

# A COURSE IN COMPLEX ANALYSIS

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These notes come from [Tay19] and [Sch14].

1. THE COMPLEX PLANE  $\mathbb{C}$ 

The field  $\mathbb{C}$  of complex numbers is obtained by adjoining  $i$  to the field  $\mathbb{R}$  of real numbers, and is a complete, algebraically closed field. The concept of completeness arises in real analysis. One can easily obtain polynomials whose coefficients lie in the field of rational numbers  $\mathbb{Q}$ , but whose solutions do not lie in  $\mathbb{Q}$ . Take for example the equation

$$(1.1) \quad x^2 = 2.$$

It is a well known fact that  $\sqrt{2}$  is not a rational number, although the coefficients of (1.1) are integers. However, it is easy to show using Newton's method that  $\sqrt{2}$  is the limit of a sequence of rational numbers. Therefore, the real numbers  $\mathbb{R}$  are defined to be the completion of the field of rational numbers, and  $\mathbb{R}$  is a complete field.

**Definition 1.** *A space is called complete if every Cauchy sequence converges. That is, if  $\{z_n\}_{n=1}^{\infty}$  is a sequence such that for any  $\epsilon > 0$  there exists  $N(\epsilon) < \infty$  where  $m, n \geq N(\epsilon)$  implies  $|z_n - z_m| < \epsilon$ .*

This is not the end of the story, however, since it is still possible to obtain a polynomial whose coefficients lie in the field of real numbers, but whose solutions do not lie in  $\mathbb{R}$ . Consider the equation

$$(1.2) \quad x^2 = -1.$$

Since  $x^2 \geq 0$  for every real number, (1.2) does not have any solutions in  $\mathbb{R}$ . Therefore, define  $i$  to be the number satisfying  $i^2 = -1$ , and let  $\mathbb{C}$  be the numbers of the form

$$(1.3) \quad z = x + iy, \quad x, y \in \mathbb{R}, \quad x = \operatorname{Re}(z), \quad y = \operatorname{Im}(z).$$

The complex conjugate of  $z$  is  $\bar{z} = x - iy$ , and we have  $|z|^2 = z\bar{z}$ . Indeed,

$$(1.4) \quad (x + iy)(x - iy) = x^2 + ixy - ixy + y^2 = x^2 + y^2,$$

which is the usual square of the length of a vector by the Pythagorean theorem.

The complex field easily inherits completeness from  $\mathbb{R}$  under the topology induced by the Pythagorean theorem. Indeed, every Cauchy sequence in  $\mathbb{R}$  converging implies that every Cauchy sequence in  $\mathbb{C}$  converges under the norm  $|z|$ . Let  $z_n = x_n + iy_n$  and  $z = x + iy$ . Then  $z_n \rightarrow z$  if and only if  $x_n \rightarrow x$  in  $\mathbb{R}$  and  $y_n \rightarrow y$  in  $\mathbb{R}$ . Similarly,  $\{z_n\}_{n=1}^{\infty}$  is Cauchy in  $\mathbb{C}$  if and only if  $\{x_n\}_{n=1}^{\infty}$  and  $\{y_n\}_{n=1}^{\infty}$  are Cauchy sequences in  $\mathbb{R}$ . Thus,  $\mathbb{C}$  is a complete field.

More generally, let  $z_1 = x_1 + iy_1$  and  $z_2 = x_2 + iy_2$ . Then,

$$(1.5) \quad z_1 + z_2 = (x_1 + x_2) + i(y_1 + y_2), \quad \text{and} \quad z_1 z_2 = x_1 x_2 - y_1 y_2 + i(x_1 y_2 + x_2 y_1).$$

Thus, the complex plane  $\mathbb{C}$  is closed under multiplication and addition. These operations obey the commutative, associative, and distributive properties. Moreover, (1.4) implies that if  $z \neq 0$ ,  $\frac{1}{z} = \frac{\bar{z}}{|z|^2}$ .

Returning to the Cartesian plane, it is possible to assign polar coordinates  $(r, \theta)$  to  $z$ . By definition,  $r = |z|$  and  $z = r(\cos \theta + i \sin \theta)$ . Thus,

$$(1.6) \quad z_1 z_2 = |z_1| |z_2| (\cos(\theta_1 + \theta_2) + i \sin(\theta_1 + \theta_2)).$$

This reveals the remarkable fact that complex numbers are multiplied by multiplying their lengths and adding their angles. In particular,  $|z_1 z_2| = |z_1| |z_2|$ .

The group of complex numbers can be represented by a two dimensional algebra of commuting matrices. Observe that for  $c \in \mathbb{R}$ , the operation  $c : v \mapsto cv$  is represented by the dilation matrix

$$(1.7) \quad \begin{pmatrix} c & 0 \\ 0 & c \end{pmatrix}.$$

Next, the action of  $i$  on  $x + iy$  is given by  $i(x + iy) = -y + ix$ , which is a ninety degree clockwise rotation, given by the matrix

$$(1.8) \quad J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

**Lemma 1.** *A matrix commutes with (1.8) if and only if it is of the form*

$$(1.9) \quad \begin{pmatrix} a & -b \\ b & a \end{pmatrix}.$$

*Proof.* By direct calculation,

$$(1.10) \quad \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = \begin{pmatrix} d & -c \\ -b & a \end{pmatrix}.$$

Therefore, a matrix commutes with (1.8) if and only if  $a = d$  and  $b = -c$ . □

By the distributive property, (1.7), and (1.8),  $a + ib$  can be represented by the matrix

$$(1.11) \quad \begin{pmatrix} a & -b \\ b & a \end{pmatrix}.$$

Indeed,

$$(1.12) \quad \begin{pmatrix} a & -b \\ b & a \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} ax - by \\ bx + ay \end{pmatrix},$$

which corresponds to

$$(1.13) \quad (a + ib)(x + iy) = (ax - by) + i(ay + bx).$$

Furthermore, since the column vectors in (1.9) are the vectors

$$(1.14) \quad \begin{pmatrix} a \\ b \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix},$$

$$(1.15) \quad \begin{pmatrix} a & -b \\ b & a \end{pmatrix} \begin{pmatrix} x & -y \\ y & x \end{pmatrix} = \begin{pmatrix} ax - by & -bx - ay \\ bx + ay & ax - by \end{pmatrix},$$

so the algebra of matrices of the form (1.9) corresponds to the algebra of complex numbers of the form  $a + ib$ .

The norm  $|z|$  also obeys the triangle inequality.

**Proposition 1** (Triangle inequality).

$$(1.16) \quad |z + w| \leq |z| + |w|.$$

*Proof.* Calculating directly,

$$(1.17) \quad \overline{z+w} = \bar{z} + \bar{w}, \quad \overline{z\bar{w}} = \bar{z}w,$$

which implies that

$$(1.18) \quad |zw|^2 = zw\bar{z}\bar{w} = |z|^2|w|^2.$$

Therefore,

$$(1.19) \quad |z+w|^2 = (z+w)(\bar{z}+\bar{w}) = |z|^2 + |w|^2 + z\bar{w} + \bar{z}w \leq (|z| + |w|)^2,$$

which implies (1.16).  $\square$

**Remark 1.** Note that  $z\bar{w} + \bar{z}w$  is guaranteed to be a real number since

$$(1.20) \quad 2\operatorname{Re}(z) = z + \bar{z}, \quad \text{and} \quad 2\operatorname{Im}(z) = z - \bar{z}.$$

**Remark 2.** By standard properties of limits, (1.3), and (1.4), if  $z_n \rightarrow z$  in  $\mathbb{C}$  and  $w_n \rightarrow w$  in  $\mathbb{C}$ , then  $z_n + w_n \rightarrow z + w$  and  $z_n w_n \rightarrow zw$  in  $\mathbb{C}$ .

The fact that  $\mathbb{C}$  is algebraically closed follows from the fundamental theorem of algebra.

**Theorem 1** (Fundamental theorem of algebra). *If  $p(z)$  is a non-constant polynomial with complex coefficients, then  $p(z)$  must have a complex root.*

*Proof.* Suppose that for some  $n \geq 1$ ,

$$(1.21) \quad p(z) = a_n z^n + \dots + a_1 z + a_0, \quad a_n \neq 0, \quad a_j \in \mathbb{C} \quad \forall 0 \leq j \leq n.$$

Therefore, as  $|z| \rightarrow \infty$ ,

$$(1.22) \quad p(z) = a_n z^n (1 + O(z^{-1})),$$

which implies that

$$(1.23) \quad \lim_{|z| \rightarrow \infty} |p(z)| = \infty,$$

so there exists  $0 < R < \infty$  such that

$$(1.24) \quad \inf_{|z| > R} |p(z)| > |p(0)|,$$

and therefore,

$$(1.25) \quad \inf_{|z| \leq R} |p(z)| = \inf_{z \in \mathbb{C}} |p(z)|.$$

Since  $p$  is continuous, there exists  $z_0 \in D_R$  which satisfies

$$(1.26) \quad |p(z_0)| = \inf_{z \in \mathbb{C}} |p(z)|,$$

where  $D_R$  refers to the disk of radius  $R$ ,  $D_R = \{z \in \mathbb{C} : |z| \leq R\}$ .

**Lemma 2.** *If  $p(z)$  is a non-constant polynomial and (1.26) holds, then  $p(z_0) = 0$ .*

*Proof.* Suppose by contradiction that  $p(z_0) = a \neq 0$ . Since a polynomial in  $z$  can easily be rewritten as a polynomial of the same degree in  $(z - z_0)$  for any  $z_0 \in \mathbb{C}$ ,

$$(1.27) \quad p(z_0 + \zeta) = a + q(\zeta), \quad \zeta = z - z_0,$$

where  $q$  is a non-constant polynomial of order  $n$ . Therefore, for some  $k \geq 1$ ,  $b \neq 0$ ,

$$(1.28) \quad q(\zeta) = b\zeta^k + \dots + b_n \zeta^n.$$

The term  $b\zeta^k$  dominates the behavior of  $q(\zeta)$  for  $|\zeta|$  small,

$$(1.29) \quad q(\zeta) = b\zeta^k + O(\zeta^{k+1}), \quad \text{as } \zeta \rightarrow 0.$$

Therefore, take  $S^1 = \{\omega : |\omega| = 1\}$ . For any fixed  $\omega \in S^1$ ,

$$(1.30) \quad p(z_0 + \epsilon\omega) = a + b\omega^k \epsilon^k + O(\epsilon^{k+1}), \quad \text{as } \epsilon \searrow 0.$$

Since  $a \neq 0$  and  $b \neq 0$ , choose  $\omega \in S^1$  such that

$$(1.31) \quad \frac{b}{|b|} \omega^k = -\frac{a}{|a|}.$$

Then,

$$(1.32) \quad p(z_0 + \epsilon\omega) = a\left(1 - \left|\frac{b}{a}\right| \epsilon^k\right) + O(\epsilon^{k+1}),$$

which contradicts the minimality of  $p(z_0)$  when  $\epsilon > 0$  is sufficiently small.  $\square$

Therefore,  $p(z)$  has the root  $p(z_0) = 0$ .  $\square$

Rewriting  $p(z)$  as a polynomial of order  $n$  in  $(z - z_0)$ , since  $p(z_0) = 0$ ,

$$(1.33) \quad p(z) = a_n(z - z_0)^n + \dots + \tilde{a}_1(z - z_0).$$

Dividing  $p(z)$  by  $(z - z_0)$  gives a polynomial of order  $n - 1$ . Using Theorem 1 and arguing by induction implies that  $p(z)$  has  $n$  roots in  $\mathbb{C}$ .

## 2. HOLOMORPHIC, ANALYTIC, AND CONFORMAL FUNCTIONS

Suppose  $f$  is defined on the open set  $\Omega \subset \mathbb{C}$ . Let  $\Omega \subset \mathbb{C}$  be an open set. A set is called open if for all  $z_0 \in \Omega$ , there exists  $\epsilon_0 > 0$  such that

$$(2.1) \quad D_{\epsilon_0}(z_0) = \{z : |z - z_0| < \epsilon_0\} \subset \Omega.$$

There are two relevant notions of derivative that will be discussed here.

**Definition 2.** Let  $f : \Omega \rightarrow \mathbb{C}$ .

(1) We say that  $f \in C^1(\Omega)$  if there exists  $df$ , a continuous  $2 \times 2$  matrix valued function, such that

$$(2.2) \quad f(z + h) = f(z) + df(z)(h) + r(z, h), \quad \forall z, z + h \in \Omega,$$

$$(2.3) \quad \lim_{h \rightarrow 0} \frac{r(z, h)}{|h|} = 0,$$

where  $df(z)(h)$  is the matrix  $df(z)$  acting on the vector  $h$ .

(2) We say that  $f$  is holomorphic on  $\Omega$  if

$$(2.4) \quad f'(z) = \lim_{w \rightarrow z} \frac{f(w) - f(z)}{w - z},$$

exists for all  $z \in \Omega$  and is continuous on  $\Omega$ . We denote this by  $f \in \mathcal{H}(\Omega)$ . A function  $f \in \mathcal{H}(\mathbb{C})$  is called entire.

Therefore, holomorphic functions are precisely those functions in  $C^1(\Omega)$  in the sense of (1) for which the differential  $df(z)$  acts as a linear map via multiplication by a complex number.

Let  $f : \mathbb{R}^2 \rightarrow \mathbb{R}^2$  be a differentiable function,  $f(x, y) = \begin{pmatrix} u(x, y) \\ v(x, y) \end{pmatrix}$ . It is known from multivariable calculus that if  $f$  is differentiable,

$$(2.5) \quad Df(x, y) = \begin{pmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{pmatrix},$$

and furthermore,

$$(2.6) \quad f(x_0 + h_1, y_0 + h_2) = f(x_0, y_0) + Df(x_0, y_0) \begin{pmatrix} h_1 \\ h_2 \end{pmatrix} + o(h).$$

Comparing (1.15) to (2.6) yields a number of important facts about holomorphic functions.

**Proposition 2** (Cauchy–Riemann equations). *If  $f : \Omega \rightarrow \mathbb{C}$  is holomorphic, then  $\frac{\partial f}{\partial x}$  and  $\frac{\partial f}{\partial y}$  exist and are continuous on  $\Omega$ , and*

$$(2.7) \quad \frac{\partial f}{\partial x} = \frac{1}{i} \frac{\partial f}{\partial y} = f'(z).$$

**Proposition 3.** *If  $f : \Omega \rightarrow \mathbb{C}$  is  $C^1$  and  $\frac{\partial f}{\partial x} = \frac{1}{i} \frac{\partial f}{\partial y}$ , then  $f$  is holomorphic.*

*Proof.* Propositions 2 and 3 follow from (1.11). □

**Proposition 4.** *If  $f \in C^1(\Omega)$ , then  $f$  is holomorphic if and only if for all  $z \in \Omega$ ,  $Df(z)$  and  $J$  commute.*

*Proof.* This follows from (1.10) and (2.6). □

The Cauchy–Riemann equations are equivalent to the property that

$$(2.8) \quad df = \begin{pmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{pmatrix} = \rho A,$$

where  $\rho \geq 0$  and  $A \in SO(2, \mathbb{R})$ . In other words, at each point where a holomorphic function has a non-vanishing derivative,  $df$  is a conformal matrix: it preserves angles and the orientations between vectors. Conversely, if  $f \in C^1(\Omega)$  has the property that  $df$  is proportional to a rotation matrix everywhere in  $\Omega$ , then  $f \in \mathcal{H}(\Omega)$ .

**Definition 3.** *A function  $f \in C^1(\Omega)$  is called conformal if  $df \neq 0$  in  $\Omega$  and  $df$  preserves the angle and orientation at each point.*

**Remark 3.** *The holomorphic functions are precisely those functions which are conformal at all points for which  $df \neq 0$ .*

**Remark 4.** *Observe that  $f(z) = \bar{z}$  is a differentiable function from  $\mathbb{R}^2 \rightarrow \mathbb{R}^2$ , but*

$$(2.9) \quad f(x, y) = \begin{pmatrix} x \\ -y \end{pmatrix},$$

so  $df = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ . Thus,  $df$  preserves angles, but not orientation.

**Lemma 3.** *If  $T : V \rightarrow W$  is an  $\mathbb{R}$ -linear map between complex vector spaces, then there is a unique representation  $T = T_1 + T_2$ , where  $T_1$  is complex linear and  $T_2$  complex anti-linear. The latter property means that  $T_2(\lambda \vec{v}) = \bar{\lambda} T_2(\vec{v})$ .*

*Proof.* Uniqueness follows since a  $\mathbb{C}$ -linear map which is simultaneously  $\mathbb{C}$ -antilinear vanishes identically. Set

$$(2.10) \quad T_1 = \frac{1}{2}(T - iTi), \quad T_2 = \frac{1}{2}(T + iTi).$$

Then,  $T_1i = \frac{1}{2}(Ti + iT) = \frac{1}{2}(iT - i^2Ti) = iT_1$  and  $T_2i = \frac{1}{2}(Ti + iT(-1)) = -i\frac{1}{2}(T + iTi) = -iT_2$ .  $\square$

Therefore, there exist complex numbers  $w_1(z)$ ,  $w_2(z)$  such that

$$(2.11) \quad df(z) = w_1(z)dz + w_2(z)d\bar{z}.$$

To find  $w_1$  and  $w_2$  observe that since  $dz = dx + idy$  and  $d\bar{z} = dx - idy$ ,

$$(2.12) \quad df = \partial_x f dx + \partial_y f dy = \partial_x f \cdot \frac{1}{2}(dz + d\bar{z}) + \partial_y f \cdot \frac{1}{2i}(dz - d\bar{z}) = \frac{1}{2}(\partial_x f - i\partial_y f)dz + \frac{1}{2}(\partial_x f + i\partial_y f)d\bar{z} = \partial_z f dz + \partial_{\bar{z}} f d\bar{z}.$$

Therefore,  $f \in \mathcal{H}(\Omega)$  if and only if  $f \in C^1(\Omega)$  and  $\partial_{\bar{z}} f = 0$  in  $\Omega$ . Thus, the Cauchy–Riemann system (2.7) is the same as  $\partial_{\bar{z}} f = 0$ .

As an application of this formalism record the following crucial fact: for any  $f \in \mathcal{H}(\Omega)$ ,

$$(2.13) \quad d(f(z)dz) = \partial_z f dz \wedge dz + \partial_{\bar{z}} f d\bar{z} \wedge dz = 0,$$

which means that  $f(z)dz$  is a closed differential form. This property is equivalent to the homotopy invariance of the Cauchy integral. We also have

$$(2.14) \quad \Delta = 4 \frac{\partial^2}{\partial z \partial \bar{z}}.$$

### 3. POWER SERIES

Next, it is straightforward to verify that the function  $f(z) = z$  is holomorphic, since

$$(3.1) \quad \frac{1}{h}[(z+h) - z] = 1.$$

On the other hand,  $f(z) = \bar{z}$  is not holomorphic, since

$$(3.2) \quad \frac{1}{h}[f(z+h) - f(z)] = \frac{\bar{h}}{h}.$$

Furthermore, since we can do algebra in  $\mathbb{C}$  in the same way that we do algebra in  $\mathbb{R}$ , the basic differentiation rules like the sum, product, quotient, and chain rules continue to hold for holomorphic functions.

**Proposition 5.** *If  $f$  and  $g$  are holomorphic on  $\Omega$ , then so are  $(fg)(z)$ ,  $f(z) + g(z)$ , and  $cf(z)$ , where  $c \in \mathbb{C}$  is a constant. Furthermore,*

$$(3.3) \quad \frac{d}{dz}(fg)(z) = f'(z)g(z) + f(z)g'(z), \quad \frac{d}{dz}(f+g)(z) = f'(z) + g'(z), \quad \frac{d}{dz}(cf(z)) = cf'(z).$$

*Proof.* The proof uses the limit definition of the derivative and the usual computations from calculus.  $\square$

A corollary of this fact is that every polynomial is holomorphic.

**Corollary 1.** *Every polynomial is holomorphic.*

**Proposition 6** (Chain rule). *Let  $\Omega, \mathcal{O}$  be open sets in  $\mathbb{C}$ . If  $f : \Omega \rightarrow \mathbb{C}$  and  $g : \mathcal{O} \rightarrow \Omega$  are holomorphic, then  $f \circ g : \mathcal{O} \rightarrow \mathbb{C}$  is holomorphic, and*

$$(3.4) \quad \frac{d}{dz} f(g(z)) = f'(g(z))g'(z).$$

*Proof.* Observe that

$$(3.5) \quad \begin{aligned} (g \circ f)(z+h) &= g(f(z) + f'(z)h + o(h)) = g(f(z)) + g'(f(z))(f'(z)h + o(h)) + o(f'(z)h + o(h)) \\ &= g(f(z)) + g'(f(z))f'(z)h + o(h). \end{aligned}$$

□

We also have the chain rule

$$(3.6) \quad \begin{aligned} \partial_z(g \circ f) &= [(\partial_w g) \circ f] \partial_z f + [(\partial_{\bar{w}} g) \circ f] \partial_z \bar{f}, \\ \partial_{\bar{z}}(g \circ f) &= [(\partial_w g) \circ f] \partial_{\bar{z}} f + [(\partial_{\bar{w}} g) \circ f] \partial_{\bar{z}} \bar{f}. \end{aligned}$$

Combining the chain rule with the computation

$$(3.7) \quad \frac{1}{z+h} - \frac{1}{z} = -\frac{h}{z(z+h)} = -\frac{h}{z^2} + o(h),$$

**Proposition 7.** *If  $f : \Omega \rightarrow \mathbb{C}$  is holomorphic, then  $\frac{1}{\bar{f}(z)}$  is holomorphic on  $\Omega \setminus S$ , where  $S = \{z \in \Omega : f(z) = 0\}$ , and on  $\Omega \setminus S$ ,*

$$(3.8) \quad \frac{d}{dz} \frac{1}{\bar{f}(z)} = -\frac{f'(z)}{f(z)^2}.$$

Moving on from polynomials, next consider the power series. First, let

$$(3.9) \quad \sum_{k=0}^{\infty} z_k$$

denote a series. Then define the sequence  $s_n = \sum_{k=0}^n z_k$ . The sequence  $s_n$  converges as a sequence if and only if  $\sum_{k=0}^{\infty} z_k$  converges.

**Lemma 4.** *Assume that*

$$(3.10) \quad \sum_{k=0}^{\infty} |z_k| < \infty.$$

*Then  $s_n$  is a Cauchy sequence.*

*Proof.* Since  $\sum_{k=0}^n |z_k| \leq A$ , for any  $n$ ,

$$(3.11) \quad s_n = \sum_{k=0}^n |z_k|$$

is a bounded, monotone sequence. Since a bounded, monotone sequence converges in  $\mathbb{R}$ , for any  $\epsilon > 0$ , there exists  $M(\epsilon) < \infty$  such that for any  $n$ ,

$$(3.12) \quad \sum_{k=M(\epsilon)}^n |z_k| < \epsilon.$$

Then by the triangle inequality, (3.12) implies that for any  $m, n \geq M(\epsilon)$ ,  $|s_n - s_m| < \epsilon$ , and therefore  $s_n$  is a Cauchy sequence.  $\square$

**Definition 4.** A series that satisfies (3.10) is absolutely convergent.

A power series has the form

$$(3.13) \quad f(z) = \sum_{n=0}^{\infty} a_n(z - z_0)^n.$$

Any such series has a radius of convergence, some  $0 \leq R \leq \infty$  such that (3.13) converges absolutely on the disk  $D_R(z_0) = \{z : |z - z_0| < R\}$  and diverges for  $z$  such that  $|z - z_0| > R$ . Let

$$(3.14) \quad \frac{1}{R} = \limsup_{n \rightarrow \infty} |a_n|^{1/n},$$

where  $R = \infty$  if (3.14) = 0, and  $R = 0$  if (3.14) =  $\infty$ .

**Proposition 8.** The series (3.13) converges whenever  $|z - z_0| < R$  and diverges whenever  $|z - z_0| > R$ , where  $R$  is given by (3.14). If  $R > 0$ , then the series converges uniformly on any  $D_{R'}(z_0)$  for  $R' < R$ . Thus, when  $R > 0$ , the series (3.13) defines a continuous function on  $D_R(z_0)$ ,

$$(3.15) \quad f : D_R(z_0) \rightarrow \mathbb{C}.$$

*Proof.* When  $R = 0$ , Proposition 8 is true. Now suppose  $0 < R \leq \infty$ . For any  $R' < R$ , by (3.14) there exists  $\epsilon > 0$  and  $N$  sufficiently large such that

$$(3.16) \quad \sup_{n \geq N} |a_n|^{1/n} \leq \frac{1}{R' + \epsilon}.$$

Doing some algebra, for any  $n \geq N$ ,

$$(3.17) \quad |a_n| \leq \frac{1}{(R' + \epsilon)^n}.$$

Therefore, for  $z \in D_{R'}(z_0)$ ,

$$(3.18) \quad |a_n(z - z_0)^n| \leq \left(\frac{R'}{R' + \epsilon}\right)^n.$$

Therefore, (3.13) converges uniformly on  $D_{R'}(z_0)$ .

Meanwhile, for  $0 \leq R < \infty$ , for any  $z \in \mathbb{C}$  satisfying  $|z - z_0| > R$ , there exists a subsequence  $m(n) \nearrow \infty$  such that

$$(3.19) \quad |a_{m(n)}(z - z_0)^{m(n)}| \geq 1,$$

so (3.13) fails to converge.  $\square$

**Theorem 2.** If the power series

$$(3.20) \quad f(z) = \sum_{k=0}^{\infty} a_k(z - z_0)^k,$$

converges for some  $z_1 \neq z_0$ , then either (3.20) converges for all  $z \in \mathbb{C}$ , or (3.20) converges on a disk of radius  $0 < R < \infty$ .

Since  $a_k(z_1 - z_0)^k \rightarrow 0$ , there exists a constant  $C$  such that  $|a_k(z_1 - z_0)^k| \leq C$ . Therefore, the series will converge for  $|z - z_0| < |z_1 - z_0|$ .

**Definition 5.** We say that  $f : \Omega \rightarrow \mathbb{C}$  is analytic, or  $f \in \mathcal{A}(\Omega)$ , if  $f$  is represented by a convergent power series expansion in a neighborhood around every point of  $\Omega$ .

**Lemma 5.**

$$(3.21) \quad \mathcal{A}(\Omega) \subset \mathcal{H}(\Omega).$$

*Proof.* Suppose without loss of generality that  $z_0 \in \Omega$  and

$$(3.22) \quad f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n, \quad \forall z \text{ such that } |z - z_0| < r(z_0),$$

where  $r(z_0) > 0$ . Suppose without loss of generality that  $z_0 = 0$  and  $r_0 = r_0(0)$ . Fix  $|z| < r_0$  and write

$$(3.23) \quad f(z+h) = f(z) + g(z)h + r(z,h), \quad \text{where} \quad g(z) = \sum_{n=1}^{\infty} n a_n z^{n-1}.$$

Following (3.14),

$$(3.24) \quad \limsup_{n \rightarrow \infty} |n a_n|^{\frac{1}{n-1}} = \lim_{n \rightarrow \infty} n^{\frac{1}{n-1}} \cdot \limsup_{n \rightarrow \infty} |a_n|^{\frac{1}{n} \cdot \frac{n}{n-1}} = \frac{1}{R}.$$

Therefore, the right hand side of (3.23) converges on  $D_R(z_0)$  and diverges for  $|z - z_0| > R$ .

Observe that the power series defining  $g(z)$  converges since  $n^{1/n} \rightarrow 1$  as  $n \rightarrow \infty$ . We need to prove  $\frac{r(z,h)}{|h|} \rightarrow 0$  as  $h \rightarrow 0$ . Let

$$(3.25) \quad r(z,h) = \sum_{n=2}^{\infty} a_n B_n(z,h),$$

$$B_n(z,h) = (z+h)^n - z^n - n z^{n-1} h = \sum_{k=2}^n \binom{n}{k} h^k z^{n-k}.$$

Now let  $|z| + |h| < r_1 < r_2 < r_0$ . Then we have  $|a_n| \leq \frac{M}{r_2^n}$  for all  $n \geq 0$  and some  $M < \infty$ . Also,

$$(3.26) \quad |B_n(z,h)| \leq |h|^2 \sum_{k=0}^{n-2} \binom{n}{k+2} |h|^k |z|^{n-2-k} \leq |h|^2 \sum_{k=0}^{n-2} n^2 \binom{n-2}{k} |h|^k |z|^{n-2-k} = |h|^2 n^2 (|z| + |h|)^{n-2},$$

for any  $n \geq 2$ . Therefore,

$$(3.27) \quad |r(z,h)| \leq \sum_{n=2}^{\infty} \frac{M}{r_2^n} n^2 |h|^2 (|z| + |h|)^{n-2} \leq \frac{|h|^2 M}{r_2^2} \sum_{n=0}^{\infty} (n+2)^2 \left(\frac{r_1}{r_2}\right)^n.$$

Therefore,  $|r(z,h)| \leq C|h|^2$ . Thus,

$$(3.28) \quad f'(z) = \sum_{n=0}^{\infty} n a_n (z - z_0)^{n-1}, \quad \forall \quad |z - z_0| < r(z_0).$$

In fact, one can differentiate any number of times, and

$$(3.29) \quad f^{(k)}(z) = \sum_{n=0}^{\infty} \binom{n}{k} a_n (z - z_0)^{n-k}, \quad \forall |z - z_0| < r(z_0),$$

where

$$(3.30) \quad (n)_k = n(n-1)\cdots(n-k+1) = \frac{n!}{(n-k)!}.$$

This establishes  $a_n = \frac{f^{(n)}(z_0)}{n!}$ . □

There are several well-known examples of power series. First,

$$(3.31) \quad \frac{1}{1-z} = \sum_{n=0}^{\infty} z^n, \quad |z| < 1.$$

In this case  $a_n = 1$  for all  $n$ , so  $R = 1$ .

Another example is the series

$$(3.32) \quad E(z) = \sum_{n=0}^{\infty} \frac{z^n}{n!}.$$

Since  $a_n = \frac{1}{n!}$ ,  $(a_n)^{1/n} \rightarrow 0$  as  $n \rightarrow \infty$ . Thus, (3.32) converges for all  $z \in \mathbb{C}$ . Furthermore, by the binomial theorem,

$$(3.33) \quad E(z_1 + z_2) = \sum_{n=0}^{\infty} \frac{1}{n!} (z_1 + z_2)^n = \sum_{n=0}^{\infty} \frac{1}{n!} \sum_{j=0}^n \frac{n!}{j!(n-j)!} z_1^j z_2^{n-j} = \sum_{j=0}^{\infty} \frac{1}{j!} z_1^j \cdot \sum_{k=0}^{\infty} \frac{1}{k!} z_2^k = E(z_1)E(z_2).$$

Therefore,  $E(z) \neq 0$  for any  $z \in \mathbb{C}$ , since  $E(z)E(-z) = 1$ .

In general, products of convergent power series also converge.

