}{g(p)}, \qquad for\ any \qquad (x, z, p) \in \Omega \times \mathbb{R} \times \mathbb{R}^n, \tag{9.42}$$

$$\int_\Omega h(x)dx < \int_{\mathbb{R}^n} g(p)dp \equiv g_\infty. \tag{9.43}$$

Then there holds

$$\sup_\Omega |u| \leq \sup_{\partial\Omega} |u| + Cdiam(\Omega), \qquad C = C(g, h). \tag{9.44}$$

Corollary 10. *Let $u \in C(\bar{\Omega}) \cap C^2(\Omega)$ satisfy*

$$det(D^2u) = f(x), \qquad in \qquad \Omega, \tag{9.45}$$

for some $f \in C(\bar{\Omega})$. Then there holds

$$\sup_\Omega |u| \leq \sup_{\partial\Omega} |u| + \frac{diam(\Omega)}{\omega_n^{1/n}} (\int_\Omega |f|^n)^{1/n}. \tag{9.46}$$

Proof. Here we take $g = 1$ and then $g_\infty = \infty$. □

Now take the prescribed Gaussian curvature condition.

Corollary 11. *Let $u \in C(\bar{\Omega}) \cap C^2(\Omega)$ satisfy*

$$det(D^2u) = K(x)(1 + |Du|^2)^{\frac{n+2}{2}}, \qquad in \qquad \Omega, \tag{9.47}$$

for some $K \in C(\bar{\Omega})$. Then if

$$K_0 \equiv \int_\Omega |K(x)| < \omega_n, \tag{9.48}$$

$$\sup_\Omega |u| \leq \sup_{\partial\Omega} |u| + C(n, K_0)diam(\Omega). \tag{9.49}$$

Proof. Again we use the fact that $\int (1 + |p|^2)^{-\frac{n+2}{n}} = \omega_n$. □

10. Surfaces with prescribed Gaussian curvature

Now let $\overline{\mathcal{M}}$ be a connected, compact, two–dimensional manifold with nonempty boundary. Let g be the metric on $\overline{\mathcal{M}}$. We wish to construct a conformally related metric whose Gaussian curvature is a given function $K(x)$ on $\overline{\mathcal{M}}$.

Now let $k(x)$ be the Gaussian curvature of g and let $g' = e^{2u}g$. Then the Gauss curvature of g' is given by

$$K(x) = (-\Delta u + k(x))e^{-2u}. \tag{10.1}$$

Then we wish to solve

$$\Delta u = k(x) - K(x)e^{2u} = f(x, u). \tag{10.2}$$

Proposition 32. *If $\overline{\mathcal{M}}$ is a connected, compact, two–dimensional manifold with nonempty boundary $\partial\mathcal{M}$, g a Riemannian metric on $\overline{\mathcal{M}}$, and $K \in C^{\infty}(\overline{\mathcal{M}})$ a given function satisfying*

$$K(x) \leq 0, \qquad on \qquad \mathcal{M}, \tag{10.3}$$

then there exists $u \in C^{\infty}(\overline{\mathcal{M}})$ such that the metric $g' = e^{2u}g$ conformal to g has Gauss curvature K. Given any $v \in C^{\infty}(\partial\mathcal{M})$, there is a unique u satisfying $u = v$ on $\partial\mathcal{M}$.

Proof. Apply Theorem 14 to the Proposition. Notice that we need $K(x) \leq 0$ to obtain $\frac{\partial f}{\partial u} \geq 0$. □

Now suppose that $\mathcal{M}$ is compact and does not have a boundary. For now, we retain the hypothesis $\frac{\partial f}{\partial u} \geq 0$. If $\mathcal{M}$ does not have a boundary, (8.16) no longer holds for constant functions on $\mathcal{M}$. Therefore, suppose that for some $a_j \in \mathbb{R}$,

$$u < a_0 \Rightarrow f(x,u) < 0, \qquad u > a_1 \Rightarrow f(x,u) > 0. \tag{10.4}$$

Then if $\frac{\partial f}{\partial u} > 0$, (10.4) is equivalent to the existence of a function $u = \varphi(x)$ such that $f(x, \varphi(x)) = 0$.

Theorem 18. *If u solves* (8.1) *and $\mathcal{M}$ is compact, then*

$$a_0 \leq u(x) \leq a_1, \tag{10.5}$$

provided (10.4) *holds.*

Proof. Since $\mathcal{M}$ is compact, u achieves a maximum at some $x_0 \in \mathcal{M}$. Then, $\Delta u(x_0) \leq 0$ on $\mathcal{M}$, so $f(x_0, u(x_0)) \leq 0$, and therefore by (10.5), $u \leq a_1$. The other inequality follows similarly. □

To obtain an existence result, we use the method of continuity. We show that for each $\tau \in [0,1]$, there is a smooth solution to

$$\Delta u = (1-\tau)(u-b) + \tau f(x,u) = f_{\tau}(x,u), \qquad b = \frac{a_0 + a_1}{2}. \tag{10.6}$$

This equation is certainly solvable when $\tau = 0$, since $u = b$ is a solution. Now let $J \subset [0,1]$ be the largest interval containing 0 such that (10.6) has a solution for all $\tau \in J$.

Claim 1. *J is closed.*

Proof. Observe that for any $\tau \in [0,1]$, $u < a_0$ implies $f_{\tau}(x,u) < 0$, and $u > a_1$ implies $f_{\tau}(x,u) < 0$. Therefore, any solution must satisfy (10.5). Now let $u_j = u_{\tau_j}$ for $\tau_j \in J$, $\tau_j \nearrow \sigma$. Therefore, $\|u_j\|_{L^{\infty}} \leq a < \infty$, so $g_j(x) = f_{\tau_j}(x, u_j(x))$ is bounded in $C(\mathcal{M})$, and therefore elliptic regularity for the Laplace operator implies

$$\|u_j\|_{C^r(\mathcal{M})} \leq b_r < \infty, \qquad \text{for any} \qquad r < 2. \tag{10.7}$$

The bound (10.7) implies a bound on the C^r norm of g_j, which implies a C^r bound on u for $r < 4$. Iterating this bound implies $u_j \in C^{\infty}(\mathcal{M})$. Any limit point $u \in C^{\infty}(\mathcal{M})$ solves (10.6) with $\tau = \sigma$, so J is closed. □

Claim 2. *J is open in* $[0,1]$.

Proof. To see this, we show that if $\tau_0 < 1$ and $\tau_0 \in J$, then there exists some $\epsilon > 0$ such that $[\tau_0, \tau_0 + \epsilon) \subset J$. To see this, fix k large and define the operator

$$\Psi : [0,1] \times H^k(\mathcal{M}) \rightarrow H^{k-2}(\mathcal{M}), \qquad \Psi(\tau, u) = \Delta u - f_{\tau}(x,u). \tag{10.8}$$

This map is C^1, and its derivative with respect to the second argument is given by

$$D_2 \Psi_{\tau_0}(\tau_0, u)v = Lv, \qquad Lv = \Delta v - A(x)v, \qquad A(x) = 1 - \tau_0 + \tau_0 \partial_u f(x,u). \tag{10.9}$$

Now then, if $\frac{\partial f}{\partial u} \geq 0$, then $A(x) \geq 1 - \tau_0 > 0$, and therefore L is invertible. By the inverse function theorem, $\Psi(\tau, u) = 0$ is solvable for $|\tau - \tau_0| < \epsilon$. □

Therefore, we have proved

Proposition 33. *If $\mathcal{M}$ is a compact manifold without boundary and if $\partial_u f(x,u) \geq 0$ and $f(x,u)$ satisfies* (10.4)*, then* (10.1) *has a unique solution. If $\partial_u f(x,u) > 0$ then the solution is unique.*

Proof. It only remains to prove uniqueness when $\partial_u f(x,u) > 0$. Let u and v be two solutions and let $w = u - v$. Then w solves

$$\Delta w = \lambda(x) w, \qquad \lambda(x) = \frac{f(x,u) - f(x,v)}{u - v} \geq 0. \tag{10.10}$$

Therefore,

$$-\|\nabla w\|_{L^2}^2 = \int \lambda(x) |w(x)|^2 dx, \tag{10.11}$$

which implies that $w = 0$ if $\lambda(x) > 0$ on $\mathcal{M}$. □

It is possible to use the continuity method to solve the Dirichlet problem. Let Ω be a bounded domain on $\mathbb{R}^n$ and let a_{ij}, b_i, and c be defined on Ω with $a_{ij} = a_{ji}$. Then consider the operator L given by

$$Lu = a_{ij}\partial_i\partial_j u + b_i \partial_i u + cu, \qquad \text{in} \qquad \Omega, \tag{10.12}$$

for any $u \in C^2(\Omega)$ and

$$\lambda|\xi|^2 \leq a_{ij}(x)\xi_i\xi_j \leq \Lambda|\xi|^2, \qquad \forall x \in \Omega, \qquad \xi \in \mathbb{R}^n, \qquad \lambda > 0. \tag{10.13}$$

Consider the general existence result for solutions to the Dirichlet problem with $C^{2,\alpha}$ boundary values for general uniformly elliptic equations with C^α coefficients.