**Theorem 3.** *Suppose the following power series converge for  $|z| < R$ :*

$$(3.34) \quad f(z) = \sum_{n=0}^{\infty} a_n z^n, \quad g(z) = \sum_{n=0}^{\infty} b_n z^n.$$

Then for  $|z| < R$ ,

$$(3.35) \quad f(z)g(z) = \sum_{n=0}^{\infty} c_n z^n, \quad c_n = \sum_{j=0}^n a_j b_{n-j}.$$

This theorem follows directly from

**Proposition 9.** *Given the absolutely convergent series*

$$(3.36) \quad A = \sum_{n=0}^{\infty} \alpha_n, \quad B = \sum_{n=0}^{\infty} \beta_n,$$

we have the absolutely convergent series

$$(3.37) \quad AB = \sum_{n=0}^{\infty} \gamma_n, \quad \gamma_n = \sum_{j=0}^n \alpha_j \beta_{n-j}.$$

*Proof.* Take  $A_k = \sum_{n=0}^k \alpha_n$  and  $B_k = \sum_{n=0}^k \beta_n$ . Then

$$(3.38) \quad A_k B_k = \sum_{n=0}^k \gamma_n + R_k,$$

where

$$(3.39) \quad R_k = \sum_{m, n \leq k, m+n > k} \alpha_m \beta_n.$$

Therefore,

$$(3.40) \quad |R_k| \leq \sum_{m \leq \frac{k}{2}} \sum_{\frac{k}{2} \leq n \leq k} |\alpha_m| |\beta_n| + \sum_{\frac{k}{2} \leq m \leq k} \sum_{n \leq k} |\alpha_m| |\beta_n| \leq \bar{A} \sum_{n \geq \frac{k}{2}} |\beta_n| + \bar{B} \sum_{m \geq \frac{k}{2}} |\alpha_m|,$$

where

$$(3.41) \quad \bar{A} = \sum_{n=0}^{\infty} |\alpha_n| < \infty, \quad \bar{B} = \sum_{n=0}^{\infty} |\beta_n| < \infty.$$

Therefore,  $R_k \rightarrow 0$  as  $k \rightarrow \infty$ . Since the left hand side converges to  $AB$  and the right hand side converges to  $\sum_{n=0}^{\infty} \gamma_n$ , the proof is complete.  $\square$

**Proposition 10.** *If (3.13) has a radius of convergence  $R > 0$  and  $z_1 \in D_R(z_0)$ , then  $f(z)$  has a convergent power series about  $z_1$ ,*

$$(3.42) \quad f(z) = \sum_{k=0}^{\infty} b_k (z - z_1)^k, \quad \text{for } |z - z_1| < R - |z_1 - z_0|.$$

*Proof.* Suppose without loss of generality that  $z_0 = 0$ . Setting  $f_{z_1}(\zeta) = f(z_1 + \zeta)$  when  $|\zeta| < R - |z_1|$ , using the binomial formula,

$$(3.43) \quad f_{z_1}(\zeta) = \sum_{n=0}^{\infty} a_n (z_1 + \zeta)^n = \sum_{n=0}^{\infty} \sum_{k=0}^n a_n \binom{n}{k} \zeta^k z_1^{n-k},$$

which converges absolutely by the binomial formula. Therefore,

$$(3.44) \quad f_{z_1}(\zeta) = \sum_{k=0}^{\infty} \left( \sum_{n=k}^{\infty} a_n \binom{n}{k} z_1^{n-k} \right) \zeta^k.$$

Therefore, (3.42) holds with

$$(3.45) \quad b_k = \sum_{n=k}^{\infty} a_n \binom{n}{k} z_1^{n-k}.$$

$\square$

We have used the general result.

**Proposition 11.** *If  $a_{j,k} \in \mathbb{C}$  and  $\sum_{j,k} |a_{j,k}| < \infty$ , then  $\sum_j a_{jk}$  is absolutely convergent for each  $k$ ,  $\sum_k a_{jk}$  is absolutely convergent for each  $j$ , and*

$$(3.46) \quad \sum_j \left( \sum_k a_{jk} \right) = \sum_k \left( \sum_j a_{jk} \right) = \sum_{j,k} a_{jk}.$$

*Proof.* From the hypothesis,  $\sum_j |a_{jk}| < \infty$  for each  $k$  and  $\sum_k |a_{jk}| < \infty$  for each  $j$ , and there exists  $B < \infty$  such that

$$(3.47) \quad S_N = \sum_{j=0}^N \sum_{k=0}^N |a_{jk}| \leq B, \quad \forall N.$$

The sequence  $S_N$  is bounded and monotone, so there exists a limit  $S_N \nearrow A < \infty$  as  $N \nearrow \infty$ . Therefore, for each  $\epsilon > 0$  there exists  $N(\epsilon)$  such that

$$(3.48) \quad \sum_{j,k:j>N \text{ or } k>N} |a_{jk}| < \epsilon.$$

Now then, for any  $M, K \geq N$ ,

$$(3.49) \quad \left| \sum_{j=0}^M \left( \sum_{k=0}^N a_{jk} \right) - \sum_{j=0}^N \sum_{k=0}^N a_{jk} \right| \leq \sum_{j,k:j>N \text{ or } k>N} |a_{jk}|,$$

so

$$(3.50) \quad \left| \sum_{j=0}^M \left( \sum_{k=0}^{\infty} a_{jk} \right) - \sum_{j=0}^N \sum_{k=0}^N a_{jk} \right| \leq \sum_{j,k:j>N \text{ or } k>N} |a_{jk}|,$$

and therefore,

$$(3.51) \quad \left| \sum_{j=0}^{\infty} \left( \sum_{k=0}^{\infty} a_{jk} \right) - \sum_{j=0}^N \sum_{k=0}^N a_{jk} \right| \leq \sum_{j,k:j>N \text{ or } k>N} |a_{jk}|.$$

The same is true with the order reversed, and therefore,

$$(3.52) \quad \left| \sum_{j=0}^{\infty} \left( \sum_{k=0}^{\infty} a_{jk} \right) - \sum_{k=0}^{\infty} \left( \sum_{j=0}^{\infty} a_{jk} \right) \right| < 2\epsilon.$$

This proves the proposition.  $\square$

Next, recall the definition of the Euler constant  $e$ . Consider the ordinary differential equation  $\dot{y} = y$ ,  $y(0) = 1$ , and let  $y(1) = e$ . By Picard iteration,

$$(3.53) \quad y(t) = 1 + \int_0^t y(s) ds.$$

Iterating,

$$(3.54) \quad y(t) = 1 + \int_0^t y(s) ds = 1 + t + \int_0^t \int_0^s y(\tau) d\tau ds = 1 + t + \int_0^t y(s)(t-s) ds.$$

Iterating,

$$(3.55) \quad = \sum_{j=0}^n \frac{t^j}{j!} + \frac{1}{n!} \int_0^t (t-s)^n y(s) ds.$$

Now then,

$$(3.56) \quad \frac{1}{n!} \left| \int_0^t (t-s)^n y(s) ds \right| \leq \frac{t^{n+1}}{(n+1)!} \max_{0 \leq s \leq t} |y(s)|.$$

This term vanishes as  $n \rightarrow \infty$ . Therefore, the presumed  $C^1$  solution is uniquely represented by the infinite series

$$(3.57) \quad y(t) = \sum_{j=0}^{\infty} \frac{t^j}{j!}.$$

Furthermore, the group property of flows implies

$$(3.58) \quad y(t_2)y(t_1) = y(t_1 + t_2),$$

which proves that  $y(t) = e^t$  for any rational  $t$ , which by continuity implies that  $y(t) = e^t$  for any  $t \in \mathbb{R}$ . By the chain rule, if  $f(z)$  is a holomorphic function,

$$(3.59) \quad \frac{d}{dz} e^{f(z)} = e^{f(z)} f'(z).$$

Observe that for  $t \in \mathbb{R}$ ,  $t > 0$ , (3.57) implies that  $e^t > 0$  for all  $t > 0$ . Since  $e^{-t} = \frac{1}{e^t}$ ,  $e^t > 0$  for all  $t \in \mathbb{R}$ . Also, by  $y'(t) = y(t)$ ,  $e^t$  is monotone increasing and convex. In particular,

$$(3.60) \quad \lim_{t \rightarrow \infty} e^t = \infty, \quad \lim_{t \rightarrow -\infty} e^t = 0.$$

Therefore,

$$(3.61) \quad \exp : \mathbb{R} \rightarrow (0, \infty),$$

is smooth, one to one, and onto, so apply the inverse function theorem. Thus, there exists a smooth inverse

$$(3.62) \quad L : (0, \infty) \rightarrow \mathbb{R},$$

which is called the natural logarithm. Applying  $\frac{d}{dt} L(e^t) = t$ , by the chain rule,

$$(3.63) \quad L'(e^t)e^t = 1, \quad \Rightarrow \quad L'(e^t) = \frac{1}{e^t}.$$

Therefore,

$$(3.64) \quad \frac{d}{dt} \log(t) = \frac{1}{t}.$$

Observe that (3.58) implies

$$(3.65) \quad \log xy = \log x + \log y, \quad x, y > 0.$$

Taking  $z$  complex gives the Euler formula

$$(3.66) \quad e^{i\theta} = \cos \theta + i \sin \theta.$$

Indeed, by the chain rule,  $\frac{d}{dt}(e^{it}) = ie^{it}$ , so  $\frac{d}{dt} e^{it}$  is orthogonal to  $e^{it}$ . Therefore,  $e^{it}$  is contained in the unit circle, and travels counterclockwise at speed 1, giving (3.66). This in turn implies

$$(3.67) \quad (\cos \theta + i \sin \theta)^n = \cos(n\theta) + i \sin(n\theta), \quad \text{for every } n \geq 1.$$

Then,

$$(3.68) \quad z = e^{\log r + i\theta}, \quad \text{or} \quad \log z = \log r + i\theta.$$

This logarithm is not well-defined since  $\theta$  and  $\theta + 2\pi n$  both satisfy (3.68). Similarly,

$$(3.69) \quad (r^{1/n} e^{i\theta/n} e^{2\pi i k/n})^n = z, \quad \forall \quad 1 \leq k \leq n.$$

This shows that there are  $n$  different possibilities for  $z^{1/n}$ . The complex exponential may be most naturally viewed as the covering map

$$(3.70) \quad \mathbb{C} \mapsto \mathbb{C}^* = \mathbb{C} \setminus \{0\}, \quad z \mapsto e^z,$$

showing that  $\mathbb{C}$  is the universal cover of  $\mathbb{C}^*$ .

The formula  $e^{it} = \cos(t) + i \sin(t)$  may be used to compute the value of  $\pi$ . Let  $\pi$  be defined to be the smallest positive number such that  $\gamma(2\pi) = 1$ . Then  $\gamma(\pi) = -1$  and  $\gamma(\frac{\pi}{2}) = i$ . Furthermore, from trigonometry,

$$(3.71) \quad \gamma\left(\frac{\pi}{3}\right) = \frac{1}{2} + \frac{\sqrt{3}}{2}i, \quad \gamma\left(\frac{\pi}{6}\right) = \frac{\sqrt{3}}{2} + \frac{1}{2}i.$$

We can use this fact to determine the value of  $\pi$ . We know from the fact that  $|\frac{d}{dt}(e^{it})| = |ie^{it}| = 1$  that the length of  $\gamma(t)$  on  $0 \leq t \leq \varphi$  is given by  $\varphi$ , so for  $0 < \varphi < \frac{\pi}{2}$ , parameterize this segment of the circle by

$$(3.72) \quad \sigma(s) = (\sqrt{1-s^2}, s), \quad 0 \leq s \leq \tau = \sin \varphi.$$

The length of this curve is given by

$$(3.73) \quad l = \int_0^\tau |\sigma'(s)| ds = \int_0^\tau \frac{ds}{\sqrt{1-s^2}} = \varphi.$$

Therefore, from (3.71),

$$(3.74) \quad \frac{\pi}{6} = \int_0^{1/2} \frac{dx}{\sqrt{1-x^2}}.$$

Making a power series expansion,

$$(3.75) \quad \frac{\pi}{6} = \sum_{n=0}^{\infty} \frac{a_n}{2n+1} \left(\frac{1}{2}\right)^{2n+1},$$

where  $a_n$  are defined recursively by

$$(3.76) \quad a_0 = 1, \quad a_{n+1} = \frac{2n+1}{2n+2} a_n.$$

#### 4. THE RIEMANN SPHERE

Now turn to the simple but important idea of extending the notion of analyticity to functions that take the value  $\infty$ , as well as making sense of functions that are analytic at  $z = \infty$ . To do this, define the one point compactification of  $\mathbb{C}$ , which we denote  $\mathbb{C}_\infty$ . The basis for this topology is that open neighborhoods of  $\infty$  are the complements of all compact sets. Thus,  $\mathbb{C}_\infty \sim S^2$  in the homeomorphic sense. Moreover,  $\mathbb{C} \sim S^2 \setminus \{p\}$  in the sense of conformal equivalence, where  $p \in S^2$  is arbitrary.

Define the stereographic map

$$(4.1) \quad \Phi : S^2 \setminus \{(0, 0, 1)\} \rightarrow \mathbb{C}, \quad X = (x_1, x_2, x_3) = \frac{x_1 + ix_2}{1 - x_3}.$$

Now then, it is clear how to extend the domain and range of holomorphic maps to

$$(4.2) \quad f : \mathbb{C} \rightarrow \mathbb{C}_\infty, \quad f : \mathbb{C}_\infty \rightarrow \mathbb{C}, \quad f : \mathbb{C}_\infty \rightarrow \mathbb{C}_\infty.$$

We need  $f$  to be continuous and holomorphic near points where it assumes values in the complex plane, as well as where  $\infty$  is attained. Near points where  $\infty$  is attained, it is natural to consider the function  $\frac{1}{f(z)}$ .

**Definition 6.** If  $f(z_0) = \infty$  for some  $z_0 \in \mathbb{C}$  then we say that  $f$  is holomorphic close to  $z_0$  if and only if  $\frac{1}{f(z)}$  is holomorphic around  $z_0$ . Likewise, to make sense of  $f(z)$  being holomorphic at  $z = \infty$  with values in  $\mathbb{C}$  we require that  $f(\frac{1}{z})$  be holomorphic around  $z = 0$ . Finally, if  $f(\infty) = \infty$ , then  $f$  is holomorphic around  $z = \infty$  if and only if  $\frac{1}{f(\frac{1}{z})}$  is holomorphic around  $z = 0$ .

Observe that  $z \mapsto \frac{1}{z}$  is conformal as a map from  $\mathbb{C}_\infty$  to  $\mathbb{C}_\infty$ . This is a tautology in view of our coordinate chart at  $z = \infty$ . Furthermore, if  $\Phi(z) = \frac{x_1 + ix_2}{1 - x_3}$ , using the fact that  $x_1^2 + x_2^2 + x_3^2 = 1$ ,

$$(4.3) \quad \frac{1}{\Phi(z)} = (1 - x_3) \frac{x_1 - ix_2}{x_1^2 + x_2^2} = \frac{x_1 - ix_2}{1 - x_3^2} \cdot (1 - x_3) = \frac{x_1 - ix_2}{1 + x_3}.$$

Therefore, the map  $z \mapsto \frac{1}{z}$  corresponds to the map  $(x_1, x_2, x_3) \mapsto (x_1, -x_2, -x_3)$ .

## 5. HARMONIC FUNCTIONS ON A PLANAR DOMAIN

Suppose  $f \in C^\infty(\Omega)$  is a holomorphic function. Applying  $(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y})$  to the Cauchy–Riemann equations, implies

$$(5.1) \quad (\frac{\partial}{\partial x} + i\frac{\partial}{\partial y})(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y})f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = 0,$$

on the open set  $\Omega \subset \mathbb{C}$ .

Such a function is called harmonic. More generally, if  $\mathcal{O}$  is an open set in  $\mathbb{R}^n$ , a function  $f \in C^2(\mathcal{O})$  is said to be harmonic on  $\mathcal{O}$  if  $\Delta f = 0$  on  $\mathcal{O}$ , where

$$(5.2) \quad \Delta f = \frac{\partial^2 f}{\partial x_1^2} + \dots + \frac{\partial^2 f}{\partial x_n^2} = 0.$$

Taking the real and imaginary parts of (5.1),  $f = u + iv$ ,

$$(5.3) \quad \Delta u = 0, \quad \Delta v = 0.$$

Therefore, if  $f \in C^\infty(\Omega)$  is a holomorphic function, the real and imaginary parts of  $f$  are harmonic functions on  $\Omega$ .

Many domains  $\Omega \subset \mathbb{C}$  have the property that if  $u \in C^2(\Omega)$  is a real-valued, harmonic function, then there exists a real-valued harmonic function  $v \in C^2(\Omega)$  such that  $f = u + iv$  is holomorphic on  $\Omega$ .

**Definition 7.**  $v$  is said to be the harmonic conjugate of  $u$ .

Given  $\alpha = a + ib$  and  $z = x + iy$ , let  $\gamma_{\alpha z}$  denote a path from  $a + ib$  to  $x + iy$ , and then the horizontal line from  $a + iy$  to  $x + iy$ . Next, let  $\sigma_{\alpha z}$  denote the horizontal line segment from  $a + ib$  to  $x + ib$ , and then the vertical line segment from  $x + ib$  to  $x + iy$ . Let  $R_{\alpha z}$  denote the rectangle bounded for the four line segments.

**Proposition 12.** Let  $\Omega \subset \mathbb{C}$  be open,  $\alpha = a + ib \in \Omega$ , and assume that the following property holds: If  $z \in \Omega$ , then  $R_{\alpha z} \subset \Omega$ . Let  $u \in C^2(\Omega)$  be harmonic. Then  $u$  has a harmonic conjugate  $v \in C^2(\Omega)$ .

*Proof.* For  $z \in \Omega$ , set

$$(5.4) \quad v(z) = \int_{\gamma_{\alpha z}} \left(-\frac{\partial u}{\partial y} dx + \frac{\partial u}{\partial x} dy\right) = \int_b^y \frac{\partial u}{\partial x}(a, s) ds - \int_a^x \frac{\partial u}{\partial y}(t, y) dt.$$

Also set

$$(5.5) \quad \tilde{v}(z) = \int_{\sigma_{\alpha z}} \left(-\frac{\partial u}{\partial y} dx + \frac{\partial u}{\partial x} dy\right) = -\int_a^x \frac{\partial u}{\partial y}(t, b) dt + \int_b^y \frac{\partial u}{\partial x}(x, s) ds.$$

By the fundamental theorem of calculus,

$$(5.6) \quad \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}(z), \quad \frac{\partial \tilde{v}}{\partial y}(z) = \frac{\partial u}{\partial x}(z).$$

Furthermore, since  $R_{\alpha z} \subset \Omega$ , by Green's theorem, since  $u$  is a harmonic function,

$$(5.7) \quad \tilde{v}(z) - v(z) = \int_{\partial R_{\alpha z}} \left(-\frac{\partial u}{\partial y} dx + \frac{\partial u}{\partial x} dy\right) = \int \int_{R_{\alpha z}} \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) dx dy = 0.$$

Therefore,  $u$  and  $v$  satisfy the Cauchy–Riemann equations.  $\square$

**Corollary 2.** *Let  $\Omega$  be simply connected and  $u$  a real-valued harmonic function on  $\Omega$ . Then  $u = \operatorname{Re}(f)$  for some  $f \in \mathcal{H}(\Omega)$  and  $f$  is unique up to an additive imaginary constant.*

*Proof.* We can use the argument in (5.7) for two path integrals in a simply connected domain.  $\square$

It is possible to prove this this proposition without Green's theorem.

**Proposition 13.** *Let  $\Omega \subset \mathbb{C}$  be open,  $\alpha = a + ib \in \Omega$ , and assume the following property holds: If also  $z \in \Omega$  then  $\gamma_{\alpha z} \subset \mathbb{C}$ .*

*Let  $u \in C^2(\Omega)$  be harmonic. Then  $u$  has a harmonic conjugate  $v \in C^2(\Omega)$ .*

*Proof.* Define  $v$  as in (5.4). Then  $\frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}$ . Also, by (5.4),

$$(5.8) \quad \frac{\partial v}{\partial y}(z) = \frac{\partial u}{\partial x}(a, y) - \int_a^x \frac{\partial^2 u}{\partial y^2}(t, y) dt = \frac{\partial u}{\partial x}(a, y) + \int_a^x \frac{\partial^2 u}{\partial x^2}(t, y) dt = \frac{\partial u}{\partial x}(z).$$

Therefore,  $u$  and  $v$  satisfy the Cauchy–Riemann equations.  $\square$

**Corollary 3.** *Let  $u$  be harmonic in  $\Omega$  and let  $f : \Omega_0 \rightarrow \Omega$  be holomorphic, where  $\Omega_0, \Omega \subset \mathbb{C}$ . Then  $u \circ f$  is harmonic in  $\Omega_0$ .*

*Proof.* It follows from (3.6) that

$$(5.9) \quad \partial_z(u \circ f) = [(\partial_w u) \circ f] \partial_z f.$$

Now then, since  $\Delta = 4\partial_{\bar{z}}\partial_z$ ,

$$(5.10) \quad \Delta(u \circ f) = 4\partial_{\bar{z}}\partial_z(u \circ f) = 4[(\partial_w u) \circ f] \partial_z \partial_z f + 4[(\partial_{\bar{w}} \partial_w u) \circ f] |\partial_z f|^2 = 0.$$

The last step uses the fact that  $\Delta f = \Delta w = 0$ .  $\square$

We can easily generalize this fact to simply connected domains. Returning to the question of harmonic conjugates, suppose  $\Omega$  is a simply connected domain.

**Proposition 14.** *Let  $\Omega \subset \mathbb{C}$  be a simply connected domain. Then each harmonic function  $u \in C^2(\Omega)$  has a harmonic conjugate.*

*Proof.* Choose  $\alpha \in \Omega$  and let  $\gamma_{\alpha z}$  denote an arbitrary path from  $\alpha$  to  $z$ . Set

$$(5.11) \quad v(z) = \int_{\gamma_{\alpha z}} \left(-\frac{\partial u}{\partial y} dx + \frac{\partial u}{\partial x} dy\right).$$

If this integral is independent of path from  $\alpha$  to  $z$ , then set

$$(5.12) \quad v(z) = \int_{\alpha}^z \left(-\frac{\partial u}{\partial y} dx + \frac{\partial u}{\partial x} dy\right).$$

For  $z = x + iy$ , there exists  $D_r(z) \subset \Omega$ , and choose  $\xi + iy, x + i\eta \in D_r(z)$ . Then

$$(5.13) \quad v(z) = \int_{\alpha}^{\xi+iy} \left(-\frac{\partial u}{\partial y} dx + \frac{\partial u}{\partial x} dy\right) + \int_{\xi+iy}^{x+iy} \left(-\frac{\partial u}{\partial y} dx + \frac{\partial u}{\partial x} dy\right).$$

Therefore,

$$(5.14) \quad \frac{\partial v}{\partial x}(z) = -\frac{\partial u}{\partial y}(z).$$

Also,

$$(5.15) \quad v(z) = \int_{\alpha}^{x+i\eta} \left(-\frac{\partial u}{\partial y} dx + \frac{\partial u}{\partial x} dy\right) + \int_{x+i\eta}^{x+iy} \left(-\frac{\partial u}{\partial y} dx + \frac{\partial u}{\partial x} dy\right).$$

Therefore,

$$(5.16) \quad \frac{\partial v}{\partial y}(z) = \frac{\partial u}{\partial x}(z).$$

This establishes that  $v$  is the harmonic conjugate of  $u$ . □

Path independence follows from Green's formula and the fact that  $u$  is harmonic. Indeed, if  $\gamma$  is a closed curve, and  $C_{\gamma}$  is the region contained by  $\gamma$ ,

$$(5.17) \quad \int_{\gamma} \left(-\frac{\partial u}{\partial y} dx + \frac{\partial u}{\partial x} dy\right) = \int_{C_{\gamma}} \Delta u dx dy = 0.$$

**Proposition 15.** *If  $u \in C^2(\Omega)$  is harmonic,  $z_0 \in \Omega$ , and  $\overline{D_r(z_0)} \subset \Omega$ , then*

$$(5.18) \quad u(z_0) = \frac{1}{2\pi} \int_0^{2\pi} u(z_0 + re^{i\theta}) d\theta.$$

*Proof.* Since  $u$  is a continuous function,

$$(5.19) \quad \lim_{r \searrow 0} \frac{1}{2\pi} \int_0^{2\pi} u(z_0 + re^{i\theta}) d\theta = u(z_0).$$

Taking the derivative with respect to  $r$ ,

$$(5.20) \quad \frac{d}{dr} \frac{1}{2\pi} \int_0^{2\pi} u(z_0 + re^{i\theta}) d\theta = \frac{1}{2\pi} \int_0^{2\pi} u_r(z_0 + re^{i\theta}) d\theta.$$

By Green's theorem,

$$(5.21) \quad \frac{1}{2\pi} \int_0^{2\pi} u_r(z_0 + re^{i\theta}) d\theta = \frac{1}{2\pi} \int_{D_r(z_0)} \Delta u dx dy = 0.$$

This proves (5.18). □

Writing (5.18) in polar coordinates,

$$(5.22) \quad u(z_0) = \frac{1}{\pi r^2} \int \int_{D_r(z_0)} u(z) dx dy.$$

With this, we can establish a maximum principle for harmonic functions.

**Proposition 16.** *Let  $\Omega \subset \mathbb{C}$  be a connected open set. If  $u : \Omega \rightarrow \mathbb{R}$  is harmonic on  $\Omega$ , then given  $z_0 \in \Omega$ ,*

$$(5.23) \quad u(z_0) = \sup_{z \in \Omega} u(z) \quad \Rightarrow \quad u \quad \text{is constant on} \quad \Omega.$$

*If, in addition,  $\Omega$  is bounded and  $u \in C(\bar{\Omega})$ , then*

$$(5.24) \quad \sup_{z \in \bar{\Omega}} u(z) = \sup_{z \in \partial\Omega} u(z).$$

*Proof.* Equation (5.24) follows from (5.23) if  $\Omega$  is bounded, since  $u$  must achieve a maximum somewhere on  $\bar{\Omega}$ . Thus, assume there exists  $z_0 \in \Omega$  such that the hypotheses of (5.23) hold. Set

$$(5.25) \quad \mathcal{O} = \{\zeta \in \Omega : u(\zeta) = u(z_0)\}.$$

Since  $z_0 \in \mathcal{O}$ ,  $\mathcal{O}$  is not empty. Moreover, by continuity,  $\mathcal{O}$  is a closed subset of  $\Omega$ . Moreover, by (5.22), if there exists a disk of radius  $\rho$ ,  $\overline{D_\rho(\zeta_0)} \subset \Omega$ , since  $u$  is the supremum,  $u(z) = u(\zeta_0)$  for all  $z \in D_\rho(\zeta_0)$ .  $\square$

**Corollary 4.** *If  $f(z)$  is a holomorphic function, and  $f \in C^\infty(\Omega)$ , given  $z_0 \in \Omega$ ,*

$$(5.26) \quad |f(z_0)| = \sup_{z \in \Omega} |f(z)| \quad \Rightarrow \quad f \quad \text{is constant on} \quad \Omega.$$

*If, in addition,  $\Omega$  is bounded, and  $f \in C(\bar{\Omega})$ , then*

$$(5.27) \quad \sup_{z \in \bar{\Omega}} |f(z)| = \sup_{z \in \partial\Omega} |f(z)|.$$

*Proof.* If  $f = u + iv$ , and  $u$  and  $v$  are harmonic functions, by the product rule,

$$(5.28) \quad \Delta(u^2 + v^2) = |\nabla u|^2 + |\nabla v|^2.$$

Plugging this fact into the proof of Proposition 15,

$$(5.29) \quad u(z_0)^2 + v(z_0)^2 \leq \frac{1}{\pi r^2} \int_{D_r(z_0)} (u(z)^2 + v(z)^2) dx dy.$$

Moreover, equality holds if and only if  $|\nabla u| = 0$  and  $|\nabla v| = 0$  on  $D_r(z_0)$ .  $\square$

Next, Liouville's theorem for harmonic functions on  $\mathbb{C}$ .

**Proposition 17.** *If  $u \in C^2(\Omega)$  is bounded and harmonic on all of  $\mathbb{C}$ , then  $u$  is constant.*

*Proof.* Choose any two points  $p, q \in \mathbb{C}$ . For all  $r > 0$ ,

$$(5.30) \quad u(p) - u(q) = \frac{1}{\pi r^2} \left[ \int \int_{D_r(p)} u(z) dx dy - \int \int_{D_r(q)} u(z) dx dy \right].$$

Hence,

$$(5.31) \quad |u(p) - u(q)| \leq \frac{1}{\pi r^2} \int \int_{\Delta(p,q,r)} |u(z)| dx dy,$$

where  $\Delta(p, q, r)$  is the set of points contained in  $D_r(p)$  or  $D_r(q)$ , but not both. Therefore,  $\Delta(p, q, r) \sim r$  as  $r \rightarrow \infty$ . Taking  $r \rightarrow \infty$  in (5.31), since  $|u|$  is bounded,  $u(p) - u(q) = 0$ .  $\square$

**Corollary 5.** *If  $f : \mathbb{C} \rightarrow \mathbb{C}$  is holomorphic and bounded, and  $f \in C^\infty(\Omega)$ , then  $f$  is constant.*

*Proof.* Since  $f$  is holomorphic,  $f = u + iv$ , where  $u$  and  $v$  are harmonic functions. Since  $|f|$  is uniformly bounded,  $|u|$  and  $|v|$  are uniformly bounded, and therefore, by Proposition 17,  $u$  and  $v$  are constant.  $\square$

If  $f \in C^\infty(\Omega)$  is a holomorphic function, then since  $f = u + iv$ , where  $u$  and  $v$  are harmonic functions, so

$$(5.32) \quad f(z_0) = \frac{1}{2\pi} \int_0^{2\pi} f(z_0 + re^{i\theta}) d\theta.$$

However, it is possible to prove (5.32) without making the a priori assumption that  $f \in C^\infty(\Omega)$ .

**Theorem 4.** *If  $f$  is holomorphic on the open set  $\Omega \subset \mathbb{C}$ , and  $\overline{D_r(z_0)} \subset \Omega$ , then*

$$(5.33) \quad f(z_0) = \frac{1}{2\pi} \int_0^{2\pi} f(z_0 + re^{i\theta}) d\theta = \frac{1}{2\pi i} \int_{\partial D_r(z_0)} \frac{f(\zeta)}{\zeta - z_0} d\zeta.$$

*Proof.* As in the proof of Proposition 15,

$$(5.34) \quad \lim_{r \searrow 0} \frac{1}{2\pi} \int_0^{2\pi} f(z_0 + re^{i\theta}) d\theta = f(z_0).$$

Taking a derivative with respect to  $r$ ,

$$(5.35) \quad \frac{d}{dr} \frac{1}{2\pi} \int_0^{2\pi} f(z_0 + re^{i\theta}) d\theta = \frac{1}{2\pi} \int_0^{2\pi} f'(z_0 + re^{i\theta}) e^{i\theta} d\theta = \frac{1}{2\pi i r} \int_{\partial D_r(z_0)} f'(z_0 + \zeta) d\zeta = 0.$$

□

**Corollary 6** (Cauchy integral formula). *If  $f$  is holomorphic on  $\Omega$ , then  $f \in C^\infty(\Omega)$ .*

*Proof.* We can compute the derivative of (5.33) directly.

$$(5.36) \quad \left(\frac{d}{dz}\right)^n f(z) = \frac{(-1)^n n!}{2\pi i} \int_{\partial D_r(z)} \frac{f(\zeta)}{(\zeta - z)^{n+1}} d\zeta.$$

This integral converges for any  $n$ .

□

## 6. THE CAUCHY INTEGRAL THEOREM

So far, we have not completely explained the definitions of the integrals in (5.33) and (5.36). For a general curve in  $\mathbb{C}$ , define the Cauchy integral using the fundamental theorem of calculus in one variable.

**Theorem 5.** *If  $f \in C^1([a, b])$ , then*

$$(6.1) \quad \int_a^b f'(t) dt = f(b) - f(a).$$

*Furthermore, if  $g \in C([a, b])$ , then*

$$(6.2) \quad \frac{d}{dt} \int_a^t g(s) ds = g(t).$$

In the study of the holomorphic functions on the open set  $\Omega \subset \mathbb{C}$ , consider the integral along the path

$$(6.3) \quad \gamma : [a, b] \rightarrow \Omega.$$

Then if  $f : \Omega \rightarrow \mathbb{C}$  is continuous,

$$(6.4) \quad \int_\gamma f(z) dz = \int_a^b f(\gamma(t)) \gamma'(t) dt.$$

**Remark 5.** We have already seen this definition in (5.33). There, we take  $z_0 + re^{i\theta} = \gamma(\theta)$ ,  $0 \leq \theta \leq 2\pi$ ,  $\gamma'(\theta) = rie^{i\theta} = d\zeta$ , and  $\zeta - z_0 = \frac{1}{re^{i\theta}}$ .

**Proposition 18.** If  $f$  is holomorphic on  $\Omega$  and  $\gamma : [a, b] \rightarrow \mathbb{C}$  is a  $C^1$  path, then

$$(6.5) \quad \int_{\gamma} f'(z)dz = f(\gamma(b)) - f(\gamma(a)).$$

*Proof.* The proof uses the following chain rule.

**Proposition 19.** If  $f : \Omega \rightarrow \mathbb{C}$  is holomorphic and  $\gamma : [a, b] \rightarrow \mathbb{C}$  is  $C^1$ , then for  $a < t < b$ ,

$$(6.6) \quad \frac{d}{dt}f(\gamma(t)) = f'(\gamma(t))\gamma'(t).$$

*Proof.* This follows from the the chain rule in (3.4). □

Then by the fundamental theorem of calculus, the proof of (6.5) is complete. □

It is possible to use these computations in connection with an antiderivative.

**Definition 8** (Anti-derivative). A holomorphic function  $g : \Omega \rightarrow \mathbb{C}$  is said to have an anti-derivative  $f$  on  $\Omega$  provided  $f : \Omega \rightarrow \mathbb{C}$  is holomorphic and  $f' = g$ .

Each holomorphic function  $g : \Omega \rightarrow \mathbb{C}$  has an antiderivative for a class of sets  $\Omega \subset \mathbb{C}$  which satisfy the following property: If  $a + ib \in \Omega$  and  $x + iy \in \Omega$ , then the vertical line from  $a + ib$  to  $a + iy$  and the horizontal line from  $a + iy$  to  $x + iy$  lie in  $\Omega$ .

**Proposition 20.** If  $\Omega \subset \mathbb{C}$  is an open set satisfying the above property, and  $g : \Omega \rightarrow \mathbb{C}$  is holomorphic, then there exists a holomorphic  $f : \Omega \rightarrow \mathbb{C}$  such that  $f' = g$ .

*Proof.* By the fundamental theorem of calculus,

$$(6.7) \quad \frac{\partial f}{\partial x}(z) = g(z).$$

Also,

$$(6.8) \quad \frac{1}{i} \frac{\partial f}{\partial y} = g(a + iy) + \lim_{h \rightarrow 0} \frac{1}{ih} \int_a^x [g(t + iy + ih) - g(t + iy)]dt.$$

Since  $g$  is holomorphic,

$$(6.9) \quad (6.8) = g(x + iy),$$

so  $f$  is holomorphic. □

This computation may be used to give a second proof of

**Proposition 21.** If  $R > 0$ , the function defined by

$$(6.10) \quad f(z) = \sum_{n=0}^{\infty} a_n(z - z_0)^n$$

is holomorphic on  $D_R(z_0)$  with derivative given by

$$(6.11) \quad f'(z) = \sum_{n=1}^{\infty} na_n(z - z_0)^{n-1}.$$

*Proof.* For any  $k$ , consider

$$(6.12) \quad f_k(z) = \sum_{n=0}^k a_n (z - z_0)^n, \quad g_k(z) = \sum_{n=1}^k n a_n (z - z_0)^{n-1}.$$

Then  $f_k \rightarrow f$  and  $g_k \rightarrow g$  locally uniformly on  $D_R(z_0)$ . Also, for each  $k$ ,  $f'_k(z) = g_k(z)$ . Therefore, for any  $z \in D_R(z_0)$ ,

$$(6.13) \quad f_k(z) = a_0 + \int_{\sigma_z} g_k(\zeta) d\zeta,$$

where  $\sigma_z$  is a path from  $z_0$  to  $z$ .

Making use of local uniform convergence,

$$(6.14) \quad f(z) = a_0 + \int_{\sigma_z} g(\zeta) d\zeta.$$

Taking  $\sigma_z$  to be a path that approaches  $z$  horizontally,  $z = x + iy$ ,  $z_0 = x_0 + iy_0$ ,

$$(6.15) \quad f(z) = a_0 + \int_{y_0}^y g(x_0 + it) i dt + \int_{x_0}^x g(t + iy) dt,$$

$$(6.16) \quad \frac{\partial f}{\partial x}(z) = g(z).$$

Meanwhile, taking  $\sigma_z$  to be a path that approaches  $z$  vertically,

$$(6.17) \quad f(z) = a_0 + \int_{x_0}^x g(t + iy_0) dt + \int_{y_0}^y g(x + it) i dt,$$

so therefore,

$$(6.18) \quad \frac{1}{i} \frac{\partial f}{\partial y}(z) = g(z).$$

Since each  $g_k$  is holomorphic, and therefore by Proposition 20 the integrals of each  $g_k$  are path independent, and  $g_k \rightarrow g$  locally uniformly, the proof is complete.  $\square$

The existence of an anti-derivative can be used to prove the following fact.

**Lemma 6.** *Let  $\gamma_r$  be the circle  $\{z : |z| = r\}$  for some  $r > 0$  fixed, with the counter-clockwise orientation. Then,*

$$(6.19) \quad \int_{\gamma_r} z^n dz = \begin{cases} 0 & \text{if } n \neq -1, \\ 2\pi i & \text{if } n = -1. \end{cases}$$

*Proof.* By direct computation, if  $n \neq -1$ ,  $z^n$  has the anti-derivative  $F(z) = \frac{1}{n+1} z^{n+1}$ . Taking for example the parameterization  $\gamma(t) = r e^{it}$ ,  $0 \leq t \leq 2\pi$ , (6.19) = 0. When  $n = -1$ ,  $dz = i r e^{it} dt$ , so in that case (6.19) =  $2\pi i$ .  $\square$

**Theorem 6.** *Let  $\gamma_0, \gamma_1 : [0, 1] \rightarrow \Omega$  be  $C^1$  curves with  $\gamma_0(0) = \gamma_1(0)$  and  $\gamma_0(1) = \gamma_1(1)$ . Assume that they are  $C^1$  homotopic in the following sense: there exists a continuous map  $H : [0, 1]^2 \rightarrow \Omega$  with  $H(t, 0) = \gamma_0(t)$  and  $H(t, 1) = \gamma_1(t)$  such that  $H(\cdot, s)$  is a  $C^1$  curve for each  $0 \leq s \leq 1$ . Moreover,  $H(s, 0) = \gamma_0(0) = \gamma_1(0)$  and  $H(s, 1) = \gamma_0(1) = \gamma_1(1)$  for any  $0 \leq s \leq 1$ . Then,*

$$(6.20) \quad \int_{\gamma_0} f(z) dz = \int_{\gamma_1} f(z) dz,$$

for all  $f \in \mathcal{H}(\Omega)$ . In particular, if  $\gamma$  is a closed curve in  $\Omega$  that is homotopic to a point, then

$$(6.21) \quad \int_{\gamma} f(z)dz = 0.$$

In particular, this is the case if  $\gamma$  is the boundary of a subregion  $\Omega_1 \subset \Omega$  such that  $\overline{\Omega_1}$  is diffeomorphic to a closed disk.

*Proof.* Since  $f(z)$  is holomorphic,  $f(z)dz$  is a closed form. Indeed,

$$(6.22) \quad d(f(z)dz) = \partial_z f(z)dz \wedge dz + \partial_{\bar{z}} f(z)d\bar{z} \wedge dz = 0,$$

since  $\partial_{\bar{z}} f = 0$ . Thus, Cauchy's theorem is a special case of the homotopy invariance of the integral over closed forms, which follows from Stokes theorem.

First note that since a closed form is locally exact,  $\int_{\eta} f(z)dz = 0$  for all closed curves that fall into sufficiently small disks. But then we can triangulate the homotopy so that

$$(6.23) \quad \int_{\gamma_0} f(z)dz - \int_{\gamma_1} f(z)dz = \sum_j \int_{\eta_j} f(z)dz = 0.$$

Similarly, recall Green's formula,

$$(6.24) \quad \int_{\partial R} a(x, y)dx + b(x, y)dy = \int \int_R (-a_y(x, y) + b_x(x, y))dxdy.$$

This formula extends to far more general regions as those diffeomorphic to a disk and bounded by finitely many  $C^1$  curves. Then,

$$(6.25) \quad \int_{\partial U} f(z)dz = \int_{\partial U} udx - vdy + i(udy + vdx) = \int \int_U (-u_y - v_x)dxdy + i \int \int_U (-v_y + u_x)dxdy = 0.$$

The final equality sign follows from the Cauchy–Riemann equations.  $\square$

**Proposition 22** (Cauchy integral theorem). *Let  $f \in \mathcal{H}(\Omega)$ , and let  $z_0 \in \Omega$  be an open set, and let  $\gamma$  be a curve in  $\Omega$  that contains  $z_0$  and is contractible. Then,*

$$(6.26) \quad f(z_0) = \frac{1}{2\pi i} \int_{\gamma} \frac{f(\zeta)}{\zeta - z_0} d\zeta.$$

*Proof.* This follows from Theorem 4 and Theorem 6.  $\square$

## 7. CONSEQUENCES OF THE CAUCHY INTEGRAL FORMULA

The Cauchy integral formula implies that  $\mathcal{A}(\Omega) = \mathcal{H}(\Omega)$ .

**Theorem 7.** *If  $f \in C^1(\bar{\Omega})$  is holomorphic, then for  $z \in D_r(z_0) \subset \Omega$ ,  $f(z)$  has the convergent power series expansion*

$$(7.1) \quad f(z) = \sum_{n=0}^{\infty} a_n(z - z_0)^n,$$

with

$$(7.2) \quad a_n = \frac{1}{2\pi i} \int_{\partial \Omega} \frac{f(\zeta)}{(\zeta - z_0)^{n+1}} d\zeta = \frac{f^{(n)}(z_0)}{n!}.$$

*Proof.* Suppose  $z \in D_r(z_0)$ . By Theorem 4,

$$(7.3) \quad f(z) = \frac{1}{2\pi i} \int_{\partial\Omega} \frac{f(\zeta)}{(\zeta - z_0) - (z - z_0)} d\zeta.$$

Making the infinite series expansion, since  $|z - z_0| < |\zeta - z_0|$ ,

$$(7.4) \quad \frac{1}{(\zeta - z_0) - (z - z_0)} = \frac{1}{\zeta - z_0} \sum_{n=0}^{\infty} \left(\frac{z - z_0}{\zeta - z_0}\right)^n.$$

Plugging this series into (5.33) gives (7.1) and (7.2).  $\square$

**Corollary 7.** (1) *Cauchy's estimates.* Let  $f \in \mathcal{H}(\Omega)$  with  $|f(z)| \leq M$  on  $\Omega$ . Then,

$$(7.5) \quad |f^{(n)}(z)| \leq \frac{Mn!}{\text{dist}(z, \partial\Omega)^n},$$

for every  $n \geq 0$  and  $z \in \Omega$ .

(2) *(Liouville's theorem)* If  $f \in \mathcal{H}(\mathbb{C}) \cap L^\infty(\mathbb{C})$  then  $f$  is constant. More generally, if  $|f(z)| \leq C(1 + |z|)^N$  for some fixed integer  $N \geq 0$  and a finite constant  $C$ , then  $f$  is a polynomial of degree at most  $N$ .

*Proof.* Equation (8.5) follows from the Cauchy integral formula. To prove Liouville's theorem, apply (8.5).  $\square$

It is possible to prove the fundamental theorem of algebra using the maximum principle.

**Theorem 8** (Fundamental theorem of algebra). If  $p(z) = a_n z^n + \dots + a_1 z + a_0$ ,  $a_n \neq 0$  for some  $n \geq 1$  is a polynomial of degree  $n$ , then  $p(z)$  must vanish somewhere on  $\mathbb{C}$ .

*Proof.* If  $p(z)$  does not vanish on  $\mathbb{C}$ , then  $f(z) = \frac{1}{p(z)}$  is an entire function on  $\mathbb{C}$ . Furthermore,

$$(7.6) \quad \frac{1}{p(z)} = \frac{1}{z^n a_n + a_{n-1} z^{-1} + \dots + a_0 z^{-n}}.$$

Then

$$(7.7) \quad \lim_{z \rightarrow \infty} \left| \frac{1}{p(z)} \right|,$$

exists and is uniformly bounded. Then, by Liouville's theorem,  $\frac{1}{p(z)}$  is constant.  $\square$

**Proposition 23** (Schwarz lemma). Suppose  $f$  is holomorphic on the unit disk  $D_1(0)$ . Assume  $|f(z)| \leq 1$  for  $|z| < 1$ , and  $f(0) = 0$ . Then,

$$(7.8) \quad |f(z)| \leq |z|.$$

Furthermore, equality holds in (7.8), for some  $z \in D_1(0) \setminus 0$ , if and only if  $f(z) = cz$  for some constant of absolute value one. If  $f$  is bijective, from  $\mathbb{D}$  onto itself, then  $f$  is a rotation.

*Proof.* The hypotheses imply that  $g(z) = \frac{f(z)}{z}$  is a holomorphic function on  $D_1(0)$ . Therefore,  $|g(z)| \leq \frac{1}{a}$  for  $z \in D_a(0)$ ,  $0 < a < 1$ . Using the maximum principle,  $|g(z)| \leq \frac{1}{a}$  for all  $z \in D_a(0)$ . Taking  $a \nearrow 1$ ,

$$(7.9) \quad |g(z)| \leq 1, \quad \forall z \in D_1(0).$$

Therefore, (7.8) holds. Next, suppose that  $|f(z_0)| = |z_0|$  at some  $z_0 \in D_1(0) \setminus 0$ . Then  $g$  attains a maximum at  $z_0$ , which implies  $g(z)$  is constant on  $D_1(0)$ , so  $f(z) = cz$ .

Finally, suppose that  $f$  is invertible. Then we can apply the same reasoning to  $f^{-1}$ , which implies  $|f^{-1}(w)| \leq |w|$ , which is equivalent to  $|f(z)| \geq |z|$ . Therefore,  $|f(z)| = |z|$  for all  $z$ , so  $f$  is a rotation.  $\square$

The Cauchy integral formula yields a locally uniform convergence result.

**Proposition 24.** *Let  $\Omega \subset \mathbb{C}$  be an open set and let  $f_\nu : \Omega \rightarrow \mathbb{C}$  be holomorphic. Assume  $f_\nu \rightarrow f$  locally uniformly (i.e. uniformly on each compact subset of  $\Omega$ ). Then  $f : \Omega \rightarrow \mathbb{C}$  is holomorphic, and  $f'_\nu \rightarrow f'$  locally uniformly on  $\Omega$ .*

*Proof.* Let  $K \subset \Omega$  be a compact set. Then choose a smoothly bounded  $\mathcal{O}$  such that  $K \subset \mathcal{O} \subset \bar{\mathcal{O}} \subset \Omega$ . Then, by the Cauchy integral formula,

$$(7.10) \quad f_\nu(z) = \frac{1}{2\pi i} \int_{\partial\mathcal{O}} \frac{f_\nu(\zeta)}{\zeta - z} d\zeta,$$

$$(7.11) \quad f'_\nu(z) = \frac{1}{2\pi i} \int_{\partial\mathcal{O}} \frac{f_\nu(\zeta)}{(\zeta - z)^2} d\zeta.$$

Since  $f_\nu \rightarrow f$  locally uniformly on  $\partial\mathcal{O}$ , the integrands in (7.10) and (7.11) converge uniformly on  $\bar{\mathcal{O}}$ . Therefore, for any  $z \in \mathcal{O}$ ,

$$(7.12) \quad f(z) = \frac{1}{2\pi i} \int_{\partial\mathcal{O}} \frac{f(\zeta)}{\zeta - z} d\zeta, \quad \forall z \in \mathcal{O},$$

so  $f$  is holomorphic on  $\mathcal{O}$ , and

$$(7.13) \quad f'(z) = \frac{1}{2\pi i} \int_{\partial\mathcal{O}} \frac{f(\zeta)}{(\zeta - z)^2} d\zeta,$$

so  $f'_\nu \rightarrow f'$  locally uniformly on  $\mathcal{O}$ .  $\square$

## 8. SQUARE ROOTS, LOGS, AND OTHER INVERSE FUNCTIONS

Now turn to inverse functions. Recall the inverse function theorem for functions from  $\mathbb{R}^n$  to  $\mathbb{R}^n$ .

**Theorem 9.** *Let  $\Omega \subset \mathbb{R}^n$  be open and let  $f : \Omega \rightarrow \mathbb{R}^n$  be a  $C^1$  map. Take  $p \in \Omega$  and assume  $Df(p)$  is an invertible linear transformation on  $\mathbb{R}^n$ . Then there exists a neighborhood  $\mathcal{O}$  of  $p$  and a neighborhood  $U$  of  $q = f(p)$  such that  $f : \mathcal{O} \rightarrow U$  is one-to-one and onto, the inverse  $g = f^{-1} : U \rightarrow \mathcal{O}$  is  $C^1$ , and for  $x \in \mathcal{O}$ ,  $y = f(x)$ ,*

$$(8.1) \quad Dg(y) = Df(x)^{-1}.$$

This result has the following consequence for holomorphic functions.

**Theorem 10.** *Let  $\Omega \subset \mathbb{C}$  be open and let  $f : \Omega \rightarrow \mathbb{C}$  be holomorphic. Take  $p \in \Omega$  and assume  $f'(p) \neq 0$ . Then there exists a neighborhood  $\mathcal{O}$  of  $p$  and a neighborhood  $U$  of  $q = f(p)$  such that  $f : \mathcal{O} \rightarrow U$  is one-to-one and onto, the inverse  $g = f^{-1} : U \rightarrow \mathcal{O}$  is holomorphic, and, for  $z \in \mathcal{O}$ ,  $w = f(z)$ ,*

$$(8.2) \quad g'(w) = \frac{1}{f'(z)}.$$

*Proof.* Taking the matrix representation of the derivative,  $Df(x)$  is of the form

$$(8.3) \quad \begin{pmatrix} a & -b \\ b & a \end{pmatrix}.$$

The inverse of this matrix is given by

$$(8.4) \quad \frac{1}{a^2 + b^2} \begin{pmatrix} a & b \\ -b & a \end{pmatrix},$$

which satisfies (8.2).  $\square$

If a holomorphic function is 1–1 on a set, it is possible to produce an integral for the inverse of a holomorphic map.

**Proposition 25.** *Suppose  $f$  is holomorphic and one-to-one on a neighborhood of  $\bar{\Omega}$ , the closure of a piecewise, smoothly bounded domain  $\Omega \subset \mathbb{C}$ . Set  $g = f^{-1} : f(\Omega) \rightarrow \Omega$ . Then*

$$(8.5) \quad g(w) = \frac{1}{2\pi i} \int_{\partial\Omega} \frac{zf'(z)}{f(z) - w} dz, \quad \forall w \in f(\Omega).$$

*Proof.* Set  $\zeta = g(w)$ , so that  $h(z) = f(z) - w$  has one zero in  $\bar{\Omega}$ , at  $z = \zeta$ , and  $h'(\zeta) \neq 0$ . Indeed, if  $h$  has a zero of order  $k$  at  $z_0$ , then

$$(8.6) \quad h(z) = (z - z_0)^k \varphi(z)^k$$

for some holomorphic function  $\varphi(z)$  that is nonvanishing in a neighborhood of  $\bar{\Omega}$ . Therefore, if  $f$  is one to one, then  $f(z) - w = (z - z_0)\varphi(z)$  for some holomorphic function  $\varphi(z)$  on a neighborhood of  $\bar{\Omega}$ . Therefore,

$$(8.7) \quad \frac{1}{2\pi i} \int_{\partial\Omega} z \frac{h'(z)}{h(z)} dz = \frac{1}{2\pi i} \int_{\partial\Omega} z \left( \frac{1}{z - z_0} + \frac{\varphi'(z)}{\varphi(z)} \right) dz = \zeta. \quad \square$$

The reason we require  $k = 1$  is due to the existence of  $k$ -th roots. Let  $h(z) = (z - z_0)\varphi(z)^{1/k}$ . By the product rule,

$$(8.8) \quad \frac{d}{dz} ((z - z_0)\varphi(z)^{1/k})|_{z=z_0} = \varphi(z_0)^{1/k} + (z - z_0) \frac{1}{k} \varphi(z)^{1/k-1} \varphi'(z)|_{z=z_0} = \varphi(z_0)^{1/k} \neq 0,$$

and therefore, by the implicit function theorem,  $h(z)$  maps a neighborhood  $\Omega$  of  $z_0$  1–1 and onto a disk  $D_\epsilon(0)$ . Therefore,  $(z - z_0)^k \varphi(z)$  maps  $\Omega$  onto the disk  $D_{\epsilon^k}(0)$ , and every point in this disk  $\neq 0$  has  $k$ -preimages.

**Proposition 26.** *Let  $\Omega \subset \mathbb{C}$  be a simply connected domain. Assume  $f : \Omega \rightarrow \mathbb{C}$  is holomorphic and nowhere vanishing. Then there exists a holomorphic function  $g$  on  $\Omega$  such that*

$$(8.9) \quad e^{g(z)} = f(z), \quad \forall z \in \Omega.$$

*Proof.* Fix  $z_0 \in \Omega$  and let  $e^{g(z_0)} = f(z_0)$ . Since  $f(z_0) \neq 0$  this is always possible. Then for any  $z \in \Omega$ , define

$$(8.10) \quad g(z) = \int_{z_0}^z \frac{f'(\zeta)}{f(\zeta)} d\zeta.$$

Note that since  $f \neq 0$ ,  $\frac{f'(\zeta)}{f(\zeta)}$  is analytic on  $\Omega$ . Since  $\Omega$  is simply connected, this integral is well-defined. Furthermore,

$$(8.11) \quad \frac{g(z+h) - g(z)}{h} = \int_0^1 \frac{f'(z+th)}{f(z+th)} dt \rightarrow \frac{f'(z)}{f(z)},$$

as  $h \rightarrow 0$ . Therefore,  $g \in \mathcal{H}(\Omega)$  and  $(fe^{-g})' = 0$ , which implies that  $e^g = cf$  for some constant  $c \neq 0$ .

Furthermore, observe that we can define the roots  $f_n(z) = e^{g(z)/n}$ . Then  $(f_n(z))^n = e^{g(z)} = f(z)$ , as desired.  $\square$

Consider for example the function  $f(z) = z^2$ . In polar coordinates, if  $z = re^{i\theta}$ ,  $z^2 = r^2e^{2i\theta}$ . Therefore,  $f(z)$  maps the right half plane

$$(8.12) \quad H = \{z \in \mathbb{C} : \operatorname{Re}(z) > 0\},$$

bijectively onto  $\mathbb{C} \setminus \mathbb{R}^-$ . Since  $f'(z) = 2z$  vanishes only at the origin, we have a holomorphic inverse

$$(8.13) \quad \operatorname{Sqrt} : \mathbb{C} \setminus \mathbb{R}^- \rightarrow H,$$

which is given by

$$(8.14) \quad \operatorname{Sqrt}(re^{i\theta}) = r^{1/2}e^{i\theta/2}, \quad r > 0, \quad -\pi < \theta < \pi.$$

We can also write

$$(8.15) \quad \sqrt{z} = z^{1/2} = \operatorname{Sqrt}(z).$$

Next, consider the inverse of the exponential function  $\exp(z) = e^z$ . Consider the strip

$$(8.16) \quad \Sigma = \{x + iy : x \in \mathbb{R}, \quad -\pi < y < \pi\}.$$

Since  $e^{x+iy} = e^x e^{iy}$ , we have a bijective map

$$(8.17) \quad \exp : \Sigma \rightarrow \mathbb{C} \setminus \mathbb{R}^-.$$

Since  $\frac{d}{dz}e^z = e^z$  is nowhere vanishing, (8.17) has a holomorphic inverse denoted as  $\log$ .

$$(8.18) \quad \log : \mathbb{C} \setminus \mathbb{R}^- \rightarrow \Sigma.$$

Taking  $\log 1 = 0$  and since

$$(8.19) \quad \frac{d}{dz}e^z = e^z \Rightarrow \frac{d}{dz} \log z = \frac{1}{z}.$$

Thus,

$$(8.20) \quad \log z = \int_1^z \frac{1}{\zeta} d\zeta,$$

where the integral is along any path from 1 to  $z$  in  $\mathbb{C} \setminus \mathbb{R}^-$ .

Then, given  $a \in \mathbb{C}$ , define

$$(8.21) \quad z^a = \operatorname{Pow}_a(z), \quad \operatorname{Pow}_a : \mathbb{C} \setminus \mathbb{R}^- \rightarrow \mathbb{C},$$

by

$$(8.22) \quad z^a = e^{a \log z}.$$

Since  $e^{u+v} = e^u e^v$ ,

$$(8.23) \quad z^{a+b} = z^a z^b.$$

In particular, (8.23) implies that for any  $n \in \mathbb{Z}$ ,  $n \neq 0$ ,

$$(8.24) \quad (z^{1/n})^n = z.$$

Then by (8.19),

$$(8.25) \quad \frac{d}{dz} z^a = a z^{a-1}.$$

Now prove a global inverse function theorem.

**Theorem 11.** *Suppose  $\Omega \subset \mathbb{C}$  is convex. Assume  $f$  is holomorphic in  $\Omega$ , and there exists  $a \in \mathbb{C}$  such that*

$$(8.26) \quad \operatorname{Re}(af'(z)) > 0, \quad \text{on } \Omega.$$

*Then  $f$  maps  $\Omega$  one to one onto its image  $f(\Omega)$ .*

*Proof.* Take two distinct points  $z_0, z_1 \in \Omega$ . By convexity,  $\sigma(t) = (1-t)z_0 + tz_1$  lies in  $\Omega$  for all  $t \in [0, 1]$ . Then

$$(8.27) \quad a \frac{f(z_1) - f(z_0)}{z_1 - z_0} = \int_0^1 af'((1-t)z_0 + tz_1) dt.$$

Then by (8.26), (8.27)  $\neq 0$ . □

For example, consider the strip

$$(8.28) \quad \tilde{\Sigma} = \{x + iy : -\frac{\pi}{2} < x < \frac{\pi}{2}, \quad y \in \mathbb{R}\}.$$

Take  $f(z) = \sin z$ ,  $f'(z) = \cos z$ . Then for  $z \in \tilde{\Sigma}$ ,

$$(8.29) \quad \operatorname{Re} \cos z = \cos x \cosh y \quad \text{for } z \in \tilde{\Sigma}.$$

Therefore,  $f$  maps  $\tilde{\Sigma}$  one to one onto its image.

**Theorem 12.** *The function  $\sin$  maps  $\tilde{\Sigma}$  one-to-one onto the set*

$$(8.30) \quad \mathbb{C} \setminus \{(-\infty, -1] \cup [1, \infty)\}.$$

*Proof.* To see this, observe that  $\sin(z) = g(e^{iz})$ , where  $g(\zeta) = \frac{1}{2i}(\zeta - \frac{1}{\zeta})$ . Observe that the image of  $\tilde{\Sigma}$  under the map  $z \mapsto e^{iz}$  is the right half plane  $H$ . Next, the image of  $H$  under  $g$  is

$$(8.31) \quad \mathbb{C} \setminus \{(-\infty, -1] \cup [1, \infty)\}.$$

**Proposition 27.** *Let*

$$(8.32) \quad h(\zeta) = g(i\zeta) = \frac{1}{2}(\zeta + \frac{1}{\zeta}).$$

*Since  $g(\zeta) = h(-i\zeta)$ ,*

$$(8.33) \quad h(-i\zeta) = \frac{1}{2}(\frac{\zeta}{i} + \frac{1}{-i\zeta}) = \frac{1}{2i}(\zeta - \frac{1}{\zeta}).$$

*The function  $h$  given by (8.32) maps both the upper half plane  $U = \{\zeta : \operatorname{Im}\zeta > 0\}$  and the lower half plane  $U^* = \{\zeta : \operatorname{Im}(\zeta) > 0\}$  one-to-one and onto*

$$(8.34) \quad \mathbb{C} \setminus \{(-\infty, -1] \cup [1, \infty)\}.$$

*Proof.* Observe that  $h : \mathbb{C} \setminus 0 \rightarrow \mathbb{C}$ , and

$$(8.35) \quad h\left(\frac{1}{\zeta}\right) = h(\zeta).$$

Solving for  $h(\zeta) = w$ , if  $\zeta \neq 0$ ,

$$(8.36) \quad \zeta^2 - 2w\zeta + 1 = 0,$$

which has the solutions

$$(8.37) \quad \zeta = w \pm \sqrt{w^2 - 1}.$$

Then for each  $w \in \mathbb{C}$ , there are two solutions, except for  $w = \pm 1$ .

Then  $h$  maps  $\mathbb{R} \setminus 0$  onto  $(-\infty, 1] \cup [1, \infty)$  two to one, except at  $x = \pm 1$ . This takes care of the two images on the real line with  $|x| \geq 1$ . Therefore, given  $\zeta \in \mathbb{C} \setminus 0$ ,  $h(\zeta) = w$  belongs to  $(-\infty, -1] \cup [1, \infty)$  if and only if  $\zeta \in \mathbb{R}$ .