Theorem 19. *Let Ω be a bounded $C^{2,\alpha}$ domain in $\mathbb{R}^n$ and let L be a uniformly elliptic operator in Ω with $c \leq 0$ in Ω and a_{ij}, b_i, $c \in C^\alpha(\bar{\Omega})$ for some $\alpha \in (0,1)$. Then for any $f \in C^\alpha(\bar{\Omega})$ and $\varphi \in C^{2,\alpha}(\bar{\Omega})$, there exists a unique solution $u \in C^{2,\alpha}(\bar{\Omega})$ of the Dirichlet problem*

$$Lu = f, \qquad in \qquad \Omega, \qquad u = \varphi, \qquad on \qquad \partial\Omega. \tag{10.14}$$

The crucial step in solving the Dirichlet problem for L to assume that the similar Dirichlet problem for the Laplace operator is solved.

Theorem 20. *Let Ω be a bounded $C^{2,\alpha}$ domain in $\mathbb{R}^n$ and let L be a uniformly elliptic operator in Ω with $c \leq 0$ in Ω and a_{ij}, b_i, $c \in C^\alpha(\bar{\Omega})$ for some $\alpha \in (0,1)$. If the Dirichlet problem for the Poisson equation,*

$$\Delta u = f, \qquad in \qquad \Omega, \qquad u = \varphi, \qquad on \qquad \partial\Omega, \tag{10.15}$$

has a $C^{2,\alpha}(\bar{\Omega})$ solution for $f \in C^\alpha(\bar{\Omega})$ and $\varphi \in C^{2,\alpha}(\bar{\Omega})$, then the Dirichlet problem,

$$Lu = f, \qquad in \qquad \Omega, \qquad u = \varphi, \qquad on \qquad \partial\Omega, \tag{10.16}$$

also has a unique $C^{2,\alpha}(\bar{\Omega})$ solution for all such f and φ.

Proof. Suppose without loss of generality that $\varphi = 0$. Indeed, we can take $Lv = f - L\varphi$ in Ω and $v = 0$ on $\partial\Omega$.

Now consider the family of equations,

$$L_t u \equiv tLu + (1-t)\Delta u = f, \qquad \text{for} \qquad t \in [0,1]. \tag{10.17}$$

Now then, $L_0 = \Delta$, $L_1 = L$,

$$L_t u = a_{ij}^t(x) D_{ij} u + b_i^t(x) D_i u + c^t(x) u, \tag{10.18}$$

$$a_{ij}^t(x)\xi_i\xi_j \geq \min(1,\lambda)|\xi|^2, \tag{10.19}$$

for any $x \in \Omega$, $\xi \in \mathbb{R}^n$, and

$$|a_{ij}^t|_{C^\alpha(\bar{\Omega})}, |b_i^t|_{C^\alpha(\bar{\Omega})}, |c^t|_{C^\alpha(\bar{\Omega})} \leq \max(1,\Lambda). \tag{10.20}$$

Therefore,

$$|L_t u|_{C^\alpha(\bar{\Omega})} \leq C(n,\alpha,\lambda,\Lambda,\Omega)|u|_{C^{2,\alpha}(\Omega)}. \tag{10.21}$$

Then for each $t \in [0,1]$, $L_t : \mathcal{X} \to C^\alpha(\Omega)$ is a bounded linear operator, where $\mathcal{X}$ is a Banach space,

$$\mathcal{X} = \{u \in C^{2,\alpha}(\bar{\Omega}) : u = 0, \qquad \text{on} \qquad \partial\Omega\}. \tag{10.22}$$

Now let I be the collection of $s \in [0,1]$ such that the Dirichlet problem

$$L_s u = f, \qquad \text{in} \qquad \Omega, \qquad u = 0, \qquad \text{on} \qquad \partial\Omega, \tag{10.23}$$

is solvable in $C^{2,\alpha}(\bar{\Omega})$ for any $f \in C^\alpha(\bar{\Omega})$. For $s \in I$, let $u = L_s^{-1} f$ be the unique solution. Then by the maximum principle and global $C^{2,\alpha}$-estimates,

$$|L_s^{-1} f|_{C^{2,\alpha}(\Omega)} \leq C|f|_{C^\alpha(\bar{\Omega})}. \tag{10.24}$$

For any $t \in [0,1]$ and $f \in C^\alpha(\bar{\Omega})$,

$$L_t u = f \qquad \Rightarrow \qquad L_s u = f + (L_s - L_t)u = f + (t-s)(\Delta u - Lu). \tag{10.25}$$

Therefore, $u \in C^{2,\alpha}(\bar{\Omega})$ is a solution to

$$L_t u = f, \qquad \text{in} \qquad \Omega, \qquad u = 0, \qquad \text{on} \qquad \partial\Omega, \tag{10.26}$$

if and only if,

$$u = L_s^{-1}(f + (t-s)(\Delta u - Lu)). \tag{10.27}$$

For $u \in \mathcal{X}$, set

$$Tu = L_s^{-1}(f + (t-s)(\Delta u - Lu)). \tag{10.28}$$

Then, $T : \mathcal{X} \to \mathcal{X}$ is an operator and for any $u, v \in \mathcal{X}$,

$$|Tu - Tv|_{C^{2,\alpha}(\bar{\Omega})} = |(t-s)L_s^{-1}((\Delta - L)(u-v))|_{C^{2,\alpha}(\bar{\Omega})} \leq C|t-s||(\Delta - L)(u-v)|_{C^\alpha(\bar{\Omega})} \leq C|t-s||u-v|_{C^{2,\alpha}(\bar{\Omega})}. \tag{10.29}$$

Therefore, $T : \mathcal{X} \to \mathcal{X}$ is a contraction if $|t-s| < \delta = C^{-1}$. Thus, for any $t \in [0,1]$ with $|t-s| < \delta$, there exists a unique $u \in \mathcal{X}$ such that $u = Tu$,

$$u = L_s^{-1}(f + (t-s)(\Delta u - Lu)), \qquad L_t u = f. \tag{10.30}$$

Therefore, if $|t-s| < \delta$, there exists a unique solution to (10.26). Partition the interval $[0,1]$ into subintervals of length $< \delta$ and argue by induction. Since $0 \in I$, $1 \in I$. $\square$

11. Fixed point theorems and existence results

Recall Brouwer's fixed point theorem.

Proposition 34. *If K is a compact, convex set in a finite dimensional vector space V, and $F : K \to K$ is a continuous map, then F has a fixed point.*

Proof. If K is the closed unit ball in $\mathbb{R}^n$, and $\psi : K \to K$ is a continuous map without fixed point, then let $F(x)$ map x to z, where z is the endpoint of the ray from x through $\psi(x)$ that intersects ∂K. Then F is a continuous retraction of B to ∂B, which violates homotopy theory.

For a general compact K, put an inner product on V and let $B \subset V$ denote a ball that contains K. Let $\psi : B \to K$ map a point x to the point in K closet to x. If x is a fixed point of $F \circ \psi : B \to K \subset B$, then $x \in K$. Since ψ is the identity on K, F has a fixed point. □

Schauder's fixed point theorem is a generalization of the Brouwer fixed point theorem.