Therefore, if  $w \in \mathbb{C} \setminus \{(-\infty, -1] \cup [1, \infty)\}$ , then  $h(\zeta) = w$  has two solutions, both in  $\mathbb{C} \setminus \mathbb{R}$ . Furthermore, the two solutions are reciprocals of each other, so given  $\zeta \in \mathbb{C} \setminus \mathbb{R}$ ,  $\zeta \in U \Leftrightarrow \frac{1}{\zeta} \in U^*$ .  $\square$

The inverse function is denoted

$$(8.38) \quad \sin^{-1} : \mathbb{C} \setminus \{(-\infty, -1] \cup [1, \infty)\} \rightarrow \tilde{\Sigma}.$$

For  $z \in \tilde{\Sigma}$ ,  $\sin^2(z) \in \mathbb{C} \setminus [1, \infty)$ , and therefore,

$$(8.39) \quad \cos(z) = (1 - \sin^2 z)^{1/2}, \quad z \in \tilde{\Sigma}.$$

Therefore, by the inverse function theorem,  $g(z) = \sin^{-1} z$  satisfies,

$$(8.40) \quad g'(z) = (1 - z^2)^{-1/2}, \quad z \in \mathbb{C} \setminus \{(-\infty, -1] \cup [1, \infty)\}.$$

Therefore,

$$(8.41) \quad \sin^{-1} z = \int_0^z (1 - \zeta^2)^{-1/2} d\zeta,$$

where the integral is along any path from 0 to  $z$  in  $\mathbb{C} \setminus \{(-\infty, -1] \cup [1, \infty)\}$ .  $\square$

## 9. MORERA'S THEOREM, SCHWARZ REFLECTION PRINCIPLE, AND GOURSAT'S THEOREM

Let  $\Omega$  be a connected open set in  $\mathbb{C}$ . If  $f : \Omega \rightarrow \mathbb{C}$  is holomorphic, then the Cauchy integral formula and Cauchy integral theorem hold for  $f$ . Here, we establish a converse of the Cauchy integral theorem, Morera's theorem.

**Theorem 13.** *Assume  $g : \Omega \rightarrow \mathbb{C}$  is continuous and*

$$(9.1) \quad \int_{\gamma} g(z) dz = 0,$$

*whenever  $\gamma = \partial R$ , where  $R$  is a rectangle with sides parallel to the real and imaginary axes. Then  $g$  is holomorphic.*

*Proof.* Holomorphicity is a local property, so assume without loss of generality that  $\Omega$  is a rectangle. Fix  $\alpha = a + ib$  in  $\Omega$ . Given  $z \in \Omega$ , let  $\gamma_{\alpha z}$  and  $\sigma_{\alpha z}$  be piecewise linear paths from  $\alpha$  to  $z$ . Then

$$(9.2) \quad f(z) = \int_{\gamma_{\alpha z}} g(\zeta) d\zeta = i \int_b^y g(a + is) ds + \int_a^x g(t + iy) dt,$$

and

$$(9.3) \quad f(z) = \int_{\sigma_{\alpha z}} g(\zeta) d\zeta = \int_a^x g(s + ib) ds + i \int_b^y g(x + it) dt.$$

By (9.1), (9.2) and (9.3) are equal. Therefore,

$$(9.4) \quad \frac{\partial f}{\partial x}(z) = \frac{1}{i} \frac{\partial f}{\partial y}(z) = g(z).$$

Thus,  $f : \Omega \rightarrow \mathbb{C}$  is  $C^1$  and satisfies the Cauchy–Riemann equations, so  $f$  is holomorphic. Therefore,  $g$  is holomorphic.  $\square$

Morera’s theorem may be used to prove the Schwarz reflection principle. Assume  $\Omega \subset \mathbb{C}$  is an open set that is symmetric across the real axis,

$$(9.5) \quad z \in \Omega \Rightarrow \bar{z} \in \Omega.$$

Set  $L = \Omega \cap \mathbb{R}$  and set  $\Omega^\pm = \{z \in \Omega : \pm \text{Im}(z) > 0\}$ .

**Proposition 28.** *Assume  $f : \Omega^+ \cup L \rightarrow \mathbb{C}$  is continuous, holomorphic in  $\Omega^+$ , and real valued on  $L$ . Then define  $g : \Omega \rightarrow \mathbb{C}$  by*

$$(9.6) \quad g(z) = f(z), \quad z \in \Omega^+ \cup L, \quad g(z) = \overline{f(\bar{z})}, \quad z \in \Omega^-.$$

*Then  $g$  is holomorphic on  $\Omega$ .*

*Proof.* It can be verified that  $g$  is  $C^1$  on  $\Omega^-$  and satisfies the Cauchy–Riemann equations, so  $g$  is holomorphic on  $\Omega \setminus L$ . Also,  $g$  is continuous on  $\Omega$ .

To show that  $g$  is holomorphic on all of  $\Omega$ ,  $g$  satisfies (9.1) when  $\gamma = \partial R$ , and  $R \subset \Omega^+$ . The same is also true if  $R \subset \Omega^-$ . Finally, if  $R$  intersects  $L$ , it is possible to split the integral on  $\partial R$  into two integrals on rectangles.  $\square$

Next, prove Goursat’s theorem, which shows that if  $f$  is merely complex differentiable,  $f$  is holomorphic.

**Theorem 14.** *If  $f : \Omega \rightarrow \mathbb{C}$  is complex differentiable at each point of  $\Omega$ , then  $f$  is holomorphic, so  $f \in C^1(\Omega)$ , and in fact  $f \in C^\infty(\Omega)$ .*

*Proof.* It is enough to show that the hypotheses yield

$$(9.7) \quad \int_{\partial R} f(z) dz = 0,$$

for every rectangle  $R \subset \Omega$ .

Given a rectangle  $R \subset \Omega$ , set  $a = \int_{\partial R} f(z) dz$ . Divide  $R$  into four rectangles of equal size. The integral over  $R$  is equal to the sum of the integrals over all four rectangles. Therefore, there must exist one rectangle  $R_1$  such that

$$(9.8) \quad \left| \int_{\partial R_1} f(z) dz \right| \geq \frac{|a|}{4}.$$

Then, divide  $R_1$  into four equal rectangles. One of them,  $R_2$ , must have the property that

$$(9.9) \quad \left| \int_{\partial R_2} f(z) dz \right| \geq 4^{-2} |a|.$$

Thus, there exists a sequence of nested rectangles  $R_k$  with perimeter  $\partial R_k$  of length  $2^{-k}l(\partial R) = 2^{-k}b$  such that

$$(9.10) \quad \left| \int_{\partial R_k} f(z) dz \right| \geq 4^{-k} |a|.$$

The rectangles shrink to a point  $p \in \Omega$ . Since  $f$  is complex differentiable,

$$(9.11) \quad f(z) = f(p) + f'(p)(z - p) + o(|z - p|).$$

Now then,

$$(9.12) \quad \int_{\partial R_k} f(p) dz = \int_{\partial R_k} f'(p)(z - p) dz = 0.$$

Therefore,

$$(9.13) \quad \left| \int_{\partial R_k} f(z) dz \right| \leq C \delta_k 2^{-k} 2^{-k}.$$

Plugging (9.13) into (9.10),  $a = 0$ . □

## 10. SINGULARITIES

Goursat's theorem allows us to classify singularities of a function. Suppose now that  $f$  is holomorphic on a region  $\Omega$ , except at a point  $z_0 \in \Omega$ . For example, the function  $f(z) = \frac{1}{z}$  is holomorphic on  $\mathbb{C} \setminus \{0\}$  and has a singularity at  $z = 0$ .

**Definition 9** (Isolated singularity). *A point  $p \in \mathbb{C}$  is an isolated singularity if there is a neighborhood  $U$  of  $p$  such that  $f$  is holomorphic on  $U \setminus \{p\}$ .*

**Remark 6.** *Other kinds of singularities are branch singularities. For example,  $z^{1/2}$  and  $\log z$  are branch singularities.*

For example, 0 is an isolated singularity for  $f(z) = \frac{1}{z}$ . An isolated singularity is said to be removable if there exists a function  $\tilde{f}$  holomorphic on  $U$ , where  $\tilde{f} = f$  on  $U \setminus \{p\}$ . If  $p$  is a removable singularity, then  $f$  is bounded near  $p$ . The converse is also true.

**Theorem 15.** *If  $p \in \Omega$  and  $f$  is holomorphic on  $\Omega \setminus \{p\}$  and bounded, then  $p$  is a removable singularity.*

*Proof.* Consider the function  $g : \Omega \rightarrow \mathbb{C}$  defined by

$$(10.1) \quad g(z) = (z - p)^2 f(z), \quad z \in \Omega \setminus \{p\}, \quad g(p) = 0.$$

Since  $f$  is bounded,  $g$  is continuous on  $\Omega$ . Also,  $g$  is complex differentiable at each point of  $\Omega$ , since

$$(10.2) \quad g'(z) = 2(z - p)f(z) + (z - p)^2 f'(z), \quad z \in \Omega \setminus \{p\}, \quad g'(p) = 0.$$

Observe that since  $f$  is bounded, the Cauchy integral formula implies

$$(10.3) \quad |f'(z)| \leq \frac{\sup_{z \in \Omega} |f(z)|}{|z - p|}, \quad \text{so} \quad |z - p|^2 |f'(z)| \leq C |z - p|.$$

Therefore, by Goursat's theorem,  $g$  is holomorphic on  $\Omega$ , so on a neighborhood  $U$  of  $p$ ,  $g$  has the convergent power series

$$(10.4) \quad g(z) = \sum_{n=0}^{\infty} a_n (z-p)^n, \quad z \in U.$$

Since  $g(p) = g'(p) = 0$ ,  $a_0 = a_1 = 0$ , so

$$(10.5) \quad g(z) = (z-p)^2 h(z), \quad h(z) = \sum_{n=0}^{\infty} a_{n+2} (z-p)^n, \quad z \in U.$$

Comparing (10.5) to (10.1),  $h(z) = f(z)$  on  $U \setminus \{p\}$ , so set

$$(10.6) \quad \tilde{f}(z) = f(z), \quad z \in \Omega \setminus \{p\}, \quad \tilde{f}(p) = h(p).$$

□

Furthermore, the function  $\tilde{f}(z)$  is unique.

**Proposition 29.** *Let  $\Omega \subset \mathbb{C}$  be open and connected, and let  $f : \Omega \rightarrow \mathbb{C}$  be holomorphic,  $f \in \mathcal{H}(\Omega)$ . The following are equivalent:*

- (1)  $f \equiv 0$ ,
- (2) For some  $z_0 \in \Omega$ ,  $f^{(n)}(z_0) = 0$  for all  $n \geq 0$ ,
- (3) The set  $\{z \in \Omega : f(z) = 0\}$  has an accumulation point in  $\Omega$ .

*Proof.* It is clear that item number one implies item number two and item number three. Furthermore, since  $\mathcal{H}(\Omega) = \mathcal{A}(\Omega)$ , if item number two is true, then the set of zeros of  $f$  must have an accumulation point in  $\Omega$ . Therefore, it only remains to prove that item number three implies item number one.

To that end, let  $p \in \Omega$  be an accumulation point of  $f(z_n) = 0$ . Observe that there exists  $R > 0$  such that the disk  $D_R(p) \subset \Omega$  and  $f$  has a convergent power series on  $D_R(p)$ ,

$$(10.7) \quad f(z) = \sum_{n=0}^{\infty} a_n (z-p)^n.$$

If  $a_n = 0$  for all  $n$ , then  $f = 0$  on  $D_R(p)$ . Otherwise, suppose  $a_j$  is the first nonzero entry of (10.7). Then,

$$(10.8) \quad f(z) = (z-p)^j g(z), \quad g(z) = \sum_{n=0}^{\infty} a_{j+n} (z-p)^n.$$

Since  $g(p) = a_j \neq 0$ ,  $g$  must be nonzero on a small neighborhood of  $p$ , which contradicts the fact that  $p$  is an accumulation point for the set of zeros.

Therefore, if  $S^\sharp$  is the set of accumulation points in  $\Omega$  of the zeros of  $f$ , then  $S^\sharp$  is open. Since  $S^\sharp$  is closed,  $S^\sharp = \emptyset$  or  $S^\sharp = \Omega$ . Since  $S^\sharp$  is not empty,  $S^\sharp = \Omega$ , so  $f = 0$  on  $\Omega$ . □

**Definition 10 (Pole).** *An isolated singularity  $p$  is said to be a pole if  $|f(z)| \rightarrow \infty$  as  $z \rightarrow p$ .*

If  $p$  is a pole, there exists a neighborhood  $U$  centered at  $p$  such that  $|f(z)| \geq 1$  on  $U \setminus \{p\}$ . Thus,  $g(z) = \frac{1}{f(z)}$  is holomorphic on  $U \setminus \{p\}$ , and  $g(z) \rightarrow 0$  as  $z \rightarrow p$ , so  $g$  has a removable singularity on  $U$ . Therefore,  $g$  has a convergent power series expansion on  $U$ ,

$$(10.9) \quad g(z) = \sum_{n=k}^{\infty} a_n (z-p)^n,$$

where  $a_k$  is the first nonzero coefficient in the power series. Therefore,

$$(10.10) \quad g(z) = (z - p)^k h(z), \quad h(p) = a_k \neq 0,$$

with  $h$  holomorphic on  $U$ . Therefore, we have proved,

**Proposition 30.** *If  $f$  is holomorphic on  $\Omega \setminus \{p\}$  with a pole at  $p$ , then there exists  $k \in \mathbb{Z}^+$  such that*

$$(10.11) \quad f(z) = (z - p)^{-k} F(z),$$

on  $\Omega \setminus \{p\}$ , with  $F$  holomorphic on  $\Omega$ , and  $F(p) \neq 0$ .

**Definition 11** (Simple pole). *If  $k = 1$ ,  $f$  has a simple pole at  $p$ .*

**Proposition 31.** *If  $f : \mathbb{C} \rightarrow \mathbb{C}$  is holomorphic and  $|f(z)| \rightarrow \infty$  as  $|z| \rightarrow \infty$ , then  $f(z)$  is a polynomial.*

*Proof.* Define the function  $g : \mathbb{C} \setminus \{0\} \rightarrow \mathbb{C}$  defined by

$$(10.12) \quad g(z) = f\left(\frac{1}{z}\right).$$

Since  $|g(z)| \rightarrow \infty$  as  $z \rightarrow \infty$ ,  $g$  has a pole at 0. Then, by Proposition 30,

$$(10.13) \quad g(z) = z^{-k} G(z),$$

on  $\mathbb{C} \setminus \{0\}$  for some  $k \in \mathbb{Z}^+$ , with  $G$  holomorphic on  $\mathbb{C}$  and  $G(0) \neq 0$ . Then,

$$(10.14) \quad G(z) = \sum_{j=0}^{k-1} g_j z^j + z^k h(z),$$

and therefore,

$$(10.15) \quad g(z) = \sum_{j=0}^{k-1} g_j z^{j-k} + h(z).$$

Therefore,

$$(10.16) \quad f(z) = \sum_{j=0}^{k-1} g_j z^{k-j} + h\left(\frac{1}{z}\right),$$

so

$$(10.17) \quad f(z) - \sum_{j=0}^{k-1} g_j z^{k-j},$$

is holomorphic on  $\mathbb{C}$ , and approaches  $h(0)$  as  $|z| \rightarrow \infty$ . Therefore, by Liouville's theorem, the difference is constant, so  $f(z)$  is a polynomial.  $\square$

**Definition 12** (Meromorphic functions). *A function holomorphic on  $\Omega$  except for a set of poles is said to be meromorphic on  $\Omega$ . One example of such a function is*

$$(10.18) \quad \tan z = \frac{\sin z}{\cos z},$$

which is meromorphic on  $\mathbb{C}$ , with poles at  $\{(k + \frac{1}{2})\pi : k \in \mathbb{Z}\}$ .

**Definition 13** (Essential singularity). *An isolated singularity of a function that is not a pole or a removable singularity is called an essential singularity.*

An example of an essential singularity is the function  $f(z) = e^{\frac{1}{z}}$ .

**Proposition 32** (Casorati-Weierstrass theorem). *Suppose  $f : \Omega \setminus \{p\} \rightarrow \mathbb{C}$  has an essential singularity at  $p$ . Then for any neighborhood  $U$  of  $p$ , the image of  $U \setminus \{p\}$  is dense in  $\mathbb{C}$ .*

*Proof.* Suppose there exists a neighborhood  $U$  of  $p$  such that the image of  $U \setminus \{p\}$  omits a neighborhood of  $w_0 \in \mathbb{C}$ . Replacing  $f(z)$  by  $f(z) - w_0$ , suppose without loss of generality  $w_0 = 0$ . Then

$$(10.19) \quad g(z) = \frac{1}{f(z)}$$

is holomorphic and bounded on  $U \setminus \{p\}$ , so  $g(z)$  has a removable singularity at  $p$ , so  $\tilde{g}(z)$  has a holomorphic extension on  $U$ . If  $\tilde{g}(p) \neq 0$ , then  $p$  is a removable singularity for  $f$ . If  $\tilde{g}(p) = 0$ , then  $p$  is a pole of  $f$ .  $\square$

## 11. LAURENT SERIES

The Laurent series is a generalization of the power series expansion, which works for functions holomorphic in an annulus. Let

$$(11.1) \quad \mathcal{A} = \{z \in \mathbb{C} : r_0 < |z - z_0| < r_1\}$$

be such an annulus, where  $0 < r_0 < r_1 < \infty$ . Let  $\gamma_j$  be the counterclockwise circles  $\{|z - z_0| = r_j\}$ , so that  $\partial\mathcal{A} = \gamma_1 - \gamma_0$ . Then for any  $f \in C^1(\mathcal{A})$  holomorphic in  $\mathcal{A}$ , the Cauchy integral formula implies that for  $z \in \mathcal{A}$ ,

$$(11.2) \quad f(z) = \frac{1}{2\pi i} \int_{\gamma_1} \frac{f(\zeta)}{\zeta - z} d\zeta - \frac{1}{2\pi i} \int_{\gamma_0} \frac{f(\zeta)}{\zeta - z} d\zeta.$$

Now then, for  $\zeta \in \gamma_1$ , since  $|z - z_0| < |\zeta - z_0|$ ,

$$(11.3) \quad \frac{1}{\zeta - z} = \frac{1}{(\zeta - z_0) - (z - z_0)} = \frac{1}{\zeta - z_0} \sum_{j=0}^{\infty} \left(\frac{z - z_0}{\zeta - z_0}\right)^j.$$

Meanwhile, for  $\zeta \in \gamma_0$ , since  $|\zeta - z_0| < |z - z_0|$ ,

$$(11.4) \quad \frac{1}{\zeta - z} = -\frac{1}{(z - z_0) - (\zeta - z_0)} = \frac{-1}{z - z_0} \sum_{j=0}^{\infty} \left(\frac{\zeta - z_0}{z - z_0}\right)^j.$$

Therefore,

$$(11.5) \quad f(z) = \sum_{n=-\infty}^{\infty} a_n (z - z_0)^n, \quad z \in \mathcal{A},$$

where for  $n \geq 0$ ,

$$(11.6) \quad a_n = \frac{1}{2\pi i} \int_{\gamma_1} \frac{f(\zeta)}{(\zeta - z_0)^{n+1}} d\zeta,$$

and for  $n < 0$ ,

$$(11.7) \quad a_n = \frac{1}{2\pi i} \int_{\gamma_0} f(\zeta) (\zeta - z_0)^{-n-1} d\zeta.$$

Therefore,

**Proposition 33.** *Given  $0 \leq r_0 < r_1 \leq \infty$ , let  $\mathcal{A}$  be the annulus (11.1). If  $f : \mathcal{A} \rightarrow \mathbb{C}$  is holomorphic, then it is given by the absolutely convergent series (11.5), with*

$$(11.8) \quad a_n = \frac{1}{2\pi i} \int_{\gamma} \frac{f(\zeta)}{(\zeta - z_0)^{n+1}} d\zeta, \quad n \in \mathbb{Z},$$

where  $\gamma$  is any counterclockwise oriented circle centered at  $z_0$  of radius  $r_0 < r < r_1$ .

*Proof.* The preceding argument can be applied to any annulus

$$(11.9) \quad \mathcal{A}^b = \{z \in \mathbb{C} : r'_0 < |z - z_0| < r'_1\},$$

where  $r_0 < r'_0 < r'_1 < r_1$ . Since  $f$  is holomorphic on  $\mathcal{A}^b$ , the integrals on  $\gamma_0$  and  $\gamma_1$  can be moved to  $\gamma$ .  $\square$

When  $f$  has an isolated singularity at  $z_0$ , we can take  $r_0 = 0$ .

**Proposition 34.** *Suppose  $f$  is holomorphic on  $D_R(z_0) \setminus \{z_0\}$ , with Laurent expansion (11.5), then  $f$  has a pole at  $z_0$  if and only if  $a_n = 0$  for all but finitely many  $n < 0$ , and  $a_n \neq 0$  for some  $n < 0$ . Therefore,  $f$  has an essential singularity at  $z_0$  if and only if  $a_n \neq 0$  for infinitely many  $n < 0$ .*

*Proof.* By Proposition 30,  $f$  has a pole at  $z_0$  if and only if  $a_n = 0$  for all but finitely many  $n < 0$ , and  $a_n \neq 0$  for some  $n < 0$ . Conversely, if  $f$  has an essential singularity at  $z_0$  then  $a_n \neq 0$  for infinitely many  $n$ .  $\square$

For example, consider the function

$$(11.10) \quad f(z) = \frac{1}{z-1},$$

on the regions

$$(11.11) \quad \{z : |z| < 1\} \quad \text{and} \quad \{z : |z| > 1\}.$$

On the region  $\{z : |z| < 1\}$ ,

$$(11.12) \quad f(z) = \frac{-1}{1-z} = -\sum_{k=0}^{\infty} z^k.$$

On the region  $\{z : |z| > 1\}$ ,

$$(11.13) \quad f(z) = \frac{1}{z-1} = \frac{1}{z} \frac{1}{1-\frac{1}{z}} = \frac{1}{z} \sum_{n=0}^{\infty} \left(\frac{1}{z}\right)^n = \sum_{n=-\infty}^{-1} z^n.$$

Similarly, consider the function

$$(11.14) \quad g(z) = \frac{1}{z-2}.$$

For the region  $\{z : |z| < 2\}$ ,

$$(11.15) \quad g(z) = \frac{-1}{2} \frac{1}{1-\frac{z}{2}} = -\frac{1}{2} \sum_{n=0}^{\infty} \left(\frac{z}{2}\right)^n.$$

Meanwhile, for the region  $\{z : |z| > 2\}$ ,

$$(11.16) \quad g(z) = \frac{1}{z-2} = \frac{1}{z} \frac{1}{1-\frac{2}{z}} = \frac{1}{z} \sum_{n=0}^{\infty} \left(\frac{2}{z}\right)^{n+1}.$$

Many of the same computations that we have done for convergent power series may also be utilized for convergent Laurent series.

**Proposition 35.** *Assume  $f(z)$  is given by the series (11.5) converging for  $z \in \mathcal{A}$ , for  $r_0 < |z - z_0| < r_1$ . Then  $f$  is holomorphic on  $\mathcal{A}$ , and*

$$(11.17) \quad f'(z) = \sum_{n=-\infty}^{\infty} n a_n (z - z_0)^{n-1}, \quad z \in \mathcal{A}.$$

*Proof.* Choose  $R_1$  such that

$$(11.18) \quad \frac{1}{R_1} = \limsup_{n \rightarrow \infty} |a_n|^{1/n},$$

and

$$(11.19) \quad \frac{1}{R_0} = \limsup_{n \rightarrow \infty} |a_{-n}|^{-1/n}.$$

As before,

$$(11.20) \quad \sum_{n=0}^{\infty} a_n (z - z_0)^n,$$

converges absolutely for  $|z - z_0| < R_1$  and diverges for  $|z - z_0| > R_1$ , and

$$(11.21) \quad \sum_{n=-\infty}^{-1} a_n (z - z_0)^n,$$

converges absolutely for  $|z - z_0| > R_0$  and diverges for  $|z - z_0| < R_0$ . Once again, since  $\lim_{n \rightarrow \infty} n^{1/n} = 1$ , the same computations may be made on  $\mathcal{A}$ .

Taking

$$(11.22) \quad f_\nu(z) = \sum_{n=-\nu}^{\nu} a_n (z - z_0)^n,$$

$f_\nu \rightarrow f$  locally uniformly on  $\mathcal{A}$ , so the limit  $f$  is holomorphic on  $\mathcal{A}$ , and  $f'_\nu$  converges locally uniformly to  $f'$ .  $\square$

Laurent series can be multiplied on annuli on which  $f$  and  $g$  have convergent series.

**Proposition 36.** *Suppose  $f$  and  $g$  are given by Laurent series*

$$(11.23) \quad f(z) = \sum_{n=-\infty}^{\infty} a_n (z - z_0)^n,$$

$$(11.24) \quad g(z) = \sum_{n=-\infty}^{\infty} b_n (z - z_0)^n,$$

which converge on the annulus (11.1). Moreover, the product is given by

$$(11.25) \quad f(z)g(z) = \sum_{n=-\infty}^{\infty} c_n(z-z_0)^n, \quad c_n = \sum_{k=-\infty}^{\infty} a_k b_{n-k},$$

which converges on  $\mathcal{A}$ .

*Proof.* This fact follows directly from a convergence result.

**Proposition 37.** *Given absolutely convergent series*

$$(11.26) \quad A = \sum_{n=-\infty}^{\infty} \alpha_n, \quad B = \sum_{n=-\infty}^{\infty} \beta_n,$$

we have the absolutely convergent series

$$(11.27) \quad AB = \sum_{n=-\infty}^{\infty} \gamma_n, \quad \gamma_n = \sum_{k=-\infty}^{\infty} \alpha_k \beta_{n-k}.$$

Indeed, for any  $\epsilon > 0$  there exists  $N(\epsilon)$  sufficiently large such that  $|A - \sum_{n=-\frac{N}{2}}^{\frac{N}{2}} \alpha_n| \leq \epsilon$  and  $|B - \sum_{n=-\frac{N}{2}}^{\frac{N}{2}} \beta_n| \leq \epsilon$ . Then,

$$(11.28) \quad AB = \left( \sum_{n=-\frac{N}{2}}^{\frac{N}{2}} \alpha_n \right) \left( \sum_{k=-\frac{N}{2}}^{\frac{N}{2}} \beta_k \right) + O(\epsilon).$$

For  $|n| > N$ ,

$$(11.29) \quad \sum_{|n| \geq N} \gamma_n \leq A \sum_{|n| \geq \frac{N}{2}} \beta_n + B \sum_{|n| \geq \frac{N}{2}} \beta_n = O(\epsilon).$$

For  $|n| \leq N$ ,

$$(11.30) \quad \gamma_n = \sum_k \alpha_k \beta_{n-k} = \sum_{|k| > \frac{N}{2}} \alpha_k \beta_{n-k} + \sum_{|n-k| > \frac{N}{2}} \alpha_k \beta_{n-k} + \sum_{|k| \leq \frac{N}{2}, |n-k| \leq \frac{N}{2}} \alpha_k \beta_{n-k}.$$

Since

$$(11.31) \quad \sum_{n=-N}^N \sum_{|k| > \frac{N}{2}} \alpha_k \beta_{n-k} = O(\epsilon),$$

and

$$(11.32) \quad \sum_{n=-N}^N \sum_{|n-k| > \frac{N}{2}} \alpha_k \beta_{n-k} = O(\epsilon),$$

$$(11.33) \quad \sum_{n=-N}^N \gamma_n = AB + O(\epsilon).$$

This proves Proposition 37. □

**Proposition 38.** *The conformal automorphisms of the plane are precisely the invertible linear conformal maps, i.e.,*

$$(11.34) \quad \{az + b \mid a, b \in \mathbb{C}, \quad a \neq 0\}.$$

*Proof.* It is clear that each such map is an automorphism. Now suppose  $f \in \text{Aut}(\mathbb{C})$  with  $f(0) = 0$ , by translation. Then  $f(z)$  is the entire function represented by a power series

$$(11.35) \quad f(z) = \sum_{n=1}^{\infty} a_n z^n, \quad \forall z \in \mathbb{C}.$$

Then  $g(w) = f(\frac{1}{w})$  exhibits an isolated singularity at  $w = 0$ . By the Casorati–Weierstrass theorem,  $g$  cannot be essential. Indeed, since  $f$  is a homeomorphism,  $|f(z)| \geq 1$  for all  $|z| > R > 0$ . Therefore, the image of  $0 < |w| < \frac{1}{R}$  under  $g$  is not dense in the plane. Therefore, the Laurent series of  $g$  around  $w$  is finite, which implies that  $f$  is a polynomial. Since  $f$  is an automorphism,  $f' \neq 0$ , so  $f'$  must be constant.  $\square$

**Lemma 7.** *The analytic maps  $\mathbb{C}_{\infty} \rightarrow \mathbb{C}_{\infty}$ , which are not identically equal to  $\infty$  are precisely the rational functions, i.e., all maps of the form  $\frac{P}{Q}$  with  $P, Q$  polynomials over  $\mathbb{C}$  with  $Q$  not identically equal to 0. In particular, the automorphism group of the Riemann sphere are all Möbius transforms.*

*Proof.* It is clear that all rational maps are analytic from the extended plane to itself. Conversely, suppose  $f(z) \in \mathbb{C}$  for all  $z \in \mathbb{C}_{\infty}$ . Then  $f$  is entire and bounded on  $\mathbb{C}_{\infty}$ , by compactness of  $\mathbb{C}_{\infty}$ , and thus constant. Therefore, assume that  $f(z_0) = \infty$  for some  $z_0 \in \mathbb{C}_{\infty}$ . Observe that  $f(z)$  is only equal to infinity at finitely many points. Indeed, if  $f(z) = \infty$  at infinitely many points, then there must be an accumulation point for zeros of  $\frac{1}{f(z)}$  in  $\mathbb{C}_{\infty}$ , and thus  $\frac{1}{f(z)} \equiv 0$ .

Now suppose  $f(z_0) = \infty$  for some  $z_0 \in \mathbb{C}$ . Then  $f$  has a pole at  $z_0$ , so by Proposition 34,  $f(z) = \sum_{j=1}^k g_j(z - z_0)^{-j} + h(z)$  for some  $k > 0$  and  $h(z)$  holomorphic near  $z_0$ . Subtracting the principal parts of the poles  $z_0 \in \mathbb{C}$ ,

$$(11.36) \quad f(z) = \sum_{m=1}^n \sum_{j=1}^{k_n} g_{j,n}(z - z_m)^{-j} + h(z),$$

where  $h(z)$  is an entire function that possibly has a pole at  $\infty$ . Then by Proposition 31,  $h(z)$  is a polynomial.  $\square$

## 12. MÖBIUS TRANSFORMATIONS

Since the meromorphic functions from  $\mathbb{C}^* \rightarrow \mathbb{C}^*$  must be rational functions, it is possible to describe fully describe the conformal automorphisms of  $\mathbb{C}^*$ .