Theorem 21 (Schauder's fixed point theorem)**.** *Let $\mathcal{G}$ be a compact, convex set in a Banach space X and let T be a continuous mapping of $\mathcal{G}$ to itself. Then T has a fixed point, that is, there exists some $x \in \mathcal{G}$ such that $Tx = x$.*

Proof. Let $k \in \mathbb{N}$. Since $\mathcal{G}$ is compact, there exists a finite set such that the balls $B_i = B_{i/k}(x_i)$, $i = 1, 2, ..., n$ cover $\mathcal{G}$. Let $\mathcal{G}_k$ be the convex hull of $\{x_1, ..., x_n\}$ and let $J_k : \mathcal{G} \to \mathcal{G}_k$ be defined by

$$J_k(x) = \frac{\sum_i dist(x, \mathcal{G} - B_i) x_i}{\sum_i dist(x, \mathcal{G} - B_i)}. \tag{11.1}$$

Then J_k is continuous on $\mathcal{G}$, and furthermore,

$$\|J_k(x) - x\| \le \frac{\sum_i dist(x, \mathcal{G} - B_i) \|x - x_i\|}{\sum_i dist(x, \mathcal{G} - B_i)} < \frac{1}{k}. \tag{11.2}$$

Now then, $J_k \circ T : \mathcal{G}_k \to \mathcal{G}_k$. Therefore, by the Brouwer fixed point theorem, there exists a $y_k \in \mathcal{G}_k$ such that $J_k \circ T(y_k) = y_k$, $k = 1, 2, ...$ Since $\mathcal{G}$ is compact, there exists some $x \in \mathcal{G}$ such that $y_k \to x \in \mathcal{G}$. For any k,

$$\|y_k - T(y_k)\| = \|J_k \circ T(y_k) - T(y_k)\| < \frac{1}{k}. \tag{11.3}$$

Since T is continuous,

$$\lim_{k \to \infty} y_k = x = Tx, \qquad \text{for some} \qquad x \in \mathcal{G}. \tag{11.4}$$

□

Corollary 12. *Let $\mathcal{G}$ be a closed, convex set in a Banach space X. Suppose T is a map from $\mathcal{G}$ to $\mathcal{G}$ such that $T\mathcal{G}$ is precompact. Then T has a fixed point in $\mathcal{G}$.*

Proof. Let K be the closed, convex hull of $T\mathcal{G}$. Since $T\mathcal{G}$ is precompact, K is compact, so consider $T|_K$, which maps K to itself. □

Corollary 13. *Let B be the open ball in a Banach space V. Let $F : \bar{B} \to V$ be a continuous map such that $F(\bar{B})$ is relatively compact and $F(\partial B) \subset B$. Then F has a fixed point.*

Proof. Define a map $G : \bar{B} \to \bar{B}$ by

$$G(x) = F(x), \qquad \text{if} \qquad \|F(x)\| \le 1, \qquad G(x) = \frac{F(x)}{\|F(x)\|} \qquad \text{if} \qquad \|F(x)\| \ge 1. \tag{11.5}$$

Then $G : \bar{B} \to \bar{B}$ is continuous and $G(\bar{B})$ is relatively compact. By Corollary 12, G has a fixed point, $G(x) = x$. Since $F(\partial B) \subset B$, $\|x\| < 1$, so $F(x) = G(x) = x$. $\square$

Theorem 22 (Leray-Schauder theorem)**.** *Let V be a Banach space and let $F : [0,1] \times V \to V$ be a continuous, compact map such that $F(0, v) = v_0$ is independent of $v \in V$. Suppose there exists $M < \infty$ such that, for all $(\sigma, x) \in [0,1] \times V$,*

$$F(\sigma, x) = x \qquad \textit{implies} \qquad \|x\| < M. \tag{11.6}$$

Then the map $F_1 : V \to V$, $F_1(v) = F(1, v)$, has a fixed point.

Remark 13. *For example, we could take the mapping $F(\sigma, x) = \sigma F(x)$.*

Proof. Suppose without loss of generality that $v_0 = 0$ and $M = 1$. Let B be the open ball in V. Given $\epsilon \in (0,1]$, define the map $G_\epsilon : \bar{B} \to V$ by

$$\begin{aligned} G_\epsilon(x) &= F(\frac{1 - \|x\|}{\epsilon}, \frac{x}{\|x\|}), \qquad \text{if} \qquad 1 - \epsilon \leq \|x\| \leq 1, \\ &= F(1, \frac{x}{1-\epsilon}), \qquad \text{if} \qquad \|x\| \leq 1 - \epsilon. \end{aligned} \tag{11.7}$$

Observe that $G_\epsilon(\partial B) = 0$, and that for each $\epsilon \in (0,1]$, $G_\epsilon(V)$ is precompact. Therefore, Corollary 13 implies that G_ϵ has a fixed point $x(\epsilon)$.

For each k, let $x_k = x(\frac{1}{k})$, and set

$$\begin{aligned} \sigma_k &= k(1 - \|x_k\|), \qquad \text{if} \qquad 1 - \frac{1}{k} \leq \|x_k\| \leq 1, \\ &= 1 \qquad \text{if} \qquad \|x_k\| \leq 1 - \frac{1}{k}. \end{aligned} \tag{11.8}$$

Therefore, $\sigma_k \in (0,1]$, and $F(\sigma_k, \frac{x_k}{C_k}) = x_k$, where C_k is the denominator of (11.7). Furthermore, since F is compact and (11.6) holds, $(\sigma_k, x_k) \to (\sigma, x)$.

Next, $\sigma = 1$. Indeed, if $\sigma < 1$, then $\|x_k\| \geq 1 - \frac{1}{k}$ for k large, which implies $\|x\| = 1$ and $F(\sigma, x) = x$, which contradicts (11.6). Therefore, $\sigma_k \to 1$ and $F(1, x) = x$. $\square$

Remark 14. *Suppose T is a compact mapping of a Banach space X into itself. For some $\sigma \in (0,1]$, the map σT possesses a fixed point. Since $T(\overline{B_1})$ is compact in X, there exists $A \geq 1$ such that $\|Tx\| \leq A$ for all $x \in \overline{B_1}$. Thus, the mapping σT with $\sigma = \frac{1}{A}$ maps $\overline{B_1}$ into itself, and therefore, by the Schauder fixed point theorem, Theorem 21, σTx has a fixed point. Also, if $\|x\| < M$ for any fixed point of σTx, then for any $\sigma \in [0,1]$, the mapping σTx has a fixed point.*

Returning to the problem

$$\Delta u = f(x, u), \tag{11.9}$$

suppose $f(x,u) < 0$ for $u < a_0$ and $f(x,u) > 0$ if $u > a_1$, but $\frac{\partial f}{\partial u} > 0$ need not hold. Alter $f(x,u)$ on $u \leq a_0$ and on $u \geq a_1$ to a smooth $g(x,u)$ satisfying $g(x,u) = -\kappa_0 < 0$ for $u \leq a_0 - \delta$ and $g(x,u) = \kappa_1 > 0$ for $u \geq a_1 + \delta$, for some $\delta > 0$.We want to show that for each $\tau \in [0,1]$,

$$\Delta u = (1 - \tau)(u - b) + \tau g(x,u) = g_\tau(x,u), \tag{11.10}$$

is solvable, with solution satisfying

$$a_0 \leq u_\tau(x) \leq a_1. \tag{11.11}$$

Doing some algebra,

$$u = (\Delta - 1)^{-1}(g_\tau(x,u) - u) = \Phi_\tau(u). \tag{11.12}$$

Each Φ_τ is a continuous and compact map on $C(\mathcal{M})$,

$$\Phi_\tau : C(\mathcal{M}) \to C(\mathcal{M}), \tag{11.13}$$

with continuous dependence on τ. For solvability, we can use the Leray–Schauder fixed point theorem, Theorem 22. Indeed, when $\tau = 0$,

$$\Phi_0(u) = -(\Delta - 1)^{-1}b = b, \tag{11.14}$$

which is independent of u. Meanwhile, if u solves (11.12), then u also solves (11.10), so (11.11) holds. Therefore,

$$u = \Phi_\tau(u) \Rightarrow \|u\|_{C(\mathcal{M})} \leq A = \max\{|a_0|, |a_1|\}. \tag{11.15}$$

Applying Theorem 22, (11.12) is solvable for all $\tau \in [0,1]$. Therefore,

Theorem 23. *If $\mathcal{M}$ is a compact manifold without boundary and if the function $f(x,u)$ satisfies $f(x,u) < 0$ when $u < a_0$ and $f(x,u) > 0$ if $u > a_1$, then (11.9) has a smooth solution satisfying $a_0 \leq u(x) \leq a_1$.*

The equation

$$\Delta u = k(x) - K(x)e^{2u} = f(x,u), \tag{11.16}$$

satisfies the hypotheses of Theorem 23 when $k(x) < 0$ and $K(x) < 0$.

In higher dimensions, when $dim\mathcal{M} = n \geq 3$, we alter the metric by

$$g' = u^{\frac{4}{n-2}}g. \tag{11.17}$$

The scalar curvatures σ and S of the metrics g and g' are related by

$$S = u^{-\alpha}(\sigma u - \gamma\Delta u), \qquad \gamma = 4\frac{n-1}{n-2}, \qquad \alpha = \frac{n+2}{n-2}, \tag{11.18}$$

where Δ is the Laplacian for the metric g. Obtaining the scalar curvature S for g' is equivalent to solving

$$\gamma\Delta u = \sigma(x)u - S(x)u^\alpha, \tag{11.19}$$

for a smooth positive function u. Since $\gamma > 1$ and $\alpha > 1$, in the case when $\sigma(x) < 0$ and $S(x) < 0$, there exist $0 < a_0 < a_1 < \infty$ that satisfy $f(x,u) < 0$ when $u < a_0$ and $f(x,u) > 0$ when $u > a_1$. Then,

Proposition 35. *Let $\mathcal{M}$ be a compact manifold of dimension $n \geq 2$. Let g be a Riemannian metric on $\mathcal{M}$ with scalar curvature σ. If both σ and S are negative functions in $C^\infty(\mathcal{M})$, then there exists a conformally equivalent metric g' on $\mathcal{M}$ with scalar curvature S.*

We can also generalize to

$$\gamma\Delta u = B(x)u^\beta + \sigma(x)u - A(x)u^\alpha, \qquad \beta < 1 < \alpha. \tag{11.20}$$

It is possible that $\beta < 0$. Then we have $f(x,u) < 0$ if $u < a_0$, $f(x,u) > 0$ if $u > a_1$, if we assume $A < 0$ on $\mathcal{M}$, but only $B \leq 0$ on $\mathcal{M}$, provided $\sigma(x) < 0$ on $\{x \in \mathcal{M} : B(x) = 0\}$.

We can apply fixed point results to the minimal surface equation. For $0 < \beta < 1$, consider the Banach space $X = C^{1,\beta}(\bar{\Omega})$ where Ω is a $C^{2,\alpha}$ bounded domain in $\mathbb{R}^n$. Now let L be an operator given by

$$Lu = a^{ij}(x, u, \nabla u) u_{x_i x_j} + b(x, u, \nabla u). \tag{11.21}$$

Assume L is elliptic in $\bar{\Omega}$, that is, $a^{ij}(x, \zeta, p)$ is positive definite for all $(x, \zeta, p) \in \bar{\Omega} \times \mathbb{R} \times \mathbb{R}^n$. Also assume that for some $\alpha \in (0, 1)$ that $a^{ij}, b \in C^{\alpha}(\bar{\Omega} \times \mathbb{R} \times \mathbb{R}^n)$, and let $\phi \in C^{2,\alpha}(\partial\Omega)$. Now, for any $v \in C^{1,\beta}(\bar{\Omega}) = X$, let $u = Tv$ be the unique solution in $C^{2,\alpha\beta}(\bar{\Omega})$ of the Dirichlet problem

$$a^{ij}(x, v, Dv) u_{x_i x_j} + b(x, v, Dv) = 0, \qquad \text{in} \qquad \Omega, \qquad u|_{\partial\Omega} = \phi, \qquad \text{on} \qquad \partial\Omega. \tag{11.22}$$

The solvability of $Lu = 0$ in Ω with $u = \phi$ on $\partial\Omega$ in the space $C^{2,\alpha}(\bar{\Omega})$ in the space $C^{2,\alpha}(\bar{\Omega})$ is equivalent to the solvability of $Tu = u$ in X. Now let

$$L_{\sigma} u = a^{ij}(x, u, Du) u_{x_i x_j} + \sigma b(x, u, \nabla u). \tag{11.23}$$

Then $u = \sigma T u$ is the same as $L_{\sigma} u = 0$ in Ω and $u = \sigma\phi$ on $\partial\Omega$. By the Leray–Schauder fixed point theorem,

Theorem 24. *Let Ω, ϕ, and L be as above. If, for some $\beta > 0$, there is a constant M independent of u and σ such that for every $C^{2,\alpha}(\bar{\Omega})$ solution of the Dirichlet problem*

$$L_{\sigma} u = 0, \qquad \text{in} \qquad \Omega, \qquad u = \sigma\phi, \tag{11.24}$$

satisfies

$$\|u\|_{C^{1,\beta}(\bar{\Omega})} < M, \tag{11.25}$$

then it follows that the Dirichlet problem $Lu = 0$ in Ω with $u = \phi$ on $\partial\Omega$ is solvable in $C^{2,\alpha}(\bar{\Omega})$.

The assumptions in the previous theorem can be verified for the minimal surface equation. Consider the case where Ω is a uniformly convex, $C^{2,\alpha}$ bounded domain in $\mathbb{R}^n$, $\phi \in C^{2,\alpha}(\Omega)$, and

$$div(\frac{\nabla u}{\sqrt{1 + |\nabla u|^2}}) = 0, \qquad \text{in} \qquad \Omega, \qquad u|_{\partial\Omega} = \phi, \qquad \text{on} \qquad \partial\Omega. \tag{11.26}$$

Suppose u is a $C^{2,\alpha}$ solution of (11.26). Then the maximum principle implies

$$\|u\|_{L^{\infty}(\bar{\Omega})} \leq \|\phi\|_{L^{\infty}(\partial\Omega)} \equiv C_0 < \infty. \tag{11.27}$$

By the uniform convexity of $\partial\Omega$ and the $C^{2,\alpha}$ regularity of ϕ, it is possible show that there exist linear functions $l_{x_0}^{\pm}(x)$ such that

$$l_{x_0}^{\pm}(x_0) = \phi(x_0), \qquad \text{and} \qquad l_{x_0}^{-}(x) \leq \phi(x) \leq l_{x_0}^{+}(x), \qquad \text{for all} \qquad x \in \partial\Omega. \tag{11.28}$$

Since linear functions are solutions of $div(\frac{\nabla u}{\sqrt{1+|\nabla u|^2}}) = 0$ in Ω, then from the maximum principle,

$$l_{x_0}^{-}(x) \leq u(x) \leq l_{x_0}^{+}(x), \qquad x \in \bar{\Omega}. \tag{11.29}$$

In particular, $|\nabla u(x_0)| \leq \max |\nabla l_{x_0}^{\pm}(x_0)| \equiv C_1 < \infty$.

If u is a $C^{2,\alpha}$ solution of (11.26), then $u_{\alpha} = \frac{\partial}{\partial x_{\alpha}} u$ satisfies

$$\frac{\partial}{\partial x_i}(F_{P_i P_j}(Du) u_{aj}) = 0. \tag{11.30}$$

Here, $\sqrt{1 + |\nabla u|^2}$, so $(F_{P_i P_j}(Du)) > 0$. Thus, u_{α} satisfies the maximum principle, so

$$\|\nabla u\|_{L^{\infty}(\Omega)} \leq \|\nabla u\|_{L^{\infty}(\partial\Omega)} \leq C_1 < \infty. \tag{11.31}$$

Therefore, by (11.26), (11.30), and (11.31), from De Giorgi–Moser theory,

$$\|\nabla u\|_{C^\beta(\bar\Omega)} \le C(C_0, C_1, C_2) < \infty, \qquad C_2 = \|\phi\|_{C^{2,\alpha}(\partial\Omega)}. \tag{11.32}$$

Rewriting (11.26),

$$\Delta u - \frac{u_i u_j}{1+|\nabla u|^2} u_{ij} = 0. \tag{11.33}$$

Combining (11.32) and (11.33) and the Schauder estimates,

$$\|u\|_{C^{2,\beta}(\bar\Omega)} \le C(C_0, C_1, C_2, \Omega), \tag{11.34}$$

where $0 < \beta \le \alpha$.

12. Direct methods in the calculus of variations

We turn now to the question of existence of minima or other stationary points of functionals of the form

$$I(u) = \int_\Omega F(x, u, \nabla u) dV(x), \tag{12.1}$$

over a set of functions $\{u \in B : u = g \quad \text{on} \quad \partial\Omega\}$, where B is a Banach space and g is a smooth function on $\partial\Omega$. Let $\bar\Omega$ be a compact Riemannian manifold with boundary and suppose

$$F : \mathbb{R}^N \times (\mathbb{R}^N \otimes T^*\bar\Omega) \to \mathbb{R}, \qquad \text{is continuous.} \tag{12.2}$$

Let

$$V = \{u \in H^1(\Omega, \mathbb{R}^N) : u = g \quad \text{on} \quad \partial\Omega\}. \tag{12.3}$$

Assume that for each $x \in \bar\Omega$,

$$F(x, \cdot, \cdot) : \mathbb{R}^N \times (\mathbb{R}^N \otimes T_x^*\bar\Omega) \to \mathbb{R}, \qquad \text{is convex.} \tag{12.4}$$

Also assume that

$$A_0|\xi|^2 - B_0|u| - C_0 \le F(x, u, \xi), \tag{12.5}$$

and also suppose that

$$|F(x, u, \xi) - F(x, v, \zeta)| \le C(|u - v| + |\xi - \zeta|)(|\xi| + |\zeta| + 1). \tag{12.6}$$

Proposition 36. *Suppose Ω is connected with nonempty boundary. Also suppose $I(u) < \infty$ for some $u \in V$. Under the hypotheses* (12.2)–(12.6)*, I has a minimum on V.*

Proof. By (12.5),

$$I(u) \ge A_0\|\nabla u\|_{L^2}^2 - B_0\|u\|_{L^1} - C_0|\Omega|. \tag{12.7}$$

Therefore, following the proof of Lemma 14, $I(u)$ is bounded below. Furthermore, by (12.6), for $\|u\|_{H^1}, \|v\|_{H^1} \le L$,

$$I(u) - I(v) \lesssim_L \|u - v\|_{H^1}. \tag{12.8}$$

Finally, by (12.4), $I(u)$ is convex. □

Proposition 36 is a special case of a more general result.