**Definition 14.** *Conformal automorphisms are bijections whose inverse is also conformal.*

A diffeomorphism  $f : \Omega \rightarrow \mathcal{O}$  between two planar domains in  $\mathbb{C}$  is called conformal if it preserves angles. That is, if  $\gamma_1$  and  $\gamma_2$  are two curves that meet at an angle  $\alpha$  at  $p$ , then  $\sigma_1 = f(\gamma_1)$  and  $\sigma_2 = f(\gamma_2)$  meet at the same angle  $\alpha$  at  $q = f(p)$ . A condition for this is that for each  $p \in \Omega$ ,

$$(12.1) \quad Df(p) = \lambda(p)R(p),$$

where  $R(p)$  is an orthogonal matrix. Then  $\det Df(p) > 0$  if and only if  $\det R(p) = 1$ , and  $\det Df(p) < 0$  if and only if  $\det R(p) = -1$ . In the former case,

$$(12.2) \quad R(p) = \begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix},$$

and in the latter case,

$$(12.3) \quad R(p) = \begin{pmatrix} \cos \theta & \sin \theta \\ \sin \theta & -\cos \theta \end{pmatrix},$$

and then  $CDf(p)$  commutes with  $J$  where

$$(12.4) \quad C = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

In this case,  $Cz = \bar{z}$ . In the former case,  $f$  is said to be orientation preserving and in the latter case the map is orientation reversing.

**Proposition 39.** *Given planar regions  $\Omega$ ,  $\mathcal{O}$ , the class of orientation-preserving conformal diffeomorphisms  $f : \Omega \rightarrow \mathcal{O}$  coincides with the class of holomorphic diffeomorphisms. The class of orientation reversing conformal diffeomorphisms  $f : \Omega \rightarrow \mathcal{O}$  coincides with the class of conjugate holomorphic diffeomorphisms.*

Indeed by the fundamental theorem of algebra,  $P$  and  $Q$  both have degree  $\leq 1$ , where  $f(z) = \frac{P(z)}{Q(z)}$ . If  $Q(z)$  has two roots,  $f(z)$  has two points which map to infinity. On the other hand, if  $Q(z)$  has a double root at some  $z_0 \in \mathbb{C}$ , then for points close to  $z_0$  the map is 2-1. On the other hand, if  $P(z)$  has degree  $\geq 2$ , then  $f(z)$  has two zeros.

**Lemma 8.** *Every  $A \in GL(2, \mathbb{C})$  defines a transformation*

$$(12.5) \quad T_A(z) = \frac{az + b}{cz + d}, \quad A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

which is holomorphic as a map from  $\mathbb{C}_\infty \rightarrow \mathbb{C}_\infty$ . It is called a fractional linear or Möbius transformation. The map  $A \mapsto T_A$  only depends on the equivalence class of  $A$  under the relation  $A \sim B$  if and only if  $A = \lambda B$  for some  $\lambda \in \mathbb{C}^*$ . Thus, the family of all Möbius transformations is the same as

$$(12.6) \quad PSL(2, \mathbb{C}) = SL(2, \mathbb{C}) / \{\pm I\}.$$

We have  $T_A \circ T_B = T_{A \circ B}$  and  $T_A^{-1} = T_{A^{-1}}$ . In particular, every Möbius transformation is an automorphism of  $\mathbb{C}_\infty$ .

*Proof.* It is clear that each  $T_A$  is a holomorphic map  $\mathbb{C}_\infty \rightarrow \mathbb{C}_\infty$ . It is also straightforward to verify the composition law  $T_A \circ T_B = T_{A \circ B}$ . In particular,  $T_A$  has a conformal inverse and is thus an automorphism of  $\mathbb{C}_\infty$ . If  $T_A = T_{\tilde{A}}$ , then

$$(12.7) \quad T'_A(z) = \frac{ad - bc}{(cz + d)^2} = T'_{\tilde{A}}(z) = \frac{\tilde{a}\tilde{d} - \tilde{b}\tilde{c}}{(\tilde{c}z + \tilde{d})^2}.$$

Thus, if  $ad - bc = \tilde{a}\tilde{d} - \tilde{b}\tilde{c} = 1$ , then  $(cz + d)^2 = (\tilde{c}z + \tilde{d})^2$ . Thus, if  $A$  and  $\tilde{A}$  are both matrices in  $SL(2, \mathbb{C})$ , then  $A = \pm \tilde{A}$ .  $\square$

Combining this fact with the Schwarz lemma, we can characterize all conformal automorphisms of the unit disk.

**Proposition 40.** *All holomorphic automorphisms of the disk are given by*

$$(12.8) \quad \{\varphi_{a,\theta}(z) = e^{i\theta} \frac{z-a}{1-z\bar{a}} : |a| < 1, \quad \theta \in \mathbb{R}\}.$$

*Proof.* Observe that  $|\varphi_{a,\theta}(z)| = 1$  when  $|z| = 1$ . Indeed, when  $|z| = 1$ ,  $\bar{z} = z^{-1}$ , so

$$(12.9) \quad \frac{z-a}{1-z\bar{a}} = \overline{\left(\frac{1-z\bar{a}}{z-a}\right)}.$$

Therefore, if  $|z| = 1$ ,  $\overline{\varphi_{a,\theta}(z)} = \varphi_{a,\theta}(z)^{-1}$ , which implies that  $|\varphi_{a,\theta}(z)| = 1$ . Furthermore,  $\varphi_{a,\theta}(a) = 0$  so  $\varphi_{a,\theta}(z) : \mathbb{D} \rightarrow \mathbb{D}$ , and by Lemma 8,  $\varphi_{a,\theta}(z)$  is an automorphism.

Now suppose that  $\psi \in \text{Aut}(\mathbb{D})$  is holomorphic and satisfies  $\psi(a) = 0$ . Then  $\varphi_{a,\theta} \circ \psi^{-1} \in \text{Aut}(\mathbb{D})$  and  $\varphi_{a,\theta} \circ \psi^{-1}(0) = 0$ . By the Schwarz lemma, this map is a rotation, which proves (12.8).  $\square$

**Lemma 9.** *Every Möbius transformation is the composition of four elementary maps:*

- (1) *Translations*  $z \mapsto z + z_0$ ,  $z_0 \in \mathbb{C}$ ,
- (2) *Dilations*  $z \mapsto \lambda z$ ,  $\lambda > 0$ ,
- (3) *Rotations*  $z \mapsto e^{i\theta} z$ ,  $\theta \in \mathbb{R}$ ,
- (4) *Inversion*  $z \mapsto \frac{1}{z}$ .

*Proof.* If  $c = 0$  then  $T_A(z) = \frac{a}{d}z + \frac{b}{d}$ , which is certainly a composition of translations, dilations, and rotations. If  $c \neq 0$ , then

$$(12.10) \quad T_A(z) = \frac{bc-ad}{c^2} \frac{1}{z + \frac{d}{c}} + \frac{a}{c} = \frac{bc-ad}{c(cz+d)} + \frac{a(cz+d)}{c(cz+d)} = \frac{az+b}{cz+d}.$$

$\square$

Now consider some examples. Since the imaginary axis is the perpendicular bisector of the line segment from  $-1$  to  $1$ ,  $|z-1| = |z+1|$  for any  $z$  lying on the imaginary axis. Thus,  $z \mapsto \frac{z-1}{z+1}$  maps the imaginary axis to the circle of radius one. Since  $1 \mapsto 0$ , the right half plane is taken to the disc  $\mathbb{D} = \{z : |z| < 1\}$ . We can similarly show that  $z \mapsto \frac{2z-1}{2-z}$  maps the unit circle to itself, since if  $|z| = 1$  and we multiply the numerator and denominator by  $\frac{\bar{z}}{\bar{z}}$ ,  $\frac{2z-1}{2-z} = \frac{2-\bar{z}}{2\bar{z}-1}$ . If we think of lines as circles that pass through  $\infty$ , we have the following lemma.

**Lemma 10.** *Fractional linear transformations take circles onto circles.*

*Proof.* In view of the previous lemma, it suffices to show that inversions take circles to circles. Let  $|z - z_0| = r$  be a circle and let  $w = \frac{1}{z}$ . Then,

$$(12.11) \quad 0 = |z|^2 - 2\text{Re}(\bar{z}z_0) + |z_0|^2 - r^2 = \frac{1}{|w|^2} - 2\frac{\text{Re}(wz_0)}{|w|^2} + |z_0|^2 - r^2.$$

Now then, if  $|z_0| = r$ , then

$$(12.12) \quad 1 = 2\text{Re}(wz_0) = 2\text{Re}(w)\text{Re}(\bar{z}_0) + 2\text{Im}(w)\text{Im}(\bar{z}_0) = 2\langle w, \bar{z}_0 \rangle.$$

Therefore, a circle that passes through the origin is mapped to a line. Otherwise, we obtain the equation,

$$(12.13) \quad 0 = (|z_0|^2 - r^2)|w|^2 - 2\text{Re}(wz_0) + 1 = (|z_0|^2 - r^2)\left\{|w - \frac{\bar{z}_0}{|z_0|^2 - r^2}|^2 - \frac{r^2}{(|z_0|^2 - r^2)^2}\right\},$$

which is a circle.

A line is given by the equation  $2\operatorname{Re}(z\bar{z}_0) = a$ , which transforms into  $2\operatorname{Re}(z_0z) = a|w|^2$ . If  $a = 0$  then we have another line through the origin. Otherwise, we have an equation  $|w - z_0/a|^2 = |z_0/a|^2$ .  $\square$

**Remark 7.** *We could also prove this result using the Riemann sphere, since a stereographic projection preserves circles. The inversion  $z \mapsto \frac{1}{z}$  corresponds to a rotation of the Riemann sphere about the  $x_1$  axis. Since such a rotation preserves circles, a fractional linear transformation does too.*

Since  $Tz = \frac{az+b}{cz+d} = z$  is a quadratic equation for any Möbius transformation  $T$ ,  $T$  can have at most two fixed points, unless it is the identity. Moreover, every Möbius transformation has at least one fixed point. Indeed, if  $c \neq 0$ ,

$$(12.14) \quad \frac{az+b}{cz+d} = z,$$

is a quadratic equation, which has at least one solution. If  $c = 0$ , then the left hand side is a translation combined with a dilation and a rotation. If  $\frac{a}{d} \neq 1$ , we can solve  $\frac{a}{d}z + \frac{b}{d} = z$ . If  $\frac{a}{d} = 1$  then  $T(z) = z + b'$ , which has the fixed point of  $\infty$ .

**Lemma 11.** *A fractional linear transformation is determined completely by its action on three distinct points. Given  $z_1, z_2, z_3 \in \mathbb{C}_\infty$ , there exists a unique fractional linear transformation  $T$  with  $Tz_1 = 0$ ,  $Tz_2 = 1$ ,  $Tz_3 = \infty$ .*

*Proof.* Suppose  $S$  and  $T$  are Möbius transformations that agree at three distinct points. Then the Möbius transformation  $S^{-1} \circ T$  has three fixed points, and thus is the identity. For a second statement, let

$$(12.15) \quad Tz = \frac{z - z_1}{z - z_3} \frac{z_2 - z_3}{z_2 - z_1}.$$

Then clearly  $Tz_1 = 0$ ,  $Tz_2 = 1$ ,  $Tz_3 = \infty$ . If any of these points is  $\infty$ , we obtain the correct formula by passing to the limit there.  $\square$

Now define the cross ratio of four points.

**Definition 15** (Cross ratio). *The cross ratio of four points  $z_0, z_1, z_2, z_3 \in \mathbb{C}_\infty$  is defined as*

$$(12.16) \quad [z_0 : z_1 : z_2 : z_3] = \frac{z_0 - z_1}{z_0 - z_3} \cdot \frac{z_2 - z_3}{z_2 - z_1}.$$

**Lemma 12.** *The cross ratio of any four distinct points is preserved under Möbius transformations. Moreover, four distinct points lie on a circle if and only if their cross ratio is real.*

*Proof.* Let  $z_1, z_2, z_3$  be distinct points and let  $Tz_j = w_j$  for  $j = 1, 2, 3$ . Then

$$(12.17) \quad \frac{w - w_1}{w - w_3} \cdot \frac{w_2 - w_3}{w_2 - w_1},$$

defines a fractional linear transformation  $S_1$  such that  $S_1w_1 = 0$ ,  $S_1w_2 = 1$ , and  $S_1w_3 = \infty$ . On the other hand,

$$(12.18) \quad \frac{z - z_1}{z - z_3} \cdot \frac{z_2 - z_3}{z_2 - z_1}$$

defines a fractional linear transformation  $S_0$  such that  $S_0z_1 = 0$ ,  $S_0z_2 = 1$ , and  $S_0z_3 = \infty$ . Furthermore,  $S_1^{-1} \circ S_0$  defines a linear transformation mapping  $z_1 \mapsto w_1$ ,  $z_2 \mapsto w_2$ ,  $z_3 \mapsto w_3$ . By uniqueness of Möbius transformations,  $S_1^{-1} \circ S_0 = T$ . Furthermore,  $w = Tz = S_1^{-1} \circ S_0z$ , so  $S_1w = S_0z$  if and only if  $Tw = z$ .

The second statement follows from the first. Any circle is the image of the real line under a Möbius transformation. But for any  $z_1, z_2, z_3 \in \mathbb{R}$ ,

$$(12.19) \quad \frac{z_0 - z_1}{z_0 - z_3} \cdot \frac{z_2 - z_3}{z_2 - z_1},$$

is real if and only if  $z_0$  is real. Indeed, if  $a$  and  $b$  are real,  $a \neq b$ ,  $\frac{z_0 - a}{z_0 - b} = \frac{(z_0 - a)(\bar{z}_0 - b)}{|z_0|^2 + b^2} = \frac{|z_0|^2 + ab - z_0 b - \bar{z}_0 a}{|z_0|^2 + b^2}$ . Since  $a \neq b$ , and  $z_0$  is not real,  $z_0 b - \bar{z}_0 a$  has a non-zero imaginary part.  $\square$

**Definition 16.** Let  $z_1, z_2, z_3 \in \Gamma$ , where  $\Gamma \subset \mathbb{C}_\infty$  is a circle. We say that  $z$  and  $z^*$  are symmetric relative to  $\Gamma$  if

$$(12.20) \quad [\bar{z} : z_1 : z_2 : z_3] = [z^* : z_1 : z_2 : z_3].$$

If  $\Gamma = \mathbb{R}$ , then  $z^*$  is the reflection of  $z$  across the real line. Now suppose that  $\Gamma$  is a circle of finite radius.

**Lemma 13.** Let  $\Gamma = \{|z - z_0| = r\}$ . Then for any  $z \in \mathbb{C}_\infty$ ,

$$(12.21) \quad z^* = \frac{r^2}{\bar{z} - \bar{z}_0} + z_0.$$

*Proof.* First let  $\Gamma = \{|z| = 1\}$  be the circle of radius 1 centered at the origin. Then

$$(12.22) \quad [\bar{z} : z_1 : z_2 : z_3] = [\bar{z} : z_1^{-1} : z_2^{-1} : z_3^{-1}] = \left[\frac{1}{\bar{z}} : z_1 : z_2 : z_3\right].$$

If  $|z_1| = |z_2| = |z_3| = 1$ ,  $z_1^{-1} = \bar{z}_1$ ,  $z_2^{-1} = \bar{z}_2$ ,  $z_3^{-1} = \bar{z}_3$ , so  $[\bar{z} : z_1 : z_2 : z_3] = [\bar{z} : z_1^{-1} : z_2^{-1} : z_3^{-1}]$ . By direct computation,

$$(12.23) \quad \frac{\bar{z} - z_1^{-1}}{\bar{z} - z_3^{-1}} \cdot \frac{z_2^{-1} - z_3^{-1}}{z_2^{-1} - z_1^{-1}} = \frac{\bar{z} - z_1^{-1}}{\bar{z} - z_3^{-1}} \cdot \frac{z_2 - z_3}{z_2 - z_1} \cdot \frac{z_1}{z_3} = \frac{\bar{z} z_1 - 1}{\bar{z} z_3 - 1} \cdot \frac{z_2 - z_3}{z_2 - z_1} = \frac{z_1 - \frac{1}{\bar{z}}}{z_3 - \frac{1}{\bar{z}}} \cdot \frac{z_2 - z_3}{z_2 - z_1}.$$

Therefore,  $[\bar{z} : z_1^{-1} : z_2^{-1} : z_3^{-1}] = \left[\frac{1}{\bar{z}} : z_1 : z_2 : z_3\right]$ .

This implies that if  $z^* = \frac{1}{\bar{z}}$ , (12.20) holds. Furthermore, if  $T$  is a dilation,  $Tz = rz$ ,  $r > 0$ , then by Lemma 12, if  $z_1, z_2, z_3$  are points on the unit circle,

$$(12.24) \quad [\overline{Tz} : Tz_1 : Tz_2 : Tz_3] = [\bar{z} : z_1 : z_2 : z_3] = \left[\frac{1}{\bar{z}} : z_1 : z_2 : z_3\right] = \left[T\left(\frac{1}{\bar{z}}\right) : Tz_1 : Tz_2 : Tz_3\right].$$

Now then, if  $w = rz = Tz$ ,

$$(12.25) \quad w^* = \frac{r^2}{r\bar{z}} = \frac{r}{\bar{z}} = Tz^*.$$

On the other hand, if  $Tz = z + z_0$  is a translation,  $T\left(\frac{1}{\bar{z}}\right) = \frac{1}{\bar{z}} + z_0 = w^*$ .  $\square$

Returning to maps that preserve the real line, if  $A$  is a real valued matrix,  $A \in Sl(2, \mathbb{R})$ , then  $L_A$  preserves  $\mathbb{R} \cup \{\infty\}$ . We can also write  $L_A = L_{A_1}$ , where  $A_1 \in Sl(2, \mathbb{R})$  and  $A \in Gl(2, \mathbb{R})$  and  $\det(A) > 0$ . Also,

$$(12.26) \quad A \in Sl(2, \mathbb{R}) \Rightarrow L_A : \mathcal{U} \rightarrow \mathcal{U},$$

where

$$(12.27) \quad \mathcal{U} = \{z : \text{Im}(z) > 0\}.$$

Then for  $a, b, c, d \in \mathbb{R}$ , if  $z = x + iy$ ,

$$(12.28) \quad \frac{az + b}{cz + d} = \frac{(az + b)(c\bar{z} + d)}{(cz + d)(c\bar{z} + d)} = \frac{ac|z|^2 + bd + (ad + bc)x}{|cz + d|^2} + i \frac{ad - bc}{|cz + d|^2} y.$$

Therefore,  $L_A = L_{-A}$ , and

$$(12.29) \quad \text{PSL}(2, \mathbb{R}) = \text{SL}(2, \mathbb{R})/(\pm I).$$

Now let  $\varphi$  be the fractional linear transformation

$$(12.30) \quad \varphi(z) = L_{A_0}(z), \quad A_0 = \begin{pmatrix} 1 & -i \\ 1 & i \end{pmatrix}.$$

Then,

$$(12.31) \quad \varphi(x + iy) = \frac{x + i(y - 1)}{x + i(y + 1)} \Rightarrow |\varphi(x + iy)|^2 = \frac{x^2 + (y - 1)^2}{x^2 + (y + 1)^2}.$$

Therefore,  $|\varphi(x + iy)| < 1$  if and only if  $y < 0$ . Therefore,

$$(12.32) \quad \varphi : \mathcal{U} \rightarrow D, \quad \varphi : \mathbb{R} \cup \{\infty\} \rightarrow S^1 = \partial D,$$

where  $D$  is the unit disk

$$(12.33) \quad D = \{z : |z| < 1\}.$$

Since  $\varphi$  is bijective on  $\mathbb{C} \cup \{\infty\}$ ,  $\varphi$  is bijective in (12.33). Therefore,

$$(12.34) \quad M_A = L_{A_0 A A_0^{-1}} : D \rightarrow D.$$

Then for  $A \in \text{SL}(2, \mathbb{R})$ ,

$$(12.35) \quad A_0 A A_0^{-1} = \frac{1}{2i} \begin{pmatrix} (a+d)i - b + c & (a-d)i + b + c \\ (a-d)i - b - c & (a+d)i + b - c \end{pmatrix} = \begin{pmatrix} \alpha & \beta \\ \bar{\beta} & \bar{\alpha} \end{pmatrix}.$$

Therefore,

$$(12.36) \quad A_0 \text{SL}(2, \mathbb{R}) A_0^{-1} = \left\{ \begin{pmatrix} \alpha & \beta \\ \bar{\beta} & \bar{\alpha} \end{pmatrix} : |\alpha|^2 - |\beta|^2 = 1 \right\} = \text{SU}(1, 1).$$

**Proposition 41.** *Given the groups  $\text{SL}(2, \mathbb{R})$  and  $\text{SU}(1, 1)$ ,  $\text{SL}(2, \mathbb{R})$  acts transitively in  $\mathcal{U}$  and  $\text{SU}(1, 1)$  acts transitively on  $D$ .*

*Proof.* Take  $p = a + ib \in \mathcal{U}$ . Then,  $L_p(z) = bz + a = \frac{b^{1/2}z + b^{-1/2}a}{b^{-1/2}}$  maps  $i$  to  $p$ . Given another  $q \in \mathcal{U}$ ,  $L_p L_q^{-1}$  maps  $q$  to  $p$ . The conjugation  $A_0 \text{SL}(2, \mathbb{R}) A_0^{-1}$  implies that transitivity on  $D$  is equivalent to transitivity on  $\mathcal{U}$ .  $\square$

**Proposition 42.** *If  $f : D \rightarrow D$  is a holomorphic diffeomorphism, then  $f = L_B$  for some  $B \in \text{SU}(1, 1)$ , then  $f = L_B$  for some  $B \in \text{SU}(1, 1)$ . Therefore, if  $F : \mathcal{U} \rightarrow \mathcal{U}$  is a holomorphic diffeomorphism, then  $F = L_A$  for some  $A \in \text{SL}(2, \mathbb{R})$ .*

*Proof.* Suppose  $f(0) = p \in D$ . Then by Proposition 41, there exists some  $B_1 \in \text{SU}(1, 1)$  such that  $L_{B_1}(p) = 0$ . Therefore,  $g = L_{B_1} \circ f : D \rightarrow D$  is a holomorphic diffeomorphism satisfying  $g(0) = 0$ . Therefore,  $h(z) = \frac{g(z)}{z}$  has a removable singularity at  $z = 0$ , as does  $\frac{z}{g(z)}$ . Also, as in the Schwarz lemma,

$$(12.37) \quad \left| \frac{g(z)}{z} \right| \leq 1, \quad \left| \frac{z}{g(z)} \right| \leq 1.$$

Therefore,  $|\frac{g(z)}{z}| \equiv 1$  on  $D$ , which implies that  $\frac{g(z)}{z}$  is a constant function, hence  $g(z) = cz$  for some  $|c| = 1$ .

Therefore,  $g = L_{B_2}$ , with

$$(12.38) \quad B_2 = a \begin{pmatrix} c & 0 \\ 0 & 1 \end{pmatrix}.$$

Taking  $a = \pm c^{-1/2}$ ,  $B_2 \in SU(1, 1)$ . Therefore,  $f = L_{B_1^{-1}B_2}$  and the proof is complete.  $\square$

### 13. THE WINDING NUMBER

Let  $u(z) = \log |z|$  be the harmonic function on  $\mathbb{C}^*$ . Recall that if we attempt to find the harmonic conjugate of  $\log |z|$  and we integrate on the curve  $\gamma_r(t) = re^{it}$ ,  $dx = -r \sin(t)dt$ ,  $dy = r \cos(t)dt$ ,  $u_y = \frac{r \sin(t)}{r^2}$ ,  $u_x = \frac{r \cos(t)}{r^2}$ ,

$$(13.1) \quad \oint -u_y dx + u_x dy = \int_0^{2\pi} (\sin^2(t) + \cos^2(t))dt = 2\pi.$$

**Remark 8.** The differential form  $\omega = -\frac{y}{r^2}dx + \frac{x}{r^2}dy$  pulls back to any circle as the form  $d\theta$ . Even though  $\theta$  is not well-defined on all of  $\mathbb{C}^*$ , but the differential  $d\theta$  is.

Thus, the local primitive of  $\frac{1}{z}$  is  $\log z$ , so integrating over a loop that encircles the origin once creates a jump by  $2\pi$ . This fact motivates the definition of the winding number.

**Lemma 14** (Winding number). *Let  $\gamma : [0, 1] \rightarrow \mathbb{C}$  be a closed curve. Then for any  $z_0 \in \mathbb{C} \setminus \gamma([0, 1])$  the integral*

$$(13.2) \quad n(\gamma; z_0) = \frac{1}{2\pi i} \int_{\gamma} \frac{dz}{z - z_0}$$

is an integer, called the index or winding number of  $\gamma$  relative to  $z_0$ .

The winding number is a homotopy invariant relative to  $\mathbb{C} \setminus \{z_0\}$ . It is constant on each connected component of  $\mathbb{C} \setminus \gamma([0, 1])$  and vanishes on the unbounded component.

*Proof.* Suppose without loss of generality that  $z_0 = 0$ , and suppose  $\gamma$  is an arbitrary continuous, piecewise  $C^1$  curve in  $\mathbb{C} \setminus \{0\}$ , say,

$$(13.3) \quad \gamma : [0, 1] \rightarrow \mathbb{C} \setminus \{0\}, \quad \gamma(t) = r(t)e^{i\theta(t)},$$

where  $r(t)$  and  $\theta(t)$  are continuous, piecewise  $C^1$ , real valued functions of  $t$ , and  $r(t) > 0$ . Then,

$$(13.4) \quad \gamma'(t) = [r'(t) + ir(t)\theta'(t)]e^{i\theta(t)}.$$

Computing,

$$(13.5) \quad \frac{1}{2\pi i} \int_{\gamma} \frac{dz}{z} = \frac{1}{2\pi i} \int_0^1 \frac{\gamma'(t)}{\gamma(t)} dt = \frac{1}{2\pi i} \int_0^1 \left[ \frac{r'(t)}{r(t)} + i\theta'(t) \right] dt.$$

Since  $r(0) = e^{i\theta(0)} = r(1)e^{i\theta(1)}$ ,

$$(13.6) \quad \int_0^{2\pi} \frac{r'(t)}{r(t)} dt = \log r(1) - \log r(0) = 0,$$

and

$$(13.7) \quad \frac{1}{2\pi} \int_0^1 \theta'(t) dt = \frac{1}{2\pi i} [\theta(1) - \theta(0)] = n(\gamma, 0) \in \mathbb{Z}.$$

Now suppose  $\gamma_0$  and  $\gamma_1$  are smoothly homotopic in  $\mathbb{C} \setminus \{0\}$ . Then,

$$(13.8) \quad n(\gamma_0, 0) = n(\gamma_1, 0).$$

Indeed, if  $\gamma_s$  is a smooth family of curves in  $\mathbb{C} \setminus \{0\}$ , for  $0 \leq s \leq 1$ , then

$$(13.9) \quad n(\gamma_s, 0) = \frac{1}{2\pi} \int_{\gamma_s} d\theta,$$

is a continuous function of  $s \in [0, 1]$ , taking values in  $\mathbb{Z}$ . Hence it is constant.

Alternatively, to show constancy on components, observe that

$$(13.10) \quad \frac{d}{dz_0} \oint_{\gamma} \frac{dz}{z - z_0} = \oint_{\gamma} \frac{dz}{(z - z_0)^2} = - \oint_{\gamma} \frac{d}{dz} \left[ \frac{1}{z - z_0} \right] dz = 0.$$

To show that the winding number vanishes on the unbounded component take  $z_0 \rightarrow \infty$  in  $\oint_{\gamma} \frac{dz}{z - z_0}$ .  $\square$

Suppose  $\Omega \subset \mathbb{C}$  is a bounded domain with piecewise smooth boundary and  $f \in C^2(\bar{\Omega})$  is holomorphic on  $\Omega$ , and nowhere zero on  $\partial\Omega$ . The number of zeros, counted with multiplicity, may be expressed in terms of the behavior of  $f$  on  $\partial\Omega$ . We say that  $p_j \in \Omega$  is a zero of multiplicity  $k$  provided,

$$(13.11) \quad f^{(l)}(p_j) = 0, \quad 0 \leq l \leq k - 1, \quad f^{(k)}(p_j) \neq 0.$$

**Proposition 43.** *Under the hypotheses stated above, the number  $\nu(f, \Omega)$  of zeros of  $f$  in  $\Omega$ , counted with multiplicity, is given by*

$$(13.12) \quad \nu(f, \Omega) = \frac{1}{2\pi i} \int_{\partial\Omega} \frac{f'(z)}{f(z)} dz.$$

*Proof.* Let  $D_j$  be small, disjoint disks around  $p_j \in \Omega$ . Then, by the Cauchy integral formula,

$$(13.13) \quad \frac{1}{2\pi i} \int_{\partial\Omega} \frac{f'(z)}{f(z)} dz = \frac{1}{2\pi i} \int_{\partial D_j} \frac{f'(z)}{f(z)} dz.$$

In a neighborhood  $\bar{D}_j$  of  $p_j$ ,

$$(13.14) \quad f(z) = (z - p_j)^{m_j} g(z),$$

with  $g(z)$  non-vanishing on  $\bar{D}_j$ . Therefore, on  $\bar{D}_j$ ,

$$(13.15) \quad \frac{f'(z)}{f(z)} = \frac{m_j}{z - p_j} + \frac{g'(z)}{g(z)}.$$

Since  $\frac{g'(z)}{g(z)}$  is holomorphic on  $\bar{D}_j$ ,

$$(13.16) \quad \frac{1}{2\pi i} \int_{\partial D_j} \frac{f'(z)}{f(z)} dz = \frac{m_j}{2\pi i} \int_{\partial D_j} \frac{dz}{z - p_j} = m_j. \quad \square$$

Proposition 43 has an interpretation in terms of winding numbers. Let  $C_j$  denote the connected components of  $\partial\Omega$ , with proper orientation, and suppose  $C_j$  is parameterized by  $\varphi_j : S^1 \rightarrow C_j$ . Then,

$$(13.17) \quad f \circ \varphi_j : S^1 \rightarrow \mathbb{C} \setminus \{0\}$$

parameterizes the image curve  $\gamma_j = f(C_j)$ .

**Proposition 44.** *With  $C_j$  and  $\gamma_j$  as above,*

$$(13.18) \quad \frac{1}{2\pi i} \int_{C_j} \frac{f'(z)}{f(z)} dz = \frac{1}{2\pi i} \int_{\gamma_j} \frac{dz}{z}.$$

*Proof.* In general,

$$(13.19) \quad \int_{C_j} u(z) dz = \int_0^{2\pi} u(\varphi_j(t)) \varphi_j'(t) dt,$$

and

$$(13.20) \quad \int_{\gamma_j} v(z) dz = \int_0^{2\pi} v(f(\varphi_j(t))) \frac{d}{dt} f \circ \varphi_j(t) dt = \int_0^{2\pi} v(f(\varphi_j(t))) f'(\varphi_j(t)) \varphi_j'(t) dt.$$

In particular, taking  $v(z) = \frac{1}{z}$ ,

$$(13.21) \quad \begin{aligned} \int_{C_j} \frac{f'(z)}{f(z)} dz &= \int_0^{2\pi} \frac{f'(\varphi_j(t))}{f(\varphi_j(t))} \varphi_j'(t) dt = \int_0^{2\pi} \frac{1}{f(\varphi_j(t))} f'(\varphi_j(t)) \varphi_j'(t) dt \\ &= \int_0^{2\pi} v(f(\varphi_j(t))) f'(\varphi_j(t)) \varphi_j'(t) dt = \int_{\gamma_j} \frac{1}{z} dz. \end{aligned}$$

□

**Proposition 45** (Argument principle). *If  $C_j$  denote the connected components of  $\partial\Omega$ ,*

$$(13.22) \quad \nu(f, \Omega) = \sum_j n(\gamma_j, 0), \quad \gamma_j = f(C_j).$$

*That is, the total number of zeros of  $f$  in  $\Omega$ , counting multiplicity, is equal to the sum of the winding numbers of  $f(C_j)$  about 0.*

*Proof.* For any  $C_j$ ,

$$(13.23) \quad \frac{1}{2\pi i} \int_{C_j} \frac{f'(z)}{f(z)} dz = n(\gamma_j, 0), \quad \gamma_j = f(C_j).$$

□

Many times, the right hand side is more readily calculable than the left hand side. For example, a useful corollary to the above result is Rouché's theorem.