Proposition 37. *Let V be a closed, convex subset of a reflexive Banach space W, and let $\Phi : V \to \mathbb{R}$ be a continuous map satisfying:*

$$\inf_V \Phi = \alpha_0 \in (-\infty, \infty), \tag{12.9}$$

$$\exists b > \alpha_0 \quad \text{such that} \quad \Phi^{-1}([\alpha_0, b]) \quad \text{is bounded in} \quad W, \tag{12.10}$$

$$\forall y \in (\alpha_0, b], \quad \Phi^{-1}([\alpha_0, y]) \quad \text{is convex.} \tag{12.11}$$

Then there exists $v \in V$ such that $\Phi(v) = \alpha_0$.

Proof. For any $0 < \epsilon \leq b - a_0$, $K_\epsilon = \{u \in V : \alpha_0 \leq I(u) \leq \alpha_0 + \epsilon\}$ is weakly compact. Then $\cap_{\epsilon > 0} K_\epsilon = K_0 \neq \emptyset$. □

It is possible to generalize the above result for the Sobolev space $W^{1,p}$, where $1 < p < \infty$.

Proposition 38. *Assume Ω is connected, with nonempty boundary. Take $1 < p < \infty$ and assume that $I(u) < \infty$ for some $u \in V$. If* (12.2), (12.4) *hold, along with*

$$V = \{u \in H^{1,p}(\Omega; \mathbb{R}^N) : u = g \quad \text{on} \quad \partial\Omega\}, \tag{12.12}$$

$$A_0|\xi|^p - B_0|u| - C_0 \leq F(x, u, \xi), \tag{12.13}$$

and

$$|F(x, u, \xi) - F(x, v, \zeta)| \leq C(|u - v| + |\xi - \zeta|)(|\xi| + |\zeta| + 1)^{p-1}. \tag{12.14}$$

We can replace (12.4) by a hypothesis of convexity in the last section.

Proposition 39. *Make the hypotheses of Proposition* 36*, or more generally of Proposition* 38*, by weaken* (12.4) *to the hypothesis that*

$$F(x, u, \cdot) : \mathbb{R}^N \otimes T_x^*\bar{\Omega} \to \mathbb{R} \qquad \text{is convex,} \tag{12.15}$$

for each $(x, u) \in \bar{\Omega} \times \mathbb{R}^N$. Then I has a minimum on V.

Proof. First observe that (12.13) combined with Poincare's inequality implies that $-\infty < \alpha_0 = \inf_V I(u)$. Also,

$$B = \{u \in V : I(u) \leq \alpha_0 + 1\} \qquad \text{is bounded in} \qquad H^{1,p}(\Omega, \mathbb{R}^N). \tag{12.16}$$

Now then, choose $u_j \in B$ such that $I(u_j) \searrow \alpha_0$. Assume

$$u_j \rightharpoonup u, \qquad \text{weakly in} \qquad H^{1,p}(\Omega, \mathbb{R}^N), \tag{12.17}$$

so therefore $u_j \to u$ strongly in $L^p(\Omega, \mathbb{R}^N)$. We need to show that $I(u) = \alpha_0$.

Set

$$\Phi(u, v) = \int_\Omega F(x, u, v) dV(x). \tag{12.18}$$

Setting $v_j = \nabla u_j$,

$$\Phi(u_j, v_j) \to \alpha_0. \tag{12.19}$$

Also, $v_j \rightharpoonup v = \nabla u$ weakly in $L^p(\Omega, \mathbb{R}^N \otimes T_x^*\bar{\Omega})$. Now then, by (12.14),

$$|\Phi(u_j, v_j) - \Phi(u, v_j)| \leq C \int_\Omega |u_j - u|(|v_j| + 1)^{p-1} dV(x) \leq C' \|u_j - u\|_{L^p(\Omega)}, \tag{12.20}$$

which by (12.19) implies that

$$\Phi(u, v_j) \to \alpha_0. \tag{12.21}$$

Now, by (12.5), (12.6) and (12.15),

$$\mathcal{K}_\epsilon = \{w \in L^p(\Omega, \mathbb{R}^N \otimes T^*\bar{\Omega}) : \Phi(u, w) \leq \alpha_0 + \epsilon\}, \tag{12.22}$$

is a closed, convex subset of $L^p(\Omega, \mathbb{R}^N \otimes T^*)$. Therefore, $\mathcal{K}_\epsilon$ is weakly compact, provided it is nonempty. Also, $v_j \in \mathcal{K}_{\epsilon_j}$ with $\epsilon_j \searrow 0$, so $v \in \mathcal{K}_0$. Therefore, $\Phi(u, v) \leq \alpha_0$ holds. □

Proposition 40. *Let* $1 < p < \infty$ *and let* $F(x, u, \xi)$ *satisfy the hypotheses of Proposition* 39*. If* S *is any subset of* V *that is closed in the weak topology of* $H^{1,p}(\Omega, \mathbb{R}^N)$*, it follows that* $I|_S$ *has a minimum in* S*.*

Proof. Same argument as in the proof of Proposition 39. □

Suppose that $X \subset \mathbb{R}^N$ is a closed subset and that

$$S = \{u \in V : u(x) \in X \qquad \text{for a.e.} \qquad x \in \Omega\}. \tag{12.23}$$

For example, we could take X to be a compact Riemannian manifold isometrically embedded into $\mathbb{R}^N$, and let $F(x, u, \nabla u) = |\nabla u|^2$. The minimum is a harmonic map, $u : \Omega \to X$,

$$\Delta u - \Gamma(u)(\nabla u, \nabla u) = 0. \tag{12.24}$$

Harmonic maps can be generalized to the study of "liquid crystals". Take

$$F(x, u, \nabla u) = a_1|\nabla u|^2 + a_2(div(u))^2 + a_3(u \cdot curl(u))^2 + a_4|u \times curl(u)|^2, \tag{12.25}$$

where the coefficients a_j are positive constants. Then we minimize the functional $\int_\Omega F(x, u, \nabla u)dV(x)$ over a set S of the form (12.23) with $X = S^2 \subset \mathbb{R}^3$,

$$S = \{u \in H^1(\Omega, \mathbb{R}^3) : |u(x)| = 1 \qquad \text{a.e. on} \qquad \Omega, \qquad u = g, \qquad \text{on} \qquad \partial\Omega\}. \tag{12.26}$$

In this case,

$$F(x, u, \xi) = \sum_{j,\alpha} b_{j\alpha}(u)\xi_{j\alpha}^2, \qquad b_{j,\alpha}(u) \geq a_1 > 0, \tag{12.27}$$

where $b_{j,\alpha}(u)$ is a polynomial of degree two in u. This function is convex in ξ, but does not satisfy (12.6). Instead,

$$|\Phi(u_j, v_j) - \Phi(u, v_j)| \leq C \int_\Omega |u_j - u||v_j|^2 dV(x). \tag{12.28}$$

Theorem 25. *Assume* Ω *is connected with nonempty boundary. Take* $1 < p < \infty$ *and set*

$$V = \{u \in H^{1,p}(\Omega, \mathbb{R}^N) : u = g, \qquad \text{on} \qquad \partial\Omega\}. \tag{12.29}$$

Assume $I(u) < \infty$ *for some* $u \in V$ *and that* $F(x, u, \xi)$ *is smooth in its arguments, satisfies the convexity condition* (12.15)*, and also,*

$$A_0|\xi|^p \leq F(x, u, \xi), \tag{12.30}$$

for some $A_0 > 0$*. Then* I *has a minimum on* V*.*

If S *is a closed subset of* V *that is closed under the weak topology of* $H^{1,p}(\Omega, \mathbb{R}^N)$*, then* $I|_S$ *has a minimum in* S*.*

Proof. Equation (12.30) clearly implies that $0 \leq \alpha_0 = \inf_S I(u)$. Choose B as in (12.16) and choose $u_j B \cap S$ such that $I(u_j) \to \alpha_0$, $u_j \rightharpoonup u$ weakly in $H^{1,p}(\Omega, \mathbb{R}^N)$. Then passing to a subsequence, assume $u_j \to u$ a.e. on Ω. We need to show that

$$\int_{\Omega} F(x, u, \nabla u) dV(x) \leq \alpha_0. \tag{12.31}$$

By Egorov's theorem, there exist measurable sets $E_\nu \supset E_{\nu+1} \supset ...$ in Ω of measure $< 2^{-\nu}$, such that $u_j \to u$ uniformly on $\Omega \setminus E_\nu$. We can also arrange that

$$|u(x)| + |\nabla u(x)| \leq C2^{\nu}, \qquad \text{for} \qquad x \in \Omega \setminus E_\nu. \tag{12.32}$$