**Proposition 46** (Rouché's theorem). *Let  $f, g \in C^1(\bar{\Omega})$  be holomorphic in  $\Omega$  and nowhere zero on  $\partial\Omega$ . Also assume that*

$$(13.24) \quad |f(z) - g(z)| < |f(z)|, \quad \forall z \in \partial\Omega.$$

*Then,*

$$(13.25) \quad \nu(f, \Omega) = \nu(g, \Omega).$$

*Proof.* Inequality (13.24) implies that  $f$  and  $g$  are smoothly homotopic as maps from  $\partial\Omega$  to  $\mathbb{C} \setminus \{0\}$ . Indeed, take

$$(13.26) \quad f_\tau(z) = f(z) - \tau[f(z) - g(z)], \quad 0 \leq \tau \leq 1.$$

Therefore,  $f|_{C_j}$  and  $g|_{C_j}$  have the same winding numbers about 0, for each boundary component  $C_j$ . □

Rouche's theorem gives another proof of the fundamental theorem of algebra. Let

$$(13.27) \quad f(z) = z^n, \quad \text{and} \quad g(z) = z^n + a_{n-1}z^{n-1} + \dots + a_0.$$

For  $z$  sufficiently large,  $|f(z) - g(z)| < |f(z)|$ , and therefore,  $f$  and  $g$  have the same number of zeros inside the disk  $\{z : |z| \leq R\}$ . It is clear that  $f$  has  $n$  zeros inside this disk, so  $g$  must have  $n$  zeros as well.

**Proposition 47.** *Suppose  $\Omega$  is as in Proposition 43 and let  $f \in C^1(\bar{\Omega})$  be holomorphic on  $\Omega$ . Suppose  $S \subset \mathbb{C}$  is connected, and  $S \cap f(\partial\Omega) = \emptyset$ . Then,*

$$(13.28) \quad \nu(f - q, \Omega) \quad \text{is independent of} \quad q \in S.$$

*Proof.* Define the function,

$$(13.29) \quad \varphi(q) = \nu(f - q, \Omega) = \frac{1}{2\pi i} \int_{\partial\Omega} \frac{f'(z)}{f(z) - q} dz.$$

This function is a continuous function of  $q$ , so since  $\varphi : S \rightarrow \mathbb{Z}$  and  $S$  is connected,  $\varphi$  must be constant.  $\square$

**Proposition 48** (Open mapping theorem). *If  $\Omega \subset \mathbb{C}$  is open and connected, and  $f : \Omega \rightarrow \mathbb{C}$  is holomorphic and nonconstant, then  $f$  maps open sets to open sets.*

*Proof.* Suppose  $p \in \Omega$  and  $q = f(p)$ . Then we have a power series expansion

$$(13.30) \quad f(z) = f(p) + \sum_{n=k}^{\infty} a_n(z - p)^n,$$

where  $a_k \neq 0$ . Therefore, there exists a disk  $D_\rho(p)$  such that  $f|_{D_\rho(p)}$  is bounded away from  $q$ . Applying Proposition 47 to  $S = D_\epsilon(q)$  for some  $\epsilon > 0$ , for all  $q' \in D_\epsilon(q)$ ,

$$(13.31) \quad \nu(f - q', D_\rho(p)) = \nu(f - q, D_\rho(p)) = k.$$

Therefore, such points  $q'$  are contained in the range of  $f$ , and are hit exactly  $k$  times, counting multiplicity.  $\square$

**Proposition 49** (Hurwitz theorem). *Assume  $f_n$  are holomorphic on each connected region  $\Omega$  and  $f_n \rightarrow f$  locally uniformly on  $\Omega$ . Assume each  $f_n$  is nowhere vanishing in  $\Omega$ . Then  $f$  is either nowhere vanishing or identically zero in  $\Omega$ .*

*Proof.* Since  $f_n \rightarrow f$  locally uniformly,  $f$  is holomorphic on  $\Omega$  and  $f'_n \rightarrow f'$  locally uniformly on  $\Omega$ . If  $f$  is not identically zero on  $\Omega$ , then the only zeros of  $f$  in  $\Omega$  are isolated. Let  $D$  be a disk in  $\Omega$  for which  $f$  has zeros in  $D$ , but not in  $\partial D$ . Then  $\frac{1}{f_n} \rightarrow \frac{1}{f}$  locally uniformly on  $\partial D$ . By (13.12),

$$(13.32) \quad \frac{1}{2\pi i} \int_{\partial D} \frac{f'_n(z)}{f_n(z)} dz = 0, \quad \text{for all} \quad n.$$

Passing to the limit,

$$(13.33) \quad \nu(f, D) = \frac{1}{2\pi i} \int_{\partial D} \frac{f'(z)}{f(z)} dz = 0,$$

so  $f$  does not have any zeros in  $D$ .  $\square$

## 14. THE GLOBAL FORM OF CAUCHY'S THEOREM

Let  $c$  be a cycle. We say that  $z \in c$  if  $z$  lies on one of the curves that make up the cycle.

**Definition 17.** *We say that a cycle in a region  $\Omega$  is a zero-homologous cycle in  $\Omega$ , or relative to  $\Omega$ , if  $n(c; z) = 0$  for all  $z \in \mathbb{C} \setminus \Omega$ .*

It is clear that a cycle homotopic to a point is homologous to zero. On the other hand, such a cycle need not be homotopic to a point via a homotopy in  $\Omega$ . For example, take the set  $\Omega = \mathbb{C} \setminus \{z_1, z_2\}$  and the 0-homologous cycle be a figure-eight around  $z_1$  and  $z_2$  followed by an outer wrap around both points.

**Theorem 16.** *Let  $c$  be a 0-homologous cycle in  $\Omega$ . Then for any  $f \in \mathcal{H}(\Omega)$ ,*

$$(14.1) \quad n(c; z_0)f(z_0) = \frac{1}{2\pi i} \oint \frac{f(z)}{z - z_0} dz,$$

for all  $z_0 \in \Omega \setminus c$ . Conversely, if (14.1) holds for all  $f \in \mathcal{H}(\Omega)$  and a fixed  $z_0 \in \Omega \setminus c$ , then  $c$  is a 0-homologous cycle in  $\Omega$ .

*Proof.* Define

$$(14.2) \quad \phi(z, w) = \begin{cases} \frac{f(z) - f(w)}{z - w} & \text{if } z \neq w \in \Omega, \\ f'(w) & \text{if } z = w \in \Omega. \end{cases}$$

By analyticity of  $f$ ,  $\phi(z, w)$  is analytic in  $z$  and jointly continuous in  $z$  and  $w$ . The set

$$(14.3) \quad \Omega' = \{z \in \mathbb{C} \setminus c : n(c, z) = 0\},$$

is open. □

**Theorem 17.** *Let  $c$  be a 0-homologous cycle in  $\Omega$ . For any  $f \in \mathcal{H}(\Omega)$ ,*

$$(14.4) \quad n(c; z_0)f(z_0) = \frac{1}{2\pi i} \oint \frac{f(z)}{z - z_0} dz,$$

for all  $z_0 \in \Omega \setminus c$ . Conversely, if (14.4) holds for all  $f \in \mathcal{H}(\Omega)$  and a fixed  $z_0 \in \Omega \setminus c$ , then  $c$  is a 0-homologous cycle in  $\Omega$ .

*Proof.* Let

$$(14.5) \quad \Omega' = \{z \in \mathbb{C} \setminus c : n(c, z) = 0\}.$$

The set  $\Omega'$  is open,  $\Omega \cup \Omega' = \mathbb{C}$ ,  $\partial\Omega \subset \Omega'$ . Let

$$(14.6) \quad g(z) = \begin{cases} \oint \phi(z, w) dw & \text{if } z \in \Omega, \\ \oint \frac{f(w)}{w - z} dw & \text{if } z \in \Omega'. \end{cases}$$

We show that  $g$  is well-defined and that  $g \in \mathcal{H}(\mathbb{C})$ . Indeed, if  $z \in \Omega$ ,

$$(14.7) \quad \oint_c \phi(z, w) dw = \oint \frac{f(w) - f(z)}{w - z} dw = \oint_c \frac{f(w)}{w - z} dw - 2\pi i f(z) n(c; z),$$

with  $n(c, z) = 0$  for all  $z \in \Omega' \cap \Omega$ .

The function  $g$  is holomorphic on  $\Omega'$ . To show that  $g$  is holomorphic in  $\Omega$ , let  $T \subset \Omega$  be a triangle in  $\Omega$ . By Fubini's theorem,

$$(14.8) \quad \oint_T g(z) dz = \oint_T \oint_c \frac{f(w)}{w - z} dw dz = \oint_c f(w) \oint_T \frac{1}{w - z} dz dw = 0.$$

Since  $g(z) \rightarrow 0$  as  $|z| \rightarrow \infty$ , then by Liouville's theorem,  $g \equiv 0$  on  $\mathbb{C}$ . Thus, for all  $z_0 \in \Omega \setminus c$ ,

$$(14.9) \quad n(c; z_0)f(z_0) = \frac{1}{2\pi i} \oint_c \frac{f(z)}{z - z_0} dz.$$

To prove the converse, fix  $z_1 \in \mathbb{C} \setminus \Omega$ . Then apply the left hand side of (14.4) to  $f(z) = \frac{1}{z - z_1}$ .

$$(14.10) \quad n(c; z_0)f(z_0) = \frac{1}{2\pi i} \oint_c \frac{1}{(z - z_0)(z - z_1)} dz = \frac{f(z_0)}{2\pi i} \oint_c \left[ \frac{1}{z - z_0} - \frac{1}{z - z_1} \right] dz = f(z_0)n(c; z_0) - f(z_0)n(c; z_1).$$

Thus,  $n(c; z_1) = 0$ . □

**Corollary 8.** *With  $c$  and  $\Omega$  as in the previous theorem,*

$$(14.11) \quad \oint_c f(z) dz = 0,$$

for all  $f \in \mathcal{H}(\Omega)$ . In particular, if  $\Omega$  is simply-connected, then (14.11) holds for all cycles in  $\Omega$  and  $f \in \mathcal{H}(\Omega)$ .

*Proof.* For  $z_0 \in \Omega \setminus c$ , let  $h(z) = (z - z_0)f(z)$ . Plugging  $h(z)$  into (14.4),

$$(14.12) \quad 0 = n(c; z_0)h(z_0) = \frac{1}{2\pi i} \oint_c f(z) dz.$$

Now then,  $\mathbb{C} \setminus \Omega$  is connected if and only if  $\Omega$  is simply connected. It is clear that

$$(14.13) \quad n(\gamma; z_0) = \frac{1}{2\pi i} \oint_\gamma \frac{dz}{z - z_0} \rightarrow 0, \quad \text{as } |z| \rightarrow \infty,$$

so if  $\Omega$  is simply connected then (14.13) holds for all cycles in  $\Omega$ . Therefore, every cycle is 0-homologous. □

**Definition 18.** *If each closed curve from  $c_1$  is homotopic to exactly one closed curve from  $c_2$ , then we say that  $c_1$  and  $c_2$  are homotopic.*

**Theorem 18.** *Let  $c_1$  and  $c_2$  be two cycles in  $\Omega$  that are  $C^1$  homotopic. Then,*

$$(14.14) \quad \oint_{c_1} f(z) dz = \oint_{c_2} f(z) dz,$$

for all  $f \in \mathcal{H}(\Omega)$ . In particular, if  $c$  is homotopic to a sum of points then  $\oint f(z) dz = 0$  for all  $f \in \mathcal{H}(\Omega)$ .

This implies that the winding number is homotopy invariant. If a curve  $c$  is 0-homologous relative to  $\Omega$  then any cycle homotopic to  $c$  relative to  $\Omega$  is also 0-homologous.

## 15. RESIDUE CALCULUS

Suppose  $f$  is holomorphic on an open set  $\Omega$ , except for isolated singularities at points  $p_j \in \Omega$ . Each  $p_j$  is contained in a disk  $D_j \subset \subset \Omega$  on a neighborhood of which  $f$  has a Laurent series

$$(15.1) \quad f(z) = \sum_{n=-\infty}^{\infty} a_n(p_j)(z - p_j)^n.$$

**Definition 19** (Residue). *The coefficient  $a_{-1}(p_j)$  of  $(z - p_j)^{-1}$  is called the residue of  $f$  at  $p_j$  and is denoted  $Res_{p_j}(f)$ . Then*

$$(15.2) \quad Res_{p_j}(f) = \frac{1}{2\pi i} \int_{\partial D_j} f(z) dz.$$

If, in addition  $\Omega$  is bounded, with piecewise smooth boundary, and  $f \in C(\bar{\Omega}, \{p_j\})$ , assuming  $\{p_j\}$  is a finite set, then by the Cauchy integral formula,

$$(15.3) \quad \int_{\partial\Omega} f(z) dz = \sum_j \int_{\partial D_j} f(z) dz = 2\pi i \sum_j Res_{p_j}(f).$$

**Theorem 19** (Residue theorem). *Suppose  $f \in \mathcal{H}(\Omega, \{z_j\}_{j=1}^J)$ . If  $c$  is a 0-homologous cycle in  $\Omega$  which does not pass through any of the  $z_j$ , then*

$$(15.4) \quad \frac{1}{2\pi i} \oint_c f(z) dz = \sum_{j=1}^J n(c; z_j) res(f; z_j).$$

*Proof.* Let  $\nu_j = n(c; z_j)$  and define a new cycle

$$(15.5) \quad c' = c - \sum_{j=1}^J \nu_j \gamma_j, \quad \gamma_j(t) = z_j + \epsilon e^{2\pi i t}.$$

Then  $n(c'; w) = 0$  for all  $w \in \mathbb{C} \setminus \Omega$  and  $n(c'; z_j) = 0$  for all  $1 \leq j \leq J$ . Then (15.5) follows from Corollary 8.  $\square$

The argument principle also holds for meromorphic functions. If  $f$  has a pole of order  $m_j$  at  $p_j$ , then (13.16) would give  $-m_j$  for a disk of small radius around  $p_j$ . If  $f$  is meromorphic on a bounded domain  $\Omega$ , and  $C^1$  in a neighborhood of  $\partial\Omega$ . Then the number of zeros of  $f$  minus the number of poles of  $f$  (counting multiplicity) in  $\Omega$  is equal to the sum of the winding numbers of  $f(C_j)$  about 0, where the  $C_j$  are connected components of  $\partial\Omega$ .

**Proposition 50.** *Let  $c$  be a 0-homologous cycle relative to  $\Omega$ . If  $f \in \mathcal{M}(\Omega)$  is such that no zero or pole of  $f$  lies on  $c$ , then*

$$(15.6) \quad n(f \circ c; 0) = \sum_{z \in \Omega: f(z)=0} n(c; z) - \sum_{\zeta \in \Omega: f(\zeta)=\infty} n(c; \zeta),$$

where zeros and poles are counted with multiplicity.

*Proof.* By definition,

$$(15.7) \quad n(f \circ c; 0) = \frac{1}{2\pi i} \oint_{f \circ c} \frac{dw}{w} = \frac{1}{2\pi i} \oint \frac{f'(z)}{f(z)} dz.$$

If  $f(z) = (z - z_0)^n g(z)$  with  $n \neq 0$ ,  $g(z_0) \neq 0$ , and  $g \in \mathcal{H}(\Omega)$ , then

$$(15.8) \quad res\left(\frac{f'}{f}; z_0\right) = n.$$

$\square$

The residue formula has a number of applications. For example, we can compute

$$(15.9) \quad \int_{-\infty}^{\infty} \frac{dx}{1+x^2}.$$

The function  $f(z) = (1+z^2)^{-1}$  is a meromorphic function, with simple poles at  $z = \pm i$ . The residue at  $i$  may be computed

$$(15.10) \quad \lim_{z \rightarrow i} (z-i) \frac{1}{z^2+1} = \lim_{z \rightarrow i} \frac{1}{z+i} = \frac{1}{2i}.$$

Therefore, if  $\gamma$  is a positively oriented contour that contains  $i$  in its interior, but not  $-i$ , then

$$(15.11) \quad \int_{\gamma} \frac{1}{z^2+1} dz = \pi.$$

In particular, let  $\gamma_R$  denote the contour from  $-R$  to  $R$ , and then the semicircle  $Re^{i\theta}$ , where  $0 \leq \theta \leq \pi$ . Then

$$(15.12) \quad \int_{\gamma_R} \frac{1}{z^2+1} dz = \pi.$$

Then,

$$(15.13) \quad \int_0^{\pi} \frac{R}{1+R^2 e^{2i\theta}} i e^{i\theta} d\theta = O\left(\frac{1}{R}\right),$$

so

$$(15.14) \quad \int_{-R}^R \frac{1}{1+x^2} dx = \pi + O\left(\frac{1}{R}\right).$$

Taking  $R \rightarrow \infty$ ,

$$(15.15) \quad \int_{-\infty}^{\infty} \frac{1}{1+x^2} dx = \pi.$$

We also know the antiderivative of  $\frac{1}{1+x^2}$ , which is  $\arctan(x)$ .

Next, consider the integral

$$(15.16) \quad \int_{-\infty}^{\infty} \frac{1}{1+x^4} dx.$$

The function  $\frac{1}{1+z^4}$  has four simple poles, at  $z = e^{i\frac{\pi}{4}}$ ,  $e^{i\frac{3\pi}{4}}$ ,  $e^{i\frac{5\pi}{4}}$ , and  $e^{i\frac{7\pi}{4}}$ . Since

$$(15.17) \quad \lim_{z \rightarrow e^{i\frac{\pi}{4}}} (z^2 + i) = 2i, \quad \text{and} \quad \lim_{z \rightarrow e^{i\frac{3\pi}{4}}} (z^2 - i) = -2i,$$

$$(15.18) \quad \lim_{z \rightarrow e^{i\frac{\pi}{4}}} \frac{1}{(z^2 + i)(z + e^{i\frac{\pi}{4}})} = \frac{1}{4i} \frac{1}{e^{i\frac{\pi}{4}}} = \frac{1}{4} e^{-\frac{3\pi i}{4}},$$

$$(15.19) \quad \lim_{z \rightarrow e^{i\frac{3\pi}{4}}} \frac{1}{(z^2 - i)(z + e^{i\frac{3\pi}{4}})} = \frac{-1}{4i} \frac{1}{e^{i\frac{\pi}{4}}} = \frac{1}{4} e^{-\frac{\pi i}{4}}.$$

Therefore, for any  $\gamma_R$ ,

$$(15.20) \quad \int_{\gamma_R} \frac{1}{z^4+1} dz = \frac{\pi}{\sqrt{2}}.$$

Since

$$(15.21) \quad \int \frac{1}{1 + R^4 e^{4i\theta}} R i e^{i\theta} d\theta = O\left(\frac{1}{R^3}\right),$$

$$(15.22) \quad \int_{-\infty}^{\infty} \frac{1}{1 + x^4} dx = \frac{\pi}{\sqrt{2}}.$$

The evaluation of Fourier transforms provides a rich source of examples to which to apply residue calculus. For example, consider the problem of computing

$$(15.23) \quad \int_{-\infty}^{\infty} \frac{e^{ix\xi}}{1 + x^2} dx.$$

This integral has simple poles at  $z = \pm i$ . Moreover, the residue at  $z = i$  is  $\frac{e^{-\xi}}{2i}$  and the residue at  $z = -i$  is  $-\frac{e^{\xi}}{2i}$ . Then for  $\xi \geq 0$ ,

$$(15.24) \quad \int_{\gamma_R} \frac{e^{iz\xi}}{1 + z^2} dz = \pi e^{-\xi}.$$

Taking  $R \rightarrow \infty$ ,

$$(15.25) \quad \int_{-\infty}^{\infty} \frac{e^{ix\xi}}{1 + x^2} dx = \pi e^{-\xi}.$$

Making the same computation for  $\xi \leq 0$ ,

$$(15.26) \quad \int_{-\infty}^{\infty} \frac{e^{ix\xi}}{1 + x^2} dx = \pi e^{-|\xi|}.$$

## 16. MORE RESIDUE THEOREM COMPUTATIONS USING ALGEBRA OF PATHS

Next, consider the Fourier transform

$$(16.1) \quad A = \int_{-\infty}^{\infty} \frac{e^{ix\xi}}{2 \cosh \frac{x}{2}} dx.$$

Take the integral over the the contour  $\gamma(x) = x + 2\pi i$ . Then compute

$$(16.2) \quad 2 \cosh \frac{x - 2\pi i}{2} = -(e^{x/2} + e^{-x/2}) = -2 \cosh\left(\frac{x}{2}\right).$$

The poles of  $\frac{1}{2 \cosh \frac{z}{2}}$  are exactly the points where  $\cosh\left(\frac{z}{2}\right) = 0$ , or  $e^{z/2} = -e^{-z/2}$ , so then  $e^z = -1$ ,  $z = i\pi + i2n\pi$ . Therefore,

$$(16.3) \quad (1 + e^{-2\pi\xi}) \int_{-\infty}^{\infty} \frac{e^{ix\xi}}{2 \cosh \frac{x}{2}} dx = 2\pi i \lim_{z \rightarrow \pi i} \frac{(z - \pi i)e^{iz\xi}}{2 \cosh\left(\frac{z}{2}\right)} = 2\pi i \frac{e^{-\xi}}{\sinh\left(\frac{\pi i}{2}\right)} = \pi e^{-\xi}.$$

Doing some algebra,

$$(16.4) \quad \int_{-\infty}^{\infty} \frac{e^{ix\xi}}{2 \cosh \frac{x}{2}} dx = \frac{\pi}{\cosh \pi\xi}.$$

Another example of this kind is the integral

$$(16.5) \quad B = \int_0^{\infty} \frac{x^\alpha}{1 + x^2} dx,$$

for some  $0 < \alpha < 1$ . Then define  $z^\alpha = r^\alpha e^{i\alpha\theta}$  for  $0 < \theta < 2\pi$ . Then  $z^\alpha$  is holomorphic on  $\mathbb{C} \setminus \mathbb{R}^+$ . Moreover,  $(x + iy)^\alpha$  has distinct boundary values when  $x > 0$  as  $y \searrow 0$  and  $y \nearrow 0$ . Then let  $\gamma_R$  be the curve going from 0 to  $Re^{i\epsilon}$  along the curve  $re^{i\epsilon}$ , followed by the counterclockwise circle from  $Re^{i\epsilon}$  to  $Re^{i(2\pi-\epsilon)}$ , and then from  $Re^{i(2\pi-\epsilon)}$  to zero along the path  $re^{i(2\pi-\epsilon)}$ ,  $r$  going from  $R$  to 0.

Then for any  $R > 1$ ,

$$(16.6) \quad \int_{\gamma_R} \frac{z^\alpha}{1+z^2} dz = 2\pi i \operatorname{Res}_{z=i} \left( \frac{z^\alpha}{1+z^2} \right) + 2\pi i \operatorname{Res}_{z=-i} \left( \frac{z^\alpha}{1+z^2} \right) = \frac{2\pi i}{2i} e^{i\alpha \frac{\pi}{2}} - \frac{2\pi i}{2i} e^{\frac{3\pi i\alpha}{2}} = \pi(e^{i\pi\alpha/2} - e^{3\pi i\alpha/2}).$$

Next,

$$(16.7) \quad \int_{\epsilon}^{2\pi-\epsilon} \frac{R^\alpha e^{i\theta\alpha}}{1+R^2 e^{2i\theta}} R i e^{i\theta} d\theta = O(R^{\alpha-1}).$$

Also, for any  $R > 0$  fixed,

$$(16.8) \quad \lim_{\epsilon \searrow 0} \int_0^R \frac{r^\alpha e^{i\epsilon\alpha}}{1+r^2 e^{2i\epsilon}} e^{i\epsilon} dr = \int_0^R \frac{x^\alpha}{1+x^2} dx.$$

Meanwhile,

$$(16.9) \quad -\lim_{\epsilon \searrow 0} \int_0^R \frac{r^\alpha e^{i(2\pi-\epsilon)\alpha}}{1+r^2 e^{2i(2\pi-\epsilon)}} dr = -e^{2\pi i\alpha} \int_0^R \frac{x^\alpha}{1+x^2} dx.$$

Therefore, taking  $R \rightarrow \infty$ ,

$$(16.10) \quad (1 - e^{2\pi i\alpha})B = \pi(e^{\pi i\alpha/2} - e^{3\pi i\alpha/2}).$$

Doing some algebra,

$$(16.11) \quad B = \frac{\pi}{2 \cos(\frac{\pi\alpha}{2})}.$$

**Remark 9.** Observe that when  $\alpha = 0$ , (16.11) agrees with (15.15). Also note that taking the derivative of (16.11) when  $\alpha = 0$  yields

$$(16.12) \quad \int_0^\infty \frac{\log x}{1+x^2} dx = 0.$$

This fact was pointed out in class by Tyler Wunder.

It is also possible to use residue calculus to evaluate integrals of the form

$$(16.13) \quad \int_0^{2\pi} R(\cos \theta, \sin \theta) d\theta.$$

Indeed, if we consider the curve  $\gamma(\theta) = e^{i\theta}$ , then

$$(16.14) \quad \int_{\gamma} R\left(\frac{z}{2} + \frac{1}{2z}, \frac{z}{2i} - \frac{1}{2iz}\right) \frac{dz}{iz}.$$

For example, consider the integral

$$(16.15) \quad I(r) = \int_0^{2\pi} \frac{d\theta}{1 - 2r \cos \theta + r^2}, \quad 0 < r < 1.$$

Then,

$$(16.16) \quad I(r) = \int_{\gamma} \frac{1}{1 - r(z + \frac{1}{z}) + r^2} \frac{dz}{iz} = i \int_{\gamma} \frac{dz}{rz^2 - (1+r^2)z + r}.$$

The polynomial  $p(z) = rz^2 - (1 + r^2)z + r$  has two roots,  $r$  and  $\frac{1}{r}$ . Then

$$(16.17) \quad rz^2 - (1 + r^2)z + r = r(z - r)\left(z - \frac{1}{r}\right).$$

Therefore, for  $0 < r < 1$ ,

$$(16.18) \quad I(r) = \frac{i}{r} \int_{\gamma} \frac{1}{z - 1/r} \frac{dz}{z - r} = \frac{i}{r} 2\pi i g(r),$$

where  $g(z) = (z - \frac{1}{r})^{-1}$ . Therefore,

$$(16.19) \quad I(r) = -\frac{2\pi}{r} \frac{1}{r - 1/r} = \frac{2\pi}{1 - r^2}, \quad \text{for } 0 < r < 1.$$

Now apply residue calculus to an integrand with a double pole. For example, consider the integral

$$(16.20) \quad u(\xi) = \int_{-\infty}^{\infty} \frac{e^{ix\xi}}{(1 + x^2)^2} dx, \quad \xi \in \mathbb{R}.$$

Then,

$$(16.21) \quad \text{Res}_i \frac{e^{iz\xi}}{(1 + z^2)^2} = g'(i),$$

where

$$(16.22) \quad g(z) = \frac{e^{iz\xi}}{(z + i)^2}.$$

Then,

$$(16.23) \quad g'(i) = -\frac{i}{4}(1 + \xi)e^{-\xi}.$$

Therefore, when  $\xi > 0$ ,

$$(16.24) \quad u(\xi) = \lim_{R \rightarrow \infty} \int_{\gamma_R} \frac{e^{iz\xi}}{(1 + z^2)^2} dz = \frac{\pi}{2}(1 + \xi)e^{-\xi}.$$

Therefore, since  $u(\xi)$  is an even function of  $\xi$ ,

$$(16.25) \quad \int_{-\infty}^{\infty} \frac{e^{ix\xi}}{(1 + x^2)^2} dx = \frac{\pi}{2}(1 + |\xi|)e^{-|\xi|}, \quad \forall \xi \in \mathbb{R}.$$

It is also possible to use residue calculus to compute a class of infinite series.

**Proposition 51.** *Let  $P(z)$  and  $Q(z)$  be polynomials of degree  $d_Q \geq d_P + 2$ , and let*

$$(16.26) \quad f(z) = \frac{P(z)}{Q(z)}.$$

*Let  $Z_Q = \{z \in \mathbb{C} : Q(z) = 0\}$ , and assume that none of these roots belong to  $\mathbb{Z}$ . Then,*

$$(16.27) \quad \sum_{k=-\infty}^{\infty} f(k) = -\pi \sum_{p \in Z_Q} \text{Res}_p f(z) \cot \pi z.$$

*Proof.* For  $N \in \mathbb{N}$ , take

$$(16.28) \quad \Omega_N = \left\{ z \in \mathbb{C} : |Re(z)| < N + \frac{1}{2}, \quad |Im(z)| \leq N + \frac{1}{2} \right\}.$$

There exists  $K < \infty$  such that  $|\cot(\pi z)| \leq K$  for all  $z \in \partial\Omega_N$ ,  $N \in \mathbb{N}$ . Indeed,  $\sin(\pi(N + \frac{1}{2})) = 1$ . Therefore,

$$(16.29) \quad \lim_{N \rightarrow \infty} \int_{\partial\Omega_N} f(z) \cot \pi z dz = 0.$$

Meanwhile,

$$(16.30) \quad \int_{\partial\Omega_N} f(z) \cot(\pi z) dz = 2\pi i \sum_{p \in (\Omega_N \cap \mathbb{Z}) \cup \mathcal{Z}_Q} Res_p f(z) \cot(\pi z).$$

The residue at  $k$  is given by

$$(16.31) \quad Res_k f(z) \cot \pi z = \frac{1}{\pi} f(k), \quad \forall k \in \mathbb{Z}.$$

Taking  $N \rightarrow \infty$  yields (16.27). □

For example, if  $p$  is a simple root of  $Q(z)$ , and  $p \notin \mathbb{Z}$ , then

$$(16.32) \quad Res_p f(z) \cot(\pi z) = \frac{P(p)}{Q'(p)} \cot(\pi p).$$

Another example would be to take

$$(16.33) \quad f(z) = \frac{1}{z^2 + a^2}, \quad a \in \mathbb{C} \setminus i\mathbb{Z}.$$

Then,

$$(16.34) \quad \sum_{k=-\infty}^{\infty} \frac{1}{k^2 + a^2} = -\pi (Res_{ia} \frac{\cot \pi z}{z^2 + a^2} + Res_{-ia} \frac{\cot \pi z}{z^2 + a^2}) = \frac{\pi}{a} \coth(\pi a).$$

## 17. ANALYTIC CONTINUATION

In this section we will discuss the subject of analytic continuation. Functions such as the gamma function and zeta function are defined on a subdomain of the complex plane by integral or series representations.

We begin with the Gaussian integral,

$$(17.1) \quad G(z) = \int_{-\infty}^{\infty} e^{-t^2 + tz} dt.$$

This integral is absolutely convergent, and for each  $z \in \mathbb{C}$ ,  $G(z)$  is a continuous function of  $z$ . Furthermore, if  $\gamma$  is a closed curve on  $\mathbb{C}$ , we can use Fubini's theorem to prove

$$(17.2) \quad \int_{\gamma} G(z) dz = \int_{-\infty}^{\infty} \int_{\gamma} e^{-t^2 + tz} dz dt = 0,$$

by the Cauchy integral theorem. Therefore,  $G$  is holomorphic on  $\mathbb{C}$ .

For  $z = x$  real, completing the square,

$$(17.3) \quad G(x) = \int_{-\infty}^{\infty} e^{-t^2 + tx} dt = e^{\frac{x^2}{4}} \int_{-\infty}^{\infty} e^{-(t - \frac{x}{2})^2} dt = e^{\frac{x^2}{4}} \int_{-\infty}^{\infty} e^{-t^2} dt = \sqrt{\pi} e^{\frac{x^2}{4}}.$$

Therefore, by Proposition 29,

$$(17.4) \quad G(z) = \sqrt{\pi} e^{-\frac{z^2}{4}}, \quad z \in \mathbb{C}.$$

In particular,  $G(iy) = \sqrt{\pi} e^{-\frac{y^2}{4}}$ ,

$$(17.5) \quad \int_{-\infty}^{\infty} e^{-t^2 + ity} dt = \sqrt{\pi} e^{-\frac{y^2}{4}}, \quad y \in \mathbb{R}.$$

This shows that the Fourier transform of the Gaussian function is the Gaussian.

Now consider the Gamma function  $\Gamma(z)$ . If  $\operatorname{Re}(z) > 0$ , then let

$$(17.6) \quad \Gamma(z) = \int_0^{\infty} e^{-t} t^{z-1} dt.$$

Once again, by Fubini's theorem,  $\Gamma(z)$  is holomorphic in the right half plane  $\operatorname{Re}(z) > 0$ . Indeed, observe that  $z \mapsto t^{z-1} = \exp((z-1)\log t)$  is an entire function for each  $t > 0$ . Furthermore, if  $x = \operatorname{Re}(z)$ ,

$$(17.7) \quad \int_0^{\infty} |e^{-t} t^{z-1}| dt = \int_0^{\infty} e^{-t} t^{x-1} dt,$$

which converges absolutely. Now let  $\gamma$  be a closed curve in  $\{z \in \mathbb{C} : x > 0\}$ . Since  $\gamma$  is a closed curve, there exists some  $A > 1$  such that  $\gamma$  lies in  $\{0 < A^{-1} \leq x \leq A\}$ . By Fubini's theorem, since  $t^{z-1}$  is analytic for any  $t > 0$ ,

$$(17.8) \quad \int_{\gamma} \int_0^{\infty} e^{-t} t^{z-1} dt dz = \int_0^{\infty} e^{-t} \int_{\gamma} t^{z-1} dz dt = 0.$$

Therefore, by Morera's theorem  $\Gamma$  is holomorphic in the right half plane.

Now then, integrating by parts, for any  $\operatorname{Re}(z) > 0$ ,

$$(17.9) \quad \Gamma(z+1) = z\Gamma(z).$$

Now then, integrating by parts, since  $\operatorname{Re}(z) > 0$ ,

$$(17.10) \quad z\Gamma(z) = \int_0^{\infty} e^{-t} z t^{z-1} dt = \int_0^{\infty} e^{-t} \frac{d}{dt}(t^z) dt = \int_0^{\infty} e^{-t} t^z dt = \Gamma(z+1).$$

Since the right hand side is defined for all  $\operatorname{Re}(z) > -1$ , set

$$(17.11) \quad \Gamma(z) = \frac{\Gamma(z+1)}{z}, \quad \text{for all } \operatorname{Re}(z) > -1.$$

Note that  $z = 0$  is a pole of first order. Iterating this identity yields, with  $k \geq 0$ ,

$$(17.12) \quad \Gamma(z) = \frac{\Gamma(z+k+1)}{z(z+1)\cdots(z+k)}, \quad \text{for all } \operatorname{Re}(z) > -k-1.$$

This allows one to analytically continue  $\Gamma(z)$  as a meromorphic function to all of  $\mathbb{C}$ , and has simple poles at  $\{n \in \mathbb{Z} : n \leq 0\}$  with residues  $\frac{(-1)^n}{n!}$ .

At this point, it is fair to ask whether or not the definition of  $\Gamma(z)$  given by (17.11) is the only such definition which is meromorphic on  $\mathbb{C}$  with poles at  $\{n \in \mathbb{Z} : n \leq 0\}$ . To show that this is in fact the case, we define a chain of disks along a continuous curve, and put analytic functions on the disks that are continuations of one another.

**Definition 20.** Suppose  $\gamma : [0, 1] \rightarrow \Omega$  is a continuous curve inside a region  $\Omega$ . We say that  $D_j = D(\gamma(t_j), r_j) \subset \Omega$ ,  $0 \leq j \leq J$  is a chain of disks along  $\gamma$  in  $\Omega$  if  $0 = t_0 < t_1 < t_2 < \dots < t_N = 1$  and  $\gamma([t_j, t_{j+1}]) \subset D_j \cap D_{j+1}$  for all  $0 \leq j \leq N - 1$ .

By uniform continuity of  $\gamma$ , there exists such a chain of disks along  $\gamma$  in  $\Omega$ .

**Definition 21.** Let  $\gamma : [0, 1] \rightarrow \Omega$  be a continuous curve inside  $\Omega$ . Suppose  $f \in \mathcal{H}(U)$  and  $g \in \mathcal{H}(V)$ , where  $U \subset \Omega$  and  $V \subset \Omega$  are neighborhoods of  $p = \gamma(0)$  and  $q = \gamma(1)$ , respectively. We say that  $g$  is an analytic continuation of  $f$  along  $\gamma$  if there exists a chain of disks  $D_j = D(\gamma(t_j), r_j)$  along  $\gamma$  in  $\Omega$  where  $0 \leq j \leq J$  and  $f_j \in \mathcal{H}(D_j)$  such that  $f_j = f_{j+1}$  on  $D_j \cap D_{j+1}$  and  $f_0 = f$  and  $f_J = g$  around  $p$  and  $q$  respectively.

Analytic continuation is unique where it exists, and does not depend on the chain of disks along  $\gamma$ , but only on  $\gamma$  itself.

**Lemma 15.** The analytic continuation  $g$  of  $f$  along  $\gamma$  depends only on  $f$  and  $\gamma$ , but not on the specific choice of the chain of circles. In particular, it is unique.

*Proof.* Suppose  $D_j$  and  $\tilde{D}_k$  are two different chains of disks along  $\gamma$  with underlying partitions  $\{t_j\}_{j=1}^J$  and  $\{s_k\}_{k=1}^K$ , respectively. Denote the chain of analytic functions defined on these disks by  $f_j$  and  $g_k$ .

**Claim 1.** For any  $j, k$  with  $t_{j-1} \leq s_k \leq t_j$ ,

$$(17.13) \quad f_j = g_k, \quad \text{on} \quad D_j \cap \tilde{D}_k.$$

Applying this claim to the endpoint of  $\gamma$  yields the desired uniqueness.

We prove the claim by induction. It is clearly true when  $j = k = 0$ . In general, if  $t_{j-1} \leq s_k \leq t_j$ , then  $\gamma(s_k) \in D_{j-1}$  and  $\gamma(s_k) \in D_j$ . Therefore, by Proposition 29,  $f_{j-1} = g_k$  on  $D_{j-1} \cap \tilde{D}_k$  implies  $f_j = g_k$  on  $D_j \cap \tilde{D}_k$ .  $\square$

A special case of the analytic continuation lemma is the anti-derivative of  $f \in \mathcal{H}(\Omega)$ . Then, locally around every point in  $\Omega$  there exists an anti-derivative, or primitive. Any such primitive can be analytically continued along an arbitrary  $C^1$ -curve  $\gamma : [0, 1] \rightarrow \Omega$  by integration:

$$(17.14) \quad F(z) = \int_{\gamma} f(\zeta) d\zeta,$$

where  $\gamma(1) = z$  and  $\gamma(0) = z_0$  is kept fixed. However, this need not lead to a global primitive  $F \in \mathcal{H}(\Omega)$ . For example, let  $\Omega = \mathbb{C} \setminus \{0\}$  and  $f(z) = \frac{1}{z}$ . On the other hand, we can obtain a global  $F$  if  $\Omega$  is simply connected.

**Theorem 20** (Monodromy theorem). Suppose  $\gamma_0$  and  $\gamma_1$  are two homotopic curves relative to some region  $\Omega \subset \mathbb{C}$  with the same initial point  $p$  and endpoint  $q$ . Let  $U$  be a neighborhood of  $p$  and assume further that  $f \in \mathcal{H}(U)$  can be analytically continued along every curve of the homotopy. Then the analytic continuation of  $f$  along  $\gamma_j$ ,  $j = 0, 1$  agree locally around  $q$ .

*Proof.* Let  $H : [0, 1]^2 \rightarrow \Omega$  be the homotopy between  $\gamma_0$  and  $\gamma_1$  that fixes the two endpoints. Thus,  $H = H(t, s)$  where  $\gamma_0(t) = H(t, 0)$  and  $\gamma_1(t) = H(t, 1)$ , respectively. Let  $g_s$  denote the continuation of  $f$  along  $H(\cdot, s)$  by  $g_s$ .

Fix any  $s_0 \in [0, 1]$  and suppose  $\{D_j\}_{j=1}^J$  is a chain of circles along  $\gamma_{s_0}$  with underlying partition  $0 = t_0 < t_1 < \dots < t_N = 1$  and functions  $f_j$  on  $D_j$  defining the analytic continuation of  $f$  along  $\gamma_{s_0}$ .

**Claim 2.** Let  $D_j(s)$  denote the largest disk centered at  $\gamma_s(t_j)$  which is contained in  $D_j$ . There exists  $\epsilon > 0$  such that for all  $s \in [0, 1]$ ,  $|s - s_0| < \epsilon$ , the  $D_j(s)$  form a chain of disks along  $\gamma_s$ .

By Lemma 15 we can use the same  $f_j$  for all  $|s - s_0| < \epsilon$ , so the  $g_s$  agree with  $g_{s_0}$  locally around  $q$ .

To prove the claim, we use uniform continuity of the homotopy  $H$ . □

Within a simply connected region, any two curves with the same initial and end points are homotopic. Suppose for a simply connected region  $\Omega$ , and  $f$  is holomorphic on some disk included in  $\Omega$ . Then suppose we may analytically continue  $f$  to any point in  $\Omega$  along any path. Then, there exists a necessarily unique, analytic function  $F$  on  $\Omega$  that agrees with  $f$  on the original disk. In other words, if there is no local obstruction to analytic continuation anywhere in  $\Omega$ , then we may obtain a global analytic function on  $\Omega$  from the germ  $f$ .

For example, the logarithm may be defined on a simply connected domain.

**Remark 10.** Any domain  $\Omega$  carries analytic functions which cannot be continued beyond any portion of the boundary  $\partial\Omega$ . Let  $\{z_n\}_{n=1}^\infty$  be dense in  $\partial\Omega$  and define

$$(17.15) \quad f(z) = \sum_{n=1}^{\infty} 2^{-n} \frac{1}{z - z_n},$$

which is analytic on  $\Omega$ . A power series with finite and positive radius of convergence also cannot be analytically continued across the entire circle of its convergence.

## 18. THE GAMMA FUNCTION

The Gamma function arises in the computation of the natural Laplace transform,

$$(18.1) \quad f(t) = t^{z-1} \Rightarrow \mathcal{L}f(s) = \Gamma(z)s^{-z},$$

for  $\operatorname{Re}(z) > 0$ , with

$$(18.2) \quad \Gamma(z) = \int_0^\infty e^{-t} t^{z-1} dt, \quad \operatorname{Re}(z) > 0.$$

Indeed, by a change of variables, if  $\operatorname{Re}(s) > 0$ ,

$$(18.3) \quad \int_0^\infty e^{-st} t^{z-1} dt = s^{-z} \int_0^\infty e^{-st} (st)^{z-1} s dt = s^{-z} \Gamma(z).$$

We may develop more properties for this special function. Observe that integrating by parts,

$$(18.4) \quad \Gamma(z+1) = \int_0^\infty e^{-t} t^z dt = - \int_0^\infty \frac{d}{dt} (e^{-t}) t^z dt = z\Gamma(z).$$

Since  $\Gamma(1) = 1$ , for any integer  $k \geq 1$ ,

$$(18.5) \quad \Gamma(k) = (k-1)\Gamma(k-1) = (k-1)!.$$

The integral (18.2) converges for  $\operatorname{Re}(z) > 0$ , so then by (18.4),  $\Gamma(z)$  is well defined for  $\operatorname{Re}(z) > -1$ , and using (18.4),  $\Gamma(z)$  is meromorphic on  $\{z : \operatorname{Re}(z) > -1\}$ , with a simple pole at  $z = 0$ . Iterating, using (18.4),  $\Gamma(z)$  can be extended to be meromorphic on  $\mathbb{C}$ , with simple poles at  $z = 0, -1, -2, \dots$

**Proposition 52.** For  $z \in \mathbb{C} \setminus \mathbb{Z}$ , we have

$$(18.6) \quad \Gamma(z)\Gamma(1-z) = \frac{\pi}{\sin(\pi z)}.$$

*Proof.* It suffices to establish the identity for  $0 < \operatorname{Re}(z) < 1$ . In this case,

$$(18.7) \quad \Gamma(z)\Gamma(1-z) = \int_0^\infty \int_0^\infty e^{-(s+t)} s^{-z} t^{z-1} ds dt.$$

Making the change of variables  $u = s + t$ ,  $v = \frac{t}{s}$ ,

$$(18.8) \quad = \int_0^\infty \int_0^\infty e^{-u} v^{z-1} (1+v)^{-1} du dv = \int_0^\infty (1+v)^{-1} v^{z-1} dv.$$

Taking  $v = e^x$ , the last integral is equal to

$$(18.9) \quad \int_{-\infty}^\infty (1+e^x)^{-1} e^{xz} dx,$$

which is holomorphic on  $0 < \operatorname{Re}(z) < 1$ . Evaluating (18.9) on the strip  $z = \frac{1}{2} + i\xi$ ,  $\xi \in \mathbb{R}$ , by (16.4),

$$(18.10) \quad = \int_{-\infty}^\infty (2 \cosh(\frac{x}{2}))^{-1} e^{ix\xi} dx = \frac{\pi}{\cosh(\pi\xi)}.$$

Since

$$(18.11) \quad \frac{\pi}{\sin(\pi(\frac{1}{2} + i\xi))} = \frac{\pi}{\cosh(\pi\xi)}.$$

Then by unique analytic continuation, the proof is complete.  $\square$

**Remark 11.** One could also use a contour integral to show that

$$(18.12) \quad \int_0^\infty \frac{x^{\alpha-1}}{1+x} dx = \frac{\pi}{\sin(\pi\alpha)}, \quad 0 < \operatorname{Re}(\alpha) < 1.$$

**Corollary 9.** The function  $\Gamma(z)$  has no zeros, so  $\frac{1}{\Gamma(z)}$  is an entire function.

Next, compute the following estimate.

**Lemma 16.** For  $n \geq 4$ ,

$$(18.13) \quad 0 \leq e^{-t} - (1 - \frac{t}{n})^n \leq \frac{t^2}{n} e^{-t}, \quad 0 \leq t \leq n.$$

*Proof.* To show that  $e^{-y} - (1 - y) \geq 0$ , observe that  $f(0) = 0$  when  $f(y) = e^{-y} - (1 - y)$ , and  $f'(y) = -e^{-y} + 1 \geq 0$  for  $0 \leq y \leq 1$ .

For the second inequality, using the Taylor series, write

$$(18.14) \quad \log(1 - \frac{t}{n})^n = n \log(1 - \frac{t}{n}) = -t - X, \quad \text{where} \quad X = \frac{t^2}{n} (\frac{1}{2} + \frac{1}{3} \frac{t}{n} + \dots).$$

Therefore,  $(1 - \frac{t}{n})^n = e^{t-X}$ , so for  $0 \leq t \leq n$ , since  $x - (1 - e^{-x}) \geq 0$ ,

$$(18.15) \quad e^{-t} - (1 - \frac{t}{n})^n = (1 - e^{-X}) e^{-t} \leq X e^{-t}.$$

It is clear from (18.14) that  $X \leq \frac{t^2}{n}$  if  $t \leq \frac{n}{2}$ . On the other hand, if  $t \geq n/2$  and  $n \geq 4$ , then  $t^2/n \geq 1$ , so  $e^{-t} \leq (t^2/n) e^{-t}$ .  $\square$

This fact yields the result of Euler,

**Proposition 53.** For  $\operatorname{Re}(z) > 0$ , we have

$$(18.16) \quad \Gamma(z) = \lim_{n \rightarrow \infty} n^z \frac{1 \cdot 2 \cdots n}{z(z+1) \cdots (z+n)}.$$

*Proof.* For  $Re(z) > 0$ ,

$$(18.17) \quad \Gamma(z) = \lim_{n \rightarrow \infty} \int_0^n \left(1 - \frac{t}{n}\right)^n t^{z-1} dt = \lim_{n \rightarrow \infty} n^z \int_0^1 (1-s)^n s^{z-1} ds.$$

Indeed,

$$(18.18) \quad \Gamma(z) = \int_0^\infty e^{-t} t^{z-1} dt = \lim_{n \rightarrow \infty} \int_0^n e^{-t} t^{z-1} dt = \lim_{n \rightarrow \infty} \left[ \int_0^n \left(1 - \frac{t}{n}\right)^n t^{z-1} dt + O\left(\int_0^n \frac{t^2}{n} e^{-t} t^{z-1} dt\right) \right].$$

The last term clearly goes to zero as  $n \rightarrow \infty$ . Repeatedly integrating by parts,

$$(18.19) \quad \Gamma(z) = \lim_{n \rightarrow \infty} n^z \frac{1 \cdot 2 \cdots n}{z(z+1) \cdots (z+n-1)} \int_0^1 s^{z+n-1} ds.$$

□

Meanwhile, using (18.4),

$$(18.20) \quad \Gamma(z) = \frac{\Gamma(z+1)}{z} = \lim_{n \rightarrow \infty} n^{z+1} \frac{1 \cdot 2 \cdots n}{z(z+1) \cdots (z+1+n)}, \quad \text{for } Re(z) > -1.$$

Therefore, (18.16) can be extended to  $Re(z) > -1$ , and one may use (18.4) to obtain further extensions.

Rewriting (18.16),

$$(18.21) \quad \Gamma(z) = \lim_{n \rightarrow \infty} n^z z^{-1} (1+z)^{-1} \left(1 + \frac{z}{2}\right)^{-1} \cdots \left(1 + \frac{z}{n}\right)^{-1}.$$

Now define the Euler constant

$$(18.22) \quad \Gamma = \lim_{n \rightarrow \infty} \left(1 + \frac{1}{2} + \cdots + \frac{1}{n} - \log n\right).$$

Then (18.21) is equivalent to

$$(18.23) \quad \Gamma(z) = \lim_{n \rightarrow \infty} e^{-\gamma z} e^{z(1+\frac{1}{2}+\cdots+\frac{1}{n})} z^{-1} (1+z)^{-1} \cdots \left(1 + \frac{z}{n}\right)^{-1}.$$

This leads to the Euler product expansion

**Proposition 54.** *For all  $z \in \mathbb{C}$ , we have*

$$(18.24) \quad \frac{1}{\Gamma(z)} = z e^{\gamma z} \prod_{n=1}^{\infty} \left(1 + \frac{z}{n}\right) e^{-z/n}.$$

Now then, observe that  $\frac{1}{\Gamma(z)} \frac{1}{\Gamma(-z)}$  has the product expansion

$$(18.25) \quad \frac{1}{\Gamma(z)\Gamma(-z)} = -z^2 \prod_{n=1}^{\infty} \left(1 - \frac{z^2}{n^2}\right).$$

Since  $\Gamma(z)\Gamma(1-z) = \frac{\pi}{\sin(\pi z)}$  and  $\Gamma(1-z) = z\Gamma(-z)$ ,

$$(18.26) \quad \sin(\pi z) = \pi z \prod_{n=1}^{\infty} \left(1 - \frac{z^2}{n^2}\right).$$

Next, observe that (18.6) implies that  $\Gamma(\frac{1}{2})^2 = \pi$ . Since  $\Gamma(\frac{1}{2}) > 0$ , this means that  $\Gamma(\frac{1}{2}) = \sqrt{\pi}$ . Meanwhile, by a change of variables,

$$(18.27) \quad \int_0^\infty e^{-x^2} dx = \frac{1}{2} \int_0^\infty e^{-t} t^{-1/2} dt = \frac{1}{2} \Gamma\left(\frac{1}{2}\right) = \frac{\sqrt{\pi}}{2},$$

which again gives  $\Gamma(\frac{1}{2}) = \sqrt{\pi}$ . Now then, by (18.4),

$$(18.28) \quad \Gamma\left(k + \frac{1}{2}\right) = \sqrt{\pi} \left(k - \frac{1}{2}\right) \left(k - \frac{3}{2}\right) \cdots \left(\frac{1}{2}\right).$$

The area of a sphere may be computed by relating Gaussian integrals to the Gamma function. Indeed, observe that

$$(18.29) \quad \int_{\mathbb{R}^n} e^{-|x|^2} dx = \left(\int_{-\infty}^\infty e^{-x^2} dx\right)^n = \pi^{n/2}.$$

On the other hand, using spherical polar coordinates and a change of variables,

$$(18.30) \quad \int_{\mathbb{R}^n} e^{-|x|^2} dx = A_{n-1} \int_0^\infty e^{-r^2} r^{n-1} dr = \frac{1}{2} A_{n-1} \int_0^\infty e^{-t} t^{n/2-1} dt.$$

The last integral is  $\Gamma(\frac{n}{2})$ , so

$$(18.31) \quad A_{n-1} = \frac{2\pi^{n/2}}{\Gamma(\frac{n}{2})}.$$

The following is a useful representation of  $\frac{1}{\Gamma(z)}$ .

**Proposition 55.** *Take  $\rho > 0$  and let  $\sigma$  be the path that goes from  $-\infty - i0$  to  $-\rho - i0$ , then counterclockwise from  $-\rho - i0$  to  $-\rho + i0$  along the circle  $|\zeta| = \rho$ , and then from  $-\rho + i0$  to  $-\infty + i0$ . Then,*

$$(18.32) \quad \frac{2\pi i}{\Gamma(z)} = \int_\sigma e^\zeta \zeta^{-z} d\zeta, \quad \forall z \in \mathbb{C}.$$

*Proof.* Let  $I(z)$  denote the right hand side of (18.32). The right hand side is independent of the choice of  $\rho$ , since  $\zeta^{-z} = e^{-z(\log(\zeta))}$  is defined by a branch cut. Both sides are holomorphic in  $z \in \mathbb{C}$ , so it suffices to establish the identity for  $z = x \in (0, 1)$ . In this case, as  $\rho \rightarrow 0$ , the contribution of the circle integral vanishes, so at the limit,

$$(18.33) \quad I(x) = \int_0^\infty e^{-t} t^{-x} e^{\pi i x} dt - \int_0^\infty e^{-t} t^{-x} e^{-\pi i x} dt = 2i(\sin(\pi x))\Gamma(1-x) = \frac{2\pi i}{\Gamma(x)}.$$

The last identity uses (18.6). □

**Remark 12.** *Read the Legendre duplication formula appendix.*

## 19. THE RIEMANN ZETA FUNCTION

The Riemann zeta function is defined by

$$(19.1) \quad \zeta(s) = \sum_{n=1}^\infty n^{-s}, \quad \operatorname{Re}(s) > 1.$$

Special cases include  $\zeta(2) = \frac{\pi^2}{6}$  and  $\zeta(4) = \frac{\pi^4}{90}$ .

**Proposition 56** (Euler's product formula). *Let  $\{p_j : j \geq 1\} = \{2, 3, 5, \dots\}$  denote the set of prime numbers in  $\mathbb{N}$ . Then for  $\operatorname{Re}(s) > 1$ ,*

$$(19.2) \quad \zeta(s) = \prod_{j=1}^{\infty} (1 - p_j^{-s})^{-1}.$$

*Proof.* Expanding out the right hand side of (19.2),

$$(19.3) \quad \prod_{j=1}^{\infty} (1 - p_j^{-s})^{-1} = \prod_{j=1}^{\infty} (1 + p_j^{-s} + p_j^{-2s} + p_j^{-3s} + \dots) = 1 + \sum_j p_j^{-s} + \sum_{j_1 \leq j_2} (p_{j_1} p_{j_2})^{-s} + \sum_{j_1 \leq j_2 \leq j_3} (p_{j_1} p_{j_2} p_{j_3})^{-s} + \dots$$

Then by the prime number theorem, (19.3) = (19.1).  $\square$

From (19.1), as  $s \searrow 1$ ,  $\zeta(s) \nearrow +\infty$ , so therefore,

$$(19.4) \quad \prod_{j=1}^{\infty} (1 - p_j^{-1}) = 0.$$

This gives the quantitative strengthening of Euler's result that there are infinitely many primes, since (19.4) implies

$$(19.5) \quad \sum_{j=1}^{\infty} \frac{1}{p_j} = \infty.$$

In fact, since  $x \leq \log \frac{1}{1-x} \leq 2x$ ,

$$(19.6) \quad \frac{1}{2} \log \zeta(s) \leq \sum_{j=1}^{\infty} \frac{1}{p_j^s} \leq \log \zeta(s), \quad \forall \quad s > 1.$$

Therefore,  $p_j \rightarrow \infty$  somewhat faster than  $j$ , but as fast as  $j(\log j)^{1+a}$  for some  $a > 0$ .

Also by the product formula,

$$(19.7) \quad \zeta(s) \neq 0, \quad \text{for} \quad \operatorname{Re}(s) > 1.$$

**Proposition 57.** *The function  $\zeta(s)$  extends to a meromorphic function on  $\mathbb{C}$ , with one simple pole at  $s = 1$ . In fact,*

$$(19.8) \quad \zeta(s) - \frac{1}{s-1},$$

*extends to an entire function of  $s$ .*

First, relate the Riemann zeta function to the function

$$(19.9) \quad g(t) = \sum_{n=1}^{\infty} e^{-n^2 \pi t}.$$

Indeed, by a change of variables,

$$(19.10) \quad \int_0^{\infty} g(t) t^{s-1} dt = \sum_{n=1}^{\infty} n^{-2s} \pi^{-s} \int_0^{\infty} e^{-t} t^{s-1} dt = \zeta(2s) \pi^{-s} \Gamma(s).$$

Using the Jacobi identity,

$$(19.11) \quad \sum_{l=-\infty}^{\infty} e^{-\pi l^2 \tau} = \sqrt{\frac{1}{\tau}} \sum_{k=-\infty}^{\infty} e^{-\pi k^2 / \tau},$$

we have

$$(19.12) \quad \sum_{l=-\infty}^{\infty} e^{-\pi l^2 t} = \sqrt{\frac{1}{t}} \sum_{k=-\infty}^{\infty} e^{-\pi k^2 / t}.$$

Therefore,

$$(19.13) \quad g(t) = \frac{1}{2} \sum_{n=-\infty}^{\infty} e^{-n^2 \pi t} - \frac{1}{2} = \frac{1}{2} \sqrt{\frac{1}{t}} \sum_{n=-\infty}^{\infty} e^{-n^2 \pi / t} - \frac{1}{2} = \sqrt{\frac{1}{t}} g\left(\frac{1}{t}\right) + \frac{1}{2} \sqrt{\frac{1}{t}} - \frac{1}{2}.$$

Now then, from (19.10), for  $\operatorname{Re}(s) > 1$ ,

$$(19.14) \quad \Gamma\left(\frac{s}{2}\right) \pi^{-s/2} \zeta(s) = \int_0^{\infty} g(t) t^{s/2-1} dt = \int_0^1 g(t) t^{s/2-1} dt + \int_1^{\infty} g(t) t^{s/2-1} dt.$$

Using (19.13),

$$(19.15) \quad \Gamma\left(\frac{s}{2}\right) \pi^{-s/2} \zeta(s) = \int_0^1 \left(-\frac{1}{2} + \frac{1}{2} t^{-1/2}\right) t^{s/2-1} dt + \int_0^1 g\left(\frac{1}{t}\right) t^{s/2-3/2} dt + \int_1^{\infty} g(t) t^{s/2-1} dt.$$

Therefore,

$$(19.16) \quad \Gamma\left(\frac{s}{2}\right) \pi^{-s/2} \zeta(s) = \frac{1}{s-1} - \frac{1}{s} + \int_1^{\infty} [t^{s/2} + t^{(1-s)/2}] g(t) t^{-1} dt.$$

Since  $g(t) \leq C e^{-\pi t}$  for  $t \in [1, \infty)$ , the integral on the right hand side of (19.16) defines an entire function of  $s$ . Since  $\frac{1}{\Gamma(\frac{s}{2})}$  is entire, with simple zeros at  $s = 0, -2, -4, \dots$ . Therefore,  $\zeta(s)$  is continued as a meromorphic function on  $\mathbb{C}$  with one simple pole at  $s = 1$ . Furthermore,  $\Gamma(\frac{1}{2}) = \pi^{1/2}$ , so (19.8) is regular at  $s = 1$ .