Now then,

$$\int_{\Omega \setminus E_\nu} F(x, u, \nabla u) = \int_{\Omega \setminus E_\nu} F(x, u_j, \nabla u_j) dV(x) + \int_{\Omega \setminus E_\nu} [F(x, u_j, \nabla u) - F(x, u_j, \nabla u_j)] dV(x) + \int_{\Omega \setminus E_\nu} [F(x, u, \nabla u) - F(x, u_j, \nabla u)] dV(x). \tag{12.33}$$

Now then, since F is convex in ξ,

$$F(x, u_j, \nabla u) - F(x, u_j, \nabla u_j) \leq D_\xi F(x, u_j, \nabla u) \cdot (\nabla u - \nabla u_j). \tag{12.34}$$

For each ν,

$$D_\xi F(x, u_j, \nabla u) \to D_\xi F(x, u, \nabla u), \qquad \text{uniformly on} \qquad \Omega \setminus E_\nu, \tag{12.35}$$

while $\nabla u - \nabla u_j$ weakly in $L^p(\Omega, \mathbb{R}^n)$, so

$$\lim_{j \to \infty} \int_{\Omega \setminus E_\nu} [F(x, u_j, \nabla u) - F(x, u_j, \nabla u_j)] dV(x) = 0. \tag{12.36}$$

Finally, $F(x, u, \nabla u) - F(x, u_j, \nabla u) \to 0$ uniformly on $\Omega \setminus E_\nu$. Therefore,

$$\int_{\Omega \setminus E_\nu} F(x, u, \nabla u) dV(x) \leq \limsup_{j \to \infty} \int_{\Omega} F(x, u_j, \nabla u_j) dV(x) \leq \alpha_0. \tag{12.37}$$

Taking $\nu \to \infty$ proves (12.31). □

There are variants of the above result.

Proposition 41. *Assume that F is smooth in (x, u, ξ),*

$$F(x, u, \xi) \geq 0, \tag{12.38}$$

and that

$$F(x, u, \cdot) : \mathbb{R}^N \otimes T_x^* \bar{\Omega} \to \mathbb{R}, \qquad \textit{is convex}, \tag{12.39}$$

for each x, u. Suppose

$$u_\nu \rightharpoonup u, \qquad \textit{weakly in} \qquad H^{1,1}_{loc}(\Omega, \mathbb{R}^N). \tag{12.40}$$

Then,

$$I(u) \leq \liminf_{\nu \to \infty} I(u_\nu). \tag{12.41}$$

13. Surfaces with negative curvature

Recall that if g and g' are conformally related,

$$g' = e^{2u} g, \tag{13.1}$$

then K and k are related by

$$K(x) = e^{-2u}(-\Delta u + k(x)), \tag{13.2}$$

where Δ is the Laplace operator for the original metric g. Then we wish to solve the PDE,

$$\Delta u = k(x) - K(x)e^{2u}. \tag{13.3}$$

By the Gauss–Bonnet formula,

$$\int_{\mathcal{M}} k(x)dV(x) = \int_{\mathcal{M}} K(x)e^{2u}dV(x) = 2\pi\chi(\mathcal{M}), \tag{13.4}$$

it is not possible to arrange that $K < 0$ be the curvature of $\mathcal{M}$ if $\mathcal{M}$ is diffeomorphic to the sphere or the torus, since

$$\chi(S^2) = 2, \qquad \chi(\mathbb{T}^2) = 0. \tag{13.5}$$

In fact, this is the only obstruction.

Theorem 26. *If $\mathcal{M}$ is a compact surface satisfying $\chi(\mathcal{M}) < 0$ with given Riemannian metric g, then for any negative $K \in C^\infty(\mathcal{M})$, the equation* (13.3) *has a solution, so $\mathcal{M}$ has a metric, conformal to g, with Gaussian curvature $K(x)$.*

We produce the solution as an element where the function,

$$F(u) = \int_{\mathcal{M}} (\frac{1}{2}|du|^2 + k(x)u)dV, \tag{13.6}$$

achieves a minimum on the set

$$S = \{u \in H^1(\mathcal{M}) : \int_{\mathcal{M}} K(x)e^{2u}dV = 2\pi\chi(\mathcal{M})\}. \tag{13.7}$$

Lemma 17. *The set S is a nonempty C^1–submanifold of $H^1(\mathcal{M})$ if $K < 0$ and $\chi(\mathcal{M}) < 0$.*

Proof. Set $\Phi(u) = e^{2u}$. By Trudinger's inequality,

$$\Phi : H^1(\mathcal{M}) \to L^p(\mathcal{M}), \tag{13.8}$$

for all $p < \infty$. Indeed, using the estimate,

$$\|u\|_{L^p(\mathbb{R}^n)} \leq C_n p^{1/2} \|u\|_{H^{n/2}(\mathbb{R}^n)}, \tag{13.9}$$

so

$$e^{2u} = \sum_{k=0}^{\infty} \frac{1}{k!} u^k, \tag{13.10}$$

which implies

$$\int_{\mathcal{M}} e^{2u} dV(x) \leq C \sum_{k=0}^{\infty} \frac{(k/2)^k}{k!} \|u\|^k_{H^1(\mathcal{M})} < \infty. \tag{13.11}$$

Since $|e^{2u}|^p = e^{2pu}$, (13.8) holds.

Now then, Φ is differentiable at each $u \in H^1(\mathcal{M})$, and

$$D\Phi(u)v = 2e^{2u}v, \qquad D\Phi(u) : H^1(\mathcal{M}) \to L^1(\mathcal{M}). \tag{13.12}$$

Furthermore,
(13.13)

$$\|(D\Phi(u) - D\Phi(w))v\|_{L^1(\mathcal{M})} \leq 2\int_{\mathcal{M}} |v||e^{2u} - e^{2w}|dV$$

$$\leq 2(\int |v|^4 dV)^{1/4}(\int |u-w|^4 dV)^{1/4}(\int e^{4|u|+4|w|}dV)^{1/2} \leq C(\|u\|_{H^1}, \|v\|_{H^1})\|v\|_{H^1}\|u-w\|_{H^1}.$$

Remark 15. *The first inequality follows from Taylor's formula.*

Therefore, the map $\Phi : H^1(\mathcal{M}) \to L^1(\mathcal{M})$ is a C^1 map. Consequently, if $J(u) = \int_{\mathcal{M}} Ke^{2u}dV$ implies $J : H^1(\mathcal{M}) \to \mathbb{R}$ is a C^1 map.

Furthermore, $DJ(u) = 2Ke^{2u}$ belongs to $H^{-1}(\mathcal{M}) \equiv \mathcal{L}(H^1(\mathcal{M}), \mathbb{R})$, so $DJ(u) \neq 0$ on S. Therefore, by the implicit function theorem, S is a C^1 submanifold of $H^1(\mathcal{M})$. Furthermore, if $K < 0$ and $\chi(\mathcal{M}) < 0$ then there is a constant function in S, so $S \neq \emptyset$. □

Theorem 27. *Suppose $F : S \to \mathbb{R}$, defined by* (13.6)*, assumes a minimum at $u \in S$. Then u solves* (13.3)*, provided that the hypothesis of Theorem* 26 *holds.*

Proof. The map $F : S \to \mathbb{R}$ is a C^1 map. If $\gamma(s)$ is a C^1 curve in S with $\gamma(0) = u$, $\gamma'(0) = v$,

$$0 = \frac{d}{ds}F(u+sv)|_{s=0} = \int_{\mathcal{M}} [(du, dv) + k(x)v]dV = \int_{\mathcal{M}} (-\Delta u + k(x))vdV. \tag{13.14}$$

Since v is tangent to S at u,

$$\int_{\mathcal{M}} Ke^{2(u+sv)}dV = 2\pi\chi(\mathcal{M}) + O(s^2), \tag{13.15}$$

which is equivalent to

$$\int_{\mathcal{M}} vK(x)e^{2u}dV(x) = 0. \tag{13.16}$$

Therefore, if $u \in S$ is a minimum for F,

$$v \in H^1(\mathcal{M}), \qquad \int_{\mathcal{M}} vKe^{2u}dV = 0 \qquad \text{implies} \qquad \int_{\mathcal{M}} (-\Delta u + k(x))vdV = 0. \tag{13.17}$$

Therefore, $-\Delta u + k(x)$ is parallel to Ke^{2u} in $H^1(\mathcal{M})$;

$$-\Delta u + k(x) = \beta Ke^{2u}, \qquad \text{for some} \qquad \beta \in \mathbb{R}. \tag{13.18}$$

Integrating both sides on $\mathcal{M}$,

$$\int_{\mathcal{M}} k(x)dV(x) = \chi(\mathcal{M}) = \beta\int_{\mathcal{M}} K(x)e^{2u}dV(x) = \beta\chi(\mathcal{M}), \tag{13.19}$$

so when $\chi(\mathcal{M}) \neq 0$, $\beta = 1$. By Trudinger's estimate, the right hand side belongs to $L^2(\mathcal{M})$, so $u \in H^2(\mathcal{M})$, which implies $e^{2u} \in H^2(\mathcal{M})$, so by induction, $u \in C^\infty(\mathcal{M})$. □

Now we show that F has a minimum on S, for $K < 0$ and $\chi(\mathcal{M}) < 0$. For $u \in H^1(\mathcal{M})$, let

$$u = u_0 + \alpha, \tag{13.20}$$

where $\alpha = (Area(\mathcal{M}))^{-1} \int_{\mathcal{M}} udV$ is the mean value of u, and

$$u_0 \in \bar{H}(\mathcal{M}) = \{v \in H^1(\mathcal{M}) : \int_{\mathcal{M}} vdV(x) = 0\}. \tag{13.21}$$

Then $u \in S$ if and only if,

$$e^{2\alpha} \int_{\mathcal{M}} K(x) e^{2u_0} dV(x) = 2\pi\chi(\mathcal{M}), \tag{13.22}$$

which is equivalent to

$$\alpha = \frac{1}{2} \log[2\pi\chi(\mathcal{M}) / \int K e^{2u_0} dV(x)]. \tag{13.23}$$

Therefore,

$$F(u) = \int_{\mathcal{M}} (\frac{1}{2}|du_0|^2 + ku_0) dV + \pi\chi(\mathcal{M})\{\log(2\pi|\chi(\mathcal{M})|) - \log|\int_{\mathcal{M}} K e^{2u_0} dV(x)|\}. \tag{13.24}$$

Lemma 18. *If* $\chi(\mathcal{M}) < 0$ *and* $K < 0$*, then* $\inf_S F(u) = a > -\infty$*.*

Proof. To prove this lemma, we need to bound

$$-\chi(\mathcal{M}) \log|\int_{\mathcal{M}} K(x) e^{2u_0} dV(x)|, \tag{13.25}$$

from below. Indeed, for $K(x) \le -\delta < 0$,

$$\int K(x) e^{2u_0} dV(x) \le -\delta \int e^{2u_0} dV. \tag{13.26}$$

Since $e^x \ge 1 + x$, $\int e^{2u_0} dV(x) \ge \int dV(x) + \int 2u_0 dV(x) = Area(\mathcal{M})$, $-\delta \int_{\mathcal{M}} e^{2u_0} dV(x) \le -\delta A$, where A is the area of $\mathcal{M}$. Therefore,

$$-\pi\chi(\mathcal{M}) \log|\int_{\mathcal{M}} K(x) e^{2u_0} dV(x)| \ge \pi|\chi(\mathcal{M})| \log(\delta A) \ge b > -\infty. \tag{13.27}$$

Therefore, for $u \in S$,

$$F(u) \ge \int_{\mathcal{M}} (\frac{1}{2}|du_0|^2 + k(x)u_0) dV(x) - C_2, \tag{13.28}$$

with C_2 independent of $u_0 \in H^1(\mathcal{M})$. Since $\|u_0\|_{L^2} \le C\|du_0\|_{L^2}$,

$$|\int_{\mathcal{M}} k(x) u_0 dV(x)| \le C_3 \epsilon \|du_0\|^2_{L^2} + \frac{C_4}{\epsilon}, \tag{13.29}$$

so for $\epsilon = \frac{1}{2C_3}$, $F(u) \ge -2C_3C_4 - C_2$. □

Now we can prove the main existence result.

Theorem 28. *If* $\mathcal{M}$ *and* K *are as in Theorem* 26*, then* F *achieves a minimum at a point* $u \in S$*, which consequently solves* (13.3)*.*

Proof. Choose $u_n \in S$ so that $F(u_n) \searrow a$, $F(u_n) \le a + 1$. By (13.28) and (13.29),

$$\frac{1}{4}\|du_{n0}\|^2_{L^2} - C_5 \le a + 1. \tag{13.30}$$

Here, $u_{n0} = u_n -$ mean value. The mean value of u_n is

$$\frac{1}{2} \log[2\pi\chi(\mathcal{M}) / \int_{\mathcal{M}} K e^{2u_{n0}} dV(x)], \tag{13.31}$$

which is bounded from above by the proof of Lemma 18. Therefore, u_n is bounded in $H^1(\mathcal{M})$, and passing to a subsequence, there exists an element $u \in H^1(\mathcal{M})$ such that

$$u_n \rightharpoonup u, \qquad \text{weakly in} \qquad H^1(\mathcal{M}). \tag{13.32}$$

By properties of weak convergence, $e^{2u_n} \to e^{2u}$ in $L^1(\mathcal{M})$ norm, so $u \in S$. Also, by (13.32), $\int_{\mathcal{M}} k(x) u_n dV \to \int_{\mathcal{M}} k(x) u dV(x)$. Also,

$$\int_{\mathcal{M}} |du|^2 dV(x) \leq \liminf_{n \to \infty} \int_{\mathcal{M}} |du_n|^2 dV(x). \tag{13.33}$$

Therefore, $F(u) \leq a = \inf_{v \in S} F(v)$, which implies that $F(u) = a$. □

Consider the special case when $K = -1$. For any compact surface with $\chi(\mathcal{M}) < 0$, given a Riemannian metric g, it is conformally equivalent to a metric for which $K = -1$. The universal covering surface

$$\tilde{\mathcal{M}} \to \mathcal{M}, \tag{13.34}$$

Definition 6 (Universal cover)**.** *The universal cover of a connected topological space X is a simply connected space Y with map $f : Y \to X$ that is a covering map.*

Theorem 29. *Any two complete, simply connected Riemannian manifolds with the same constant curvature and the same dimension are isometric.*

Proof. Differential geometry. □

One model surface of curvature -1 is the Poincare disk,

$$\mathcal{D} = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 < 1\} = \{z \in \mathbb{C} : |z| < 1\}, \tag{13.35}$$

with metric

$$ds^2 = 4(1 - x^2 - y^2)^{-2}(dx^2 + dy^2). \tag{13.36}$$

Any compact surface $\mathcal{M}$ with negative Euler characteristic is conformally equivalent to the quotient of $\mathcal{D}$ by a discrete group Γ of isometries. If $\mathcal{M}$ is orientable, all the elements of Γ preserve orientation.

Next, consider the case $\chi(\mathcal{M}) = 0$. We claim that any metric g on such $\mathcal{M}$ is conformally equivalent to a flat metric g', that is, one for which $K = 0$. In this case (13.2) is

$$\Delta u = k(x). \tag{13.37}$$

This equation can be solved on $\mathcal{M}$ if and only if

$$\int_{\mathcal{M}} k(x) dV(x) = 0. \tag{13.38}$$

By the Gauss–Bonnet formula, (13.4), (13.38) holds precisely when $\chi(\mathcal{M}) = 0$. Then the universal covering surface $\tilde{\mathcal{M}}$ of $\mathcal{M}$ inherits a flat metric, and must be isometric to Euclidean space.

Proposition 42. *If $\mathcal{M}$ is a compact Riemannian surface, $\chi(\mathcal{M}) = 0$, then $\mathcal{M}$ is holomorphically equivalent to the quotient of $\mathbb{C}$ by a discrete group of transformations.*

If $\mathcal{M}$ is a compact, connected Riemann surface, $\chi(\mathcal{M}) \leq 2$. If $\chi(\mathcal{M}) = 2$, then $\mathcal{M}$ is conformally equivalent to the standard sphere S^2.

Proposition 43. *If $\mathcal{M}$ is a compact Riemannian manifold homeomorphic to S^2, with Riemannian metric tensor g, then $\mathcal{M}$ has metric tensor conformal to g, with Gauss curvature $\equiv 1$.*

In other words, it is possible to solve for $u \in C^\infty(\mathcal{M})$ the equation

$$\Delta u = k(x) - e^{2u}, \tag{13.39}$$

which does not follow from Theorem 26.

14. Local solvability of nonlinear elliptic equations

An elliptic differential operator of order m is an operator that in local coordinates has the form

$$P(x,D)u = \sum_{|\alpha|\le m} a_\alpha(x) D^\alpha u, \tag{14.1}$$

which has principal symbol

$$P_m(x,\xi) = \sum_{|\alpha|=m} a_\alpha(x)\xi^\alpha, \tag{14.2}$$

which is invertible for nonzero $\xi \in \mathbb{R}^n$.

Theorem 30. *If $P(x,D)$ is elliptic of order m and $u \in \mathcal{D}'(\mathcal{M})$, $P(x,D)u = f \in H^s(\mathcal{M})$, then $u \in H^{s+m}_{loc}(\mathcal{M})$, and for each $U \subset\subset V \subset\subset \mathcal{M}$, $\sigma < s+m$, then there is an estimate*

$$\|u\|_{H^{s+m}(U)} \le C\|P(x,D)u\|_{H^s(V)} + C\|u\|_{H^\sigma(V)}. \tag{14.3}$$

Now consider the nonlinear partial differential equation,

$$f(x, D^m u) = g(x), \tag{14.4}$$

where

$$D^m u = \{D^\alpha u : |\alpha| \le m\}. \tag{14.5}$$

Suppose $f(x,\zeta)$ is smooth in its operators, $x \in \Omega$, and $\zeta = \{\zeta_\alpha : |\alpha| \le m\}$. Then define

$$F(u) = f(x, D^m u), \qquad F : C^\infty(\Omega) \to C^\infty(\Omega). \tag{14.6}$$

Definition 7. *Suppose F is a nonlinear differential operator and $u_0 \in C^m(\Omega)$. The linearization of F at u_0 is $DF(u_0)$, which is a linear map from $C^m(\Omega)$ to $C(\Omega)$.*

$$DF(u_0)v = \frac{\partial}{\partial s}F(u_0 + sv)|_{s=0} = \sum_{|\beta|\le m} \frac{\partial f}{\partial \zeta_\beta}(x, D^m u_0) D^\beta v. \tag{14.7}$$

Thus, $DF(u_0)$ is a linear differential operator of order m. The operator F is elliptic if $DF(u_0)$ is an elliptic, linear differential operator.

An operator of the form

$$f(x, D^m u) = \sum_{|\alpha|=m} a_\alpha(x, D^{m-1}u) D^\alpha u + f_1(x, D^{m-1}u), \tag{14.8}$$

is said to be quasi-linear. If $F(u) = f(x, D^m u)$, where f is in the form of (14.8), then

$$DF(u_0) = \sum_{|\alpha|=m} a_\alpha(x, D^{m-1}u_0) D^\alpha v + Lv, \tag{14.9}$$

where L is a linear differential operator of order $m-1$ with coefficients depending on $D^{m-1}u_0$.

Definition 8 (Fully nonlinear)**.** *A nonlinear operator that is not quasilinear is called completely nonlinear.*

Definition 9 (Monge-Ampere operator)**.** *One example of a completely nonlinear operator is the Monge-Ampere operator,*

$$F(u) = det \begin{pmatrix} u_{xx} & u_{xy} \\ u_{yx} & u_{yy} \end{pmatrix} = u_{xx}u_{yy} - u_{xy}u_{yx}, \tag{14.10}$$

where $(x, y) \in \Omega \subset \mathbb{R}^2$.

In this case,

$$DF(u)v = Tr[\begin{pmatrix} v_{xx} & v_{xy} \\ v_{xy} & v_{yy} \end{pmatrix} \begin{pmatrix} u_{yy} & -u_{xy} \\ -u_{xy} & u_{xx} \end{pmatrix}] = u_{yy}v_{xx} - 2u_{xy}v_{xy} + u_{xx}v_{yy}. \tag{14.11}$$

Then the principal symbol for (14.11) is

$$u_{yy}\xi_1^2 - 2u_{xy}\xi_1\xi_2 + u_{xx}\xi_2^2 = \langle \vec{\xi}, \begin{pmatrix} u_{yy} & -u_{xy} \\ -u_{xy} & u_{xx} \end{pmatrix} \vec{\xi} \rangle. \tag{14.12}$$

Thus, $DF(u)$ is elliptic if the matrix

$$\begin{pmatrix} u_{yy} & -u_{xy} \\ -u_{xy} & u_{xx} \end{pmatrix}, \tag{14.13}$$

is either positive definite or negative definite. This condition holds precisely when $F(u) > 0$.

Now, for $\Omega \subset \mathbb{R}^n$, consider the Monge–Ampere operator

$$F(u) = detH(u), \qquad H(u) = (\partial_j\partial_k u). \tag{14.14}$$

Then,

$$DF(u)v = Tr[\mathcal{C}(u)H(v)], \tag{14.15}$$

where $H(v)$ is the Hessian matrix for v and $\mathcal{C}(u)$ is the cofactor matrix of $H(u)$,

$$H(u)\mathcal{C}(u) = [detH(u)]I. \tag{14.16}$$

Then $DF(u)$ is a linear, second-order differential operator, and is elliptic provided $\mathcal{C}(u)$ is either positive definite or negative definite. This holds provided the Hessian matrix $H(u)$ is either positive-definite or negative-definite.

Theorem 31. *Let* $g \in C^{\infty}(\Omega)$ *and let* $u_1 \in C^{\infty}(\Omega)$ *satisfy*

$$F(u_1) = g(x), \qquad at \qquad x = x_0, \tag{14.17}$$

where $F(u)$ *is of the form* $F(u) = f(x, D^m u)$. *Suppose that* F *is elliptic at* u_1. *Then, for any* l, *there exists* $u \in C^l(\Omega)$ *such that*

$$F(u) = g, \tag{14.18}$$

on a neighborhood of x_0.

We start with a lemma.

Lemma 19. *Under the hypotheses of Theorem* 31, *there exists* $u_0 \in C^{\infty}(\Omega)$ *such that*

$$F(u_0) - g(x) = O(|x - x_0|^{\infty}), \tag{14.19}$$

and

$$(u_0 - u_1)(x) = O(|x - x_0|^{m+1}). \tag{14.20}$$

Proof. Suppose without loss of generality that $x_0 = 0$. Then denote the coordinates near $x_0 = 0$ by $(x, y) = (x_1, ..., x_{n-1}, y)$. Then write $u_0(x, y)$ as a formal power series in y:

$$u_0(x,y) = v_0(x) + v_1(x)y + ... + \frac{1}{k!}v_k(x)y^k + ... \tag{14.21}$$

Then set

$$v_0(x) = u_1(x,0), \qquad v_1(x) = \partial_y u_1(x,0), \qquad \cdots \qquad , v_{m-1}(x) = \partial_y^{m-1}u_1(x,0). \tag{14.22}$$

Then the PDE $F(u) = g$ can be rewritten in the form,

$$\frac{\partial^m u}{\partial y^m} = F^\sharp(x, y, D_x^m u, D_x^{m-1}D_y u, ..., D_x^1 D_y^{m-1} u). \tag{14.23}$$

Then the equation for $v_m(x)$ becomes

$$v_m(x) = f^\sharp(x, 0, D_x^m v_0(x), ..., D_x^1 v_{m-1}(x)). \tag{14.24}$$

Now then, since $F(u) = g(x)$, $v_m(0) = \partial_y^m u_1(0,0)$, so (14.20) is satisfied. Taking y-derivatives inductively yields the other coefficients, and the lemma follows by construction. □

If F is elliptic at u_1, then F continues to be elliptic at u_0, at least in a neighborhood of x_0, and we can shrink Ω appropriately. For $k > m + 1 + \frac{n}{2}$,

$$F : H^k(\Omega) \to H^{k-m}(\Omega), \tag{14.25}$$

is a C^1 map. Then,

$$\mathcal{L} = DF(u_0) : H^k(\Omega) \to H^{k-m}(\Omega). \tag{14.26}$$

Since $\mathcal{L}$ is an elliptic operator of order m, the Dirichlet problem is a regular boundary problem for the strongly elliptic operator $\mathcal{L}\mathcal{L}^*$.

Definition 10 (Strongly elliptic)**.** *A strongly elliptic operator is an operator*

$$\sum_{|\beta|=2m} a_\beta(x)\xi^\beta \geq C(x)|\xi|^{2m}, \qquad \textit{for some} \qquad C(x) > 0, \qquad \xi \in \mathbb{R}^d. \tag{14.27}$$

Furthermore, if Ω is a sufficiently small neighborhood of x_0, the map

$$\mathcal{L}\mathcal{L}^* : H^{k+m}(\Omega) \cap H_0^m(\Omega) \to H^{k-m}(\Omega), \tag{14.28}$$

is invertible. Therefore, $\mathcal{L}$ is surjective, so we can apply the implicit function theorem. For any neighborhood $\mathcal{B}_k$ of $u_0 \in H^k(\Omega)$, the image of $\mathcal{B}_k$ under the map F contains a neighborhood $\mathcal{C}_k$ of $F(u_0)$ in $H^{k-m}(\Omega)$. If (14.19) holds, then any neighborhood of $r(x) = F(u_0) - g$ in $H^{k-m}(\Omega)$ contains functions that vanish on a neighborhood of x_0. Therefore, any neighborhood $\mathcal{C}_k$ of $F(u_0)$ contains functions equal to $g(x)$ on a neighborhood of x_0. This establishes local solvability asserted in Theorem 31.

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