Note that (19.16) is invariant upon replacing  $s$  by  $1 - s$ . Therefore,

**Proposition 58** (Riemann's functional equation). *For  $s \neq 1$ ,*

$$(19.17) \quad \Gamma\left(\frac{s}{2}\right) \pi^{-s/2} \zeta(s) = \Gamma\left(\frac{1-s}{2}\right) \pi^{-(1-s)/2} \zeta(1-s).$$

**Remark 13.** *This formula is an important strengthening of (19.7).*

**Proposition 59.** *For all real  $t \neq 0$ ,  $\zeta(1+it) \neq 0$ . Hence,*

$$(19.18) \quad \zeta(s) \neq 0, \quad \text{for } \operatorname{Re}(s) \geq 1.$$

**Lemma 17.** *For all  $\sigma > 1$ ,  $t \in \mathbb{R}$ ,*

$$(19.19) \quad \zeta(\sigma)^3 |\zeta(\sigma + it)|^4 |\zeta(\sigma + 2it)| \geq 1.$$

*Proof.* Equation (19.19) is equivalent to

$$(19.20) \quad 3 \log \zeta(\sigma) + 4 \operatorname{Re} \log \zeta(\sigma + it) + \operatorname{Re} \log \zeta(\sigma + 2it) \geq 0.$$

Then by (19.2),

$$(19.21) \quad \log \zeta(s) = \sum_{p \in \mathcal{P}} \sum_{k=1}^{\infty} \frac{1}{k} \frac{1}{p^{ks}},$$

where  $\mathcal{P}$  is the set of prime numbers in  $\mathbb{N}$ . Therefore,

$$(19.22) \quad \log \zeta(s) = \sum_{n \geq 1} \frac{a(n)}{n^s}, \quad a(n) \geq 0, \quad \text{for any } n.$$

Therefore, the left hand side of (19.20) is given by

$$(19.23) \quad \sum_{n=1}^{\infty} \frac{a(n)}{n^\sigma} \operatorname{Re}(3 + 4n^{-it} + n^{-2it}).$$

Taking  $\theta_n = t \log n$ ,

$$(19.24) \quad \operatorname{Re}(3 + 4n^{-it} + n^{-2it}) = 3 + 4 \cos \theta_n + \cos 2\theta_n = 2 + 4 \cos \theta_n + 2 \cos^2 \theta_n = 2(1 + \cos \theta_n)^2.$$

This term is  $\geq 0$  for each  $n$ , which implies the lemma.  $\square$

*Proof of Proposition 59:* Recall that since  $\zeta(s) - \frac{1}{s-1}$  is an entire function, if  $t \neq 0$  and  $\zeta(1+it) = 0$ , then

$$(19.25) \quad \lim_{\sigma \searrow 1} \frac{\zeta(\sigma + it)}{\sigma - 1} = \zeta'(1 + it).$$

Now then,

$$(19.26) \quad (19.19) = ((\sigma - 1)\zeta(\sigma))^3 \left( \frac{|\zeta(\sigma + it)|}{\sigma - 1} \right)^4 |(\zeta - 1)\zeta(\sigma + 2it)|.$$

However,

$$(19.27) \quad \lim_{\sigma \searrow 1} (\sigma - 1)\zeta(\sigma) = 1,$$

so if (19.25) holds, then (19.26) converges to zero as  $\sigma \searrow 1$ , which contradicts (19.19).  $\square$

The trivial zeros of  $\zeta(s)$  are the zeros at the points  $s \in \{-2, -4, \dots\}$ . It also follows from Proposition 59 that all the other zeros of  $\zeta(s)$  lie in the critical strip

$$(19.28) \quad \Omega = \{s \in \mathbb{C} : 0 < \operatorname{Re}(s) < 1\}.$$

The Riemann hypothesis is the famous conjecture that all zeros in  $\Omega$  of  $\zeta(s)$  lie on the critical line  $\{\frac{1}{2} + it : t \in \mathbb{R}\}$ .

Now then, we would like to prove a quantitative refinement of Proposition 59. To do this, it is useful to obtain a representation of  $\zeta(s)$  that is valid beyond  $\operatorname{Re}(s) > 1$ . Indeed, observe that for  $\operatorname{Re}(s) > 1$ ,

$$(19.29) \quad \zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} = \int_0^{\infty} x^{-s} d\nu(x),$$

where

$$(19.30) \quad \nu(x) = [x] = n, \quad \text{for } x \in [n, n+1).$$

The equation (19.29) is a Stieltjes integral. Then for any  $N \in \mathbb{N}$ ,

$$(19.31) \quad \begin{aligned} \zeta(s) &= \sum_{n=1}^N \frac{1}{n^s} + \sum_{n=N+1}^{\infty} \frac{1}{n^s} = \sum_{n=1}^N \frac{1}{n^s} + \int_N^{\infty} x^{-s} d\nu(x) = \sum_{n=1}^N \frac{1}{n^s} + \int_N^{\infty} x^{-s} dx \\ &= \sum_{n=1}^N \frac{1}{n^s} + s \int_N^{\infty} x^{-s-1} [x] dx - N^{-s} \nu(N). \end{aligned}$$

The last equality follows by integrating by parts. Now note that  $N^{-s}\nu(N) = N^{1-s}$ . Adding and subtracting

$$(19.32) \quad s \int_N^\infty x^{-s-1} x dx = \frac{s}{s-1} N^{1-s},$$

for  $\operatorname{Re}(s) > 1$ ,

$$(19.33) \quad \zeta(s) = \sum_{n=1}^N \frac{1}{n^s} - s \int_N^\infty \frac{x - [x]}{x^{s+1}} dx + \frac{N^{1-s}}{s-1}.$$

The integral on the right hand side is holomorphic on  $\operatorname{Re}(s) > 0$ , so

**Proposition 60.** *For each  $N \in \mathbb{N}$ , the identity (19.33) holds for  $\operatorname{Re}(s) > 0$ .*

*Proof.* Both sides are holomorphic on  $\{s \in \mathbb{C} : \operatorname{Re}(s) > 0\}$ , except for a pole at  $s = 1$ , and the identity holds for  $\operatorname{Re}(s) > 1$ .  $\square$

## 20. THE PRIME NUMBER THEOREM

**Proposition 61.** *There exists  $K < \infty$  such that*

$$(20.1) \quad |\zeta(s)| \leq K \log t, \quad |\zeta'(s)| \leq K(\log t)^2,$$

for  $s = \sigma + it$ ,  $\sigma \geq 1$ ,  $t \geq e$ .

*Proof.* If  $\operatorname{Re}(s) \geq 2$ , then it follows from

$$(20.2) \quad \zeta(s) = \sum_{n=1}^{\infty} n^{-s},$$

that  $|\zeta(s)| \leq \zeta(2)$  and  $|\zeta'(s)| \leq |\zeta'(2)|$ . Therefore, assume  $1 \leq \sigma \leq 2$  and  $t \geq e$ . In that case,

$$(20.3) \quad |s| \leq \sigma + t \leq 2t, \quad |s-1| \geq t, \quad \frac{1}{|s-1|} \leq \frac{1}{t}.$$

Then by (19.33),

$$(20.4) \quad |\zeta(s)| \leq \sum_{n=1}^N \frac{1}{n^\sigma} + 2t \int_N^\infty x^{-(\sigma+1)} dx + \frac{N^{1-\sigma}}{t} = \sum_{n=1}^N \frac{1}{n^\sigma} + \frac{2t}{\sigma N^\sigma} + \frac{N^{1-\sigma}}{t}.$$

Now take  $N = [t]$  so that  $N \leq t \leq N+1$  and  $\log(n) \leq \log(t)$  if  $n \leq N$ . Therefore,

$$(20.5) \quad \frac{2t}{\sigma N^\sigma} \leq 2 \frac{N+1}{N}, \quad \text{and} \quad \frac{N^{1-\sigma}}{t} = \frac{N}{t} \frac{1}{N^\sigma}.$$

Therefore,

$$(20.6) \quad |\zeta(s)| = O\left(\sum_{n=1}^N \frac{1}{n}\right) + O(1) = O(\log N) + O(1) = O(\log t).$$

To compute  $\zeta'(s)$  from (19.33),

$$(20.7) \quad \zeta'(s) = - \sum_{n=1}^N \frac{\log n}{n^s} + s \int_N^\infty \frac{x - [x]}{x^{s+1}} (\log x) dx - \int_N^\infty \frac{x - [x]}{x^{s+1}} dx - \frac{N^{1-s}}{s-1} (\log N) - \frac{N^{1-s}}{(s-1)^2}.$$

Applying the above argument to  $\zeta'(s)$  and picking up an extra factor of  $\log N = O(\log t)$  proves the proposition.  $\square$

**Proposition 62.** *There is a constant  $C < \infty$  such that*

$$(20.8) \quad \left| \frac{1}{\zeta(s)} \right| \leq C(\log t)^7,$$

for  $s = \sigma + it$ ,  $\sigma \geq 1$ ,  $t \geq e$ .

*Proof.* For  $\operatorname{Re}(s) > 1$ ,

$$(20.9) \quad \frac{1}{\zeta(s)} = \sum_{n=1}^{\infty} \frac{\mu(n)}{n^s}, \quad \mu(n) \in \{-1, 0, 1\}.$$

Indeed, from (19.2),

$$(20.10) \quad \frac{1}{\zeta(s)} = \prod_{j=1}^{\infty} \left(1 - \frac{1}{p_j^s}\right),$$

so by the prime number theorem, (20.9) holds, where  $\mu(n) = 0$  if  $n$  contains a square, otherwise  $\mu(n) = (-1)^m$ , where  $m$  is the number of prime factors of  $n$ . Therefore, for  $\operatorname{Re}(s) \geq 2$ ,

$$(20.11) \quad \left| \frac{1}{\zeta(s)} \right| \leq \sum_{n=1}^{\infty} \frac{1}{n^2} = \zeta(2).$$

Now suppose that  $\sigma \in [1, 2]$  and  $t \geq e$ . In this case, by (19.19), for  $\sigma > 1$ ,  $t \in \mathbb{R}$ ,

$$(20.12) \quad \frac{1}{|\zeta(\sigma + it)|} \leq \zeta(\sigma)^{3/4} |\zeta(\sigma + 2it)|^{1/4}.$$

Since  $(\sigma - 1)\zeta(\sigma)$  is bounded for  $\sigma \in [1, 2]$  and  $\zeta(\sigma + 2it) = O(\log t)$ ,

$$(20.13) \quad \frac{1}{|\zeta(\sigma + it)|} \leq C \frac{(\log t)^{1/4}}{(\sigma - 1)^{3/4}}.$$

Equivalently, for  $1 \leq \sigma \leq 2$  and  $t \geq e$ ,

$$(20.14) \quad |\zeta(\sigma + it)| \geq B \frac{(\sigma - 1)^{3/4}}{(\log t)^{1/4}}.$$

Now take  $\alpha \in (1, 2)$ . If  $1 \leq \sigma \leq \alpha$  and  $t \geq e$ , by Proposition 61,

$$(20.15) \quad |\zeta(\sigma + it) - \zeta(\alpha + it)| \leq \int_{\sigma}^{\alpha} |\zeta'(u + it)| du \leq K(\alpha - \sigma)(\log t)^2,$$

so by the triangle inequality,

$$(20.16) \quad |\zeta(\sigma + it)| \geq |\zeta(\alpha + it)| - |\zeta(\sigma + it) - \zeta(\alpha + it)| \geq B \frac{(\alpha - 1)^{3/4}}{(\log t)^{1/4}} - K(\alpha - 1)(\log t)^2.$$

This estimate also holds for  $\sigma \in [\alpha, 2]$  since  $(\sigma - 1)^{3/4} \geq (\alpha - 1)^{3/4}$ , so for any  $\alpha \in (1, 2)$ ,

$$(20.17) \quad |\zeta(\sigma + it)| \geq B \frac{(\alpha - 1)^{3/4}}{(\log t)^{1/4}} - K(\alpha - 1)(\log t)^2.$$

Choose

$$(20.18) \quad \alpha = 1 + \left(\frac{B}{2K}\right)^4 \frac{1}{(\log t)^9}.$$

Clearly,  $\alpha > 1$ . Also, there exists  $T_0 > e$  such that  $\alpha < 2$  if  $\alpha$  is given by (20.18). Plugging (20.18) into (20.16), for all  $t \geq T_0$ ,

$$(20.19) \quad |\zeta(\sigma + it)| \geq K(\alpha - 1)(\log t)^2 = \frac{C}{(\log t)^7}, \quad \forall \sigma \in [1, 2].$$

Therefore, (20.7) holds for all  $t \geq e$ , since  $\frac{1}{\zeta(s)}$  is holomorphic.  $\square$

**Corollary 10.** *There exists  $C < \infty$  such that*

$$(20.20) \quad \left| \frac{\zeta'(s)}{\zeta(s)} \right| \leq C(\log t)^9,$$

for  $s = \sigma + it$ ,  $\sigma \geq 1$ ,  $t \geq e$ .

Now take the prime counting function

$$(20.21) \quad \pi(x) = \#\{p \in \mathcal{P} : p \leq x\},$$

where  $\mathcal{P}$  is the set of positive integers that are primes.

**Theorem 21** (Prime number theorem). *The prime counting function  $\pi(x)$  has the property that*

$$(20.22) \quad \pi(x) \sim \frac{x}{\log(x)}, \quad \text{as } x \rightarrow \infty.$$

To prove this theorem, recall that for  $\operatorname{Re}(s) > 1$ ,

$$(20.23) \quad \log \zeta(s) = - \sum_{p \in \mathcal{P}} \log(1 - p^{-s}) = \sum_{p \in \mathcal{P}} \sum_{k=1}^{\infty} \frac{1}{k} p^{-ks} = \int_0^{\infty} x^{-s} dJ(x),$$

where

$$(20.24) \quad J(x) = \sum_{k \geq 1} \frac{1}{k} \pi(x^{1/k}).$$

For each  $x$ , the sum contains at most  $\log_2 x$  terms, so applying  $\frac{d}{ds}$  to (20.23),

$$(20.25) \quad -\frac{\zeta'(s)}{\zeta(s)} = \int_0^{\infty} (\log x) x^{-s} dJ(x).$$

Then introduce  $\psi(x)$  so that

$$(20.26) \quad d\psi(x) = (\log x) dJ(x).$$

Equivalently,

$$(20.27) \quad \psi(x) = \int_0^x (\log y) dJ(y) = (\log x) J(x) - \int_0^x \frac{J(y)}{y} dy.$$

Equivalently,  $dJ(x) = (\frac{1}{\log x}) d\psi(x)$ , so

$$(20.28) \quad J(x) = \int_0^x \frac{1}{\log x} d\psi(x) = \frac{\psi(x)}{\log x} + \int_0^x \frac{1}{(\log y)^2} \frac{\psi(y)}{y} dy.$$

Also by (20.26), integrating by parts,

$$(20.29) \quad -\frac{\zeta'(s)}{\zeta(s)} = \int_1^{\infty} x^{-s} d\psi(x) = s \int_1^{\infty} \psi(x) x^{-s-1} dx.$$

**Proposition 63.** *If one shows that  $\psi(x) \sim x$  as  $x \rightarrow \infty$ , it follows that  $J(x) \sim \frac{x}{\log x}$  as  $x \rightarrow \infty$ . This in turn implies the prime number theorem.*

*Proof.* Suppose  $\psi(x) = x + E(x)$ , where  $E(x) = o(x)$  as  $x \rightarrow \infty$ . Then by (20.28),

$$(20.30) \quad J(x) = \frac{x}{\log x} + \frac{E(x)}{\log x} + \int_2^x \frac{dy}{(\log y)^2} + \int_2^\infty \frac{1}{(\log y)^2} \frac{E(y)}{y} dy \sim \frac{x}{\log x}.$$

To deduce the prime number theorem, observe that (20.24) implies

$$(20.31) \quad \pi(x) < J(x) < \pi(x) + \pi(x^{1/2}) \sum_{1 \leq k \leq \log_2(x)} \frac{1}{k} \leq \pi(x) + \pi(x^{1/2}) \log(\log_2 x).$$

Since  $\pi(x^{1/2}) \leq x^{1/2}$ ,

$$(20.32) \quad J(x) - \pi(x) = O(x^{1/2} \log(\log_2 x)).$$

□

Now bring in the function

$$(20.33) \quad \psi_1(x) = \int_0^x \psi_1(y) dy.$$

**Proposition 64.** *Let  $\psi : [0, \infty) \rightarrow [0, \infty)$  be an increasing function and set  $\psi_1(x) = \int_0^x \psi(y) dy$ . Taking  $B \in (0, \infty)$ ,  $\alpha \in [1, \infty)$ , then*

$$(20.34) \quad \psi_1(x) \sim Bx^\alpha, \quad \text{as } x \rightarrow \infty, \quad \Rightarrow \quad \psi(x) \sim \alpha Bx^{\alpha-1}, \quad \text{as } x \rightarrow \infty.$$

Therefore, to complete the proof of the prime number theorem, it suffices to show that

**Proposition 65.** *The function  $\psi_1$  satisfies*

$$(20.35) \quad \lim_{x \rightarrow \infty} \frac{\psi_1(x)}{x^2} = \frac{1}{2}.$$

*Proof.* Make the crude estimate  $\psi_1(x) \leq 2x^2 \log(x)$ . Therefore,

$$(20.36) \quad -\frac{\zeta'(s)}{\zeta(s)} = s \int_1^\infty \psi_1'(x) x^{-s-1} dx = s(s+1) \int_1^\infty \psi_1(x) x^{-s-2} dx.$$

Now then, for  $\operatorname{Re}(s) > 1$ ,

$$(20.37) \quad \int_1^\infty x^{-s} dx = \frac{1}{s-1}.$$

Therefore,

$$(20.38) \quad \frac{1}{2} \int_1^\infty \left(1 - \frac{1}{x}\right)^2 x^{-s} dx = \frac{1}{(s-1)s(s+1)}.$$

Therefore,

$$(20.39) \quad -\frac{1}{s(s+1)} \left( \frac{\zeta'(s)}{\zeta(s)} + \frac{1}{s-1} \right) = \int_1^\infty \left[ \frac{\psi_1(x)}{x^2} - \frac{1}{2} \left(1 - \frac{1}{x}\right)^2 \right] x^{-s} dx,$$

is holomorphic in  $\operatorname{Re}(s) > 0$ .

Now set

$$(20.40) \quad \Phi(x) = \left[ \frac{\psi_1(x)}{x^2} - \frac{1}{2} \left(1 - \frac{1}{x}\right)^2 \right] \chi_{[1, \infty)}(x).$$

Making a change of variables, let  $x = e^y$  and set  $s = 1 + \sigma + it$ ,  $\sigma > 0$ , the right hand side of (20.39) is given by

$$(20.41) \quad \int_{-\infty}^{\infty} \Phi(e^y) e^{-\sigma y} e^{-ity} dy, \quad \sigma > 0.$$

Since  $|\Phi(x)| \leq C \log(2 + |x|)$ ,

$$(20.42) \quad |\Phi(e^y)| \leq B(1 + |y|)^C.$$

Also, since  $\Phi(e^y) = 0$  for  $y \leq 0$ ,  $\Phi(e^y)e^{-\sigma y}$  is integrable for each  $\sigma > 0$ . For  $\sigma > 0$ , rewrite the left hand side as

$$(20.43) \quad \Psi_\sigma(t) = -\frac{1}{s(s+1)} \left( \frac{\zeta'(s)}{\zeta(s)} + \frac{1}{s-1} \right) \Big|_{s=1+\sigma+it}.$$

**Lemma 18.** *There exists  $C < \infty$  such that for  $\sigma \geq 0$ ,  $t \in \mathbb{R}$ ,*

$$(20.44) \quad |\Psi_\sigma(t)| \leq \frac{C}{1+t^2} (\log(2 + |t|))^9.$$

Therefore,  $|\Psi_\sigma|$  is uniformly bounded by a function that is integrable on  $\mathbb{R}$ . Moreover, (20.39) gives

$$(20.45) \quad \Psi_\sigma(t) = \int_{-\infty}^{\infty} \Phi(e^y) e^{-\sigma y} e^{-ity} dy, \quad \forall \sigma > 0,$$

with the integrand supported on  $\mathbb{R}^+$ . Since  $\Psi$  is integrable in  $t$ , applying the Fourier inversion formula,

$$(20.46) \quad \Phi(e^y) e^{-\sigma y} = \frac{1}{2\pi} \int_{-\infty}^{\infty} \Psi_\sigma(t) e^{ity} dt.$$

Passing to the limit  $\sigma \searrow 0$ ,

$$(20.47) \quad \Phi(e^y) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \Psi_0(t) e^{ity} dt.$$

By the Riemann–Lebesgue lemma,  $\Phi(e^y) \rightarrow 0$  as  $y \rightarrow \infty$ , which is equivalent to  $\Phi(x) \rightarrow 0$  as  $x \rightarrow \infty$ . Then by (20.40), this proves the proposition.  $\square$

## 21. NORMAL FAMILIES

Suppose  $\Omega$  is a locally compact metric space, that is,  $\Omega$  is a metric space and each  $p \in \Omega$  has a closed neighborhood that is compact. Suppose  $S$  is a complete metric space and  $C(\Omega, S)$  is the set of continuous maps  $f : \Omega \rightarrow S$ .

**Definition 22** (Normal family). *A subset  $\mathcal{F} \subset C(\Omega, S)$  is a normal family with respect to  $(\Omega, S)$  if and only if every sequence  $f_\nu \in \mathcal{F}$  has a locally uniformly convergent subsequence  $f_{\nu_k} \rightarrow f \in C(\Omega, S)$ .*

**Proposition 66** (Arzela-Ascoli theorem). *Let  $X$  and  $Y$  be compact metric spaces and fix a modulus of continuity  $\omega(\delta)$ . Then*

$$(21.1) \quad C_\omega = \{f \in C(X, Y) : d(f(x), f(y)) \leq \omega(d(x, y)), \quad \forall x, y \in X\},$$

*is a compact subset of  $C(X, Y)$ , hence a normal family.*

The defining condition

$$(21.2) \quad d(f(x), f(y)) \leq \omega(d(x, y)), \quad \forall x, y \in X, \quad f \in \mathcal{F},$$

is called equicontinuity of  $\mathcal{F}$ .

**Proposition 67.** *Assume there exists a countable family  $\{K_j\}$  of compact subsets of  $\Omega$  such that any compact  $K \subset \Omega$  is contained in some finite union of  $K_j$ . Consider a family  $\mathcal{F} \subset C(\Omega, S)$ . Suppose that for each  $j$ , there exist compact  $L_j \subset S$  such that  $f : K_j \rightarrow L_j$  for all  $f \in \mathcal{F}$  and that  $\{f|_{K_j} : f \in \mathcal{F}\}$  is equicontinuous. Then  $\mathcal{F}$  is a normal family.*

*Proof.* Let  $f_\nu$  be a sequence in  $\mathcal{F}$ . By Proposition 66, there is a uniformly convergent subsequence  $f_{\nu_k} : K_1 \rightarrow L_1$ . This has a further subsequence converging uniformly on  $K_2$ , and so on, making a diagonal argument.  $\square$

**Proposition 68** (Montel's little theorem). *Let  $\Omega \subset \mathbb{C}$  be open. A family  $\mathcal{F}$  of holomorphic functions  $f_\alpha : \Omega \rightarrow \mathbb{C}$  is normal with respect to  $(\Omega, \mathbb{C})$  if and only if this family is uniformly bounded on each compact subset of  $\Omega$ .*

*Proof.* Write  $\Omega = \cup_j \bar{D}_j$  for a countable family of closed disks  $\bar{D}_j \subset \Omega$  and satisfy the hypotheses of Proposition 67. Suppose  $\text{dist}(z, \partial\Omega) \geq 2\epsilon_j > 0$  for all  $z \in \bar{D}_j$ . Let the disk  $\tilde{D}_j$  be concentric with  $\bar{D}_j$  and have radius  $\epsilon_j$  greater. The local uniform bounds imply

$$(21.3) \quad |f_\alpha| \leq A_j \quad \text{on} \quad \tilde{D}_j, \quad \forall f_\alpha \in \mathcal{F}.$$

Therefore, by Cauchy's estimate,

$$(21.4) \quad |f'_\alpha| \leq \frac{A_j}{\epsilon_j}, \quad \text{on} \quad \bar{D}_j, \quad \forall f_\alpha \in \mathcal{F}.$$

Therefore,

$$(21.5) \quad |f_\alpha(z) - f_\alpha(w)| \leq \frac{A_j}{\epsilon_j} |z - w|, \quad \forall z, w \in \bar{D}_j, \quad f_\alpha \in \mathcal{F}.$$

The reverse implication is straightforward.  $\square$

It is also possible to prove an analogue of Proposition 67 for meromorphic functions.

**Theorem 22.** *Let  $\mathcal{F}$  be a family of meromorphic functions in  $\Omega$  such that for each  $z_0 \in \Omega$ , there is a neighborhood  $U(z_0) \subset \Omega$  and a constant  $M(z_0)$  such that either*

$$(21.6) \quad |f(z)| < M(z_0), \quad \text{or} \quad \frac{1}{|f(z)|} < M(z_0), \quad \text{for all} \quad z \in U(z_0), f \in \mathcal{F}.$$

*Then  $\mathcal{F}$  is normal.*

*Proof.* If  $f(z)$  satisfies  $\frac{1}{|f(z)|} < M(z_0)$  in some neighborhood  $U(z_0)$ , then by Propositions 67 and 68,  $\frac{1}{f_n(z)}$  has a convergent subsequence that converges to  $g \in \mathcal{H}(\Omega)$ . If  $g \equiv 0$ , then  $f(z)$  is a function that is identically  $\infty$ . Otherwise,  $f$  is a meromorphic function.  $\square$

## 22. THE HYPERBOLIC PLANE AND THE POINCARÉ DISK

Möbius transformations give an important connection to geometry. Consider the half-plane model of the hyperbolic plane given by

$$(22.1) \quad \mathbb{H} = \{z \in \mathbb{C} : \text{Im}(z) > 0\}, \quad ds^2 = \frac{dx^2 + dy^2}{y^2} = \frac{d\bar{z}dz}{(\text{Im}(z))^2}.$$

The subgroup of  $PSL(2, \mathbb{C})$  that preserves the upper half plane is precisely  $PSL(2, \mathbb{R})$ . Indeed,

$$(22.2) \quad z \mapsto \frac{az + b}{cz + d} \quad \text{preserves} \quad \mathbb{R}_\infty = \mathbb{R} \cup \{\infty\},$$

if and only if  $a, b, c, d \in \lambda\mathbb{R}$  for some  $\lambda \in \mathbb{C}^*$ .

The group  $PSL(2, \mathbb{R})$  preserves the upper half-plane, whereas matrices with negative determinant interchange the upper and lower half plane. The group  $PSL(2, \mathbb{R})$  operates transitively on  $\mathbb{H}$  and preserves the metric.

$$(22.3) \quad w = \frac{az + b}{cz + d} \Rightarrow \frac{d\bar{w}dw}{(\text{Im}(w))^2} = \frac{d\bar{z}dz}{(\text{Im}(z))^2}.$$

Since the metric does not depend on  $x$ , the vertical lines are geodesics. Now consider the stabilizer of  $i$ , the Möbius transformations that fix  $i$ . Thus,

$$(22.4) \quad \frac{ai + b}{ci + d} = i.$$

Then  $a = d$  and  $b = -c$ . Since  $a^2 + b^2 = 1$ ,  $a + bi = e^{i\theta}$ . But then

$$(22.5) \quad T'(\theta) = \frac{1}{(ci + d)^2} = e^{2i\theta},$$

acts as a rotation in the tangent space of  $\mathbb{H}$  at  $z_0 = i$ . This property carries over to other  $z_0 \in \mathbb{H}$ . Since isometries preserve geodesics, the latter are precisely all circles that intersect the real line at a right angle, with vertical lines counting as circles of infinite radius.

**Theorem 23.** *Let  $T$  be a geodesic triangle with angles  $\alpha_1, \alpha_2, \alpha_3$ . Then  $\text{Area}(T) = \pi - (\alpha_1 + \alpha_2 + \alpha_3)$ .*

**Remark 14.** *This is a special case of the Gauss–Bonnet theorem.*

*Proof.* Let us start with a geodesic triangle with exactly two vertices belonging to  $\mathbb{R}_\infty$ . Suppose without loss of generality that one vertex coincides with 1 and the projection of the second finite vertex onto the real axis is  $x_0$ . Then,

$$(22.6) \quad \text{Area}(T) = \int_{x_0}^1 \int_{y(x)}^\infty \frac{dx dy}{y^2} = \int_{x_0}^1 \frac{dx}{\sqrt{1-x^2}} = \int_{\alpha_0}^0 \frac{d \cos(\phi)}{\sqrt{1-\cos^2(\phi)}} = \alpha_0 = \pi - \alpha_1.$$

□

## 23. THE RIEMANN MAPPING THEOREM

The Riemann mapping theorem proves that any simply connected domain is conformally equivalent to the disk. Since the disk is simply connected and simple connectivity is homeomorphism invariant, it is clear that imposing this condition is necessary. Furthermore, by Liouville's theorem, there can be no isomorphism from the disk onto  $\mathbb{C}$ . Thus, the Riemann mapping theorem shows that these well-known restrictions are the only restrictions.

**Proposition 69.** *If  $\Omega \subset \mathbb{C}$  is simply connected and  $\Omega \neq \mathbb{C}$ , then there is a holomorphic diffeomorphism  $f : \Omega \rightarrow \mathcal{O}$ , where  $\mathcal{O} \subset \mathbb{C}$  is a bounded, simply connected domain.*

*Proof.* Choose  $p \in \mathbb{C} \setminus \Omega$  and define a holomorphic branch on  $\Omega$  of

$$(23.1) \quad g(z) = (z - p)^{1/2}.$$

Now then,  $g$  is one-to-one on  $\Omega$  and maps  $\Omega$  diffeomorphically to a simply connected region  $\tilde{\Omega}$  that has the property that  $z \in \tilde{\Omega}$  implies that  $-z \notin \tilde{\Omega}$ . Therefore, there is a disk  $D \subset \mathbb{C} \setminus \tilde{\Omega}$ . Composing with inversion across  $\partial D$  gives the desired holomorphic diffeomorphism.  $\square$

**Theorem 24.** *Assume  $\Omega \subset \mathbb{C}$  is a simply connected domain, and that  $\Omega \neq \mathbb{C}$ . Then there exists a holomorphic diffeomorphism*

$$(23.2) \quad f : \Omega \rightarrow D$$

of  $\Omega$  onto the unit disk  $D = \{z \in \mathbb{C} : |z| < 1\}$ .

Recall that from Proposition 69, the domain  $\Omega$  is conformally equivalent to a bounded domain in  $\mathbb{C}$ , so suppose without loss of generality that  $\Omega$  is bounded. Fix  $p \in \Omega$ .

**Definition 23.** *Let  $\mathcal{F}$  be the set of holomorphic maps  $g : \Omega \rightarrow D$  that have the following three properties:*

- (1)  $g$  is one-to-one, univalent.
- (2)  $g(p) = 0$ ,
- (3)  $g'(p) > 0$ .

Observe that  $g$  being one-to-one implies that  $g'$  is nowhere zero on  $\Omega$ . For  $\Omega \subset \mathbb{C}$  bounded,  $b(z - p)$  belongs to  $\mathcal{F}$  for  $b > 0$  sufficiently small, so  $\mathcal{F}$  is nonempty.

If  $R = \text{dist}(p, \partial\Omega)$ , then the Cauchy integral formula implies

$$(23.3) \quad |g'(p)| \leq \frac{1}{R}, \quad \forall g \in \mathcal{F},$$

so set

$$(23.4) \quad A = \sup\{g'(p) : g \in \mathcal{F}\},$$

and  $A < \infty$ . Now choose  $g_\nu \in \mathcal{F}$  such that  $g'_\nu(p) \rightarrow A$  as  $\nu \rightarrow \infty$ . Since  $\mathcal{F}$  is a normal family, there exists a subsequence  $g_\nu \rightarrow f$  locally uniformly on  $\Omega$ , and

$$(23.5) \quad f : \Omega \rightarrow D,$$

is holomorphic and satisfies

$$(23.6) \quad f(p) = 0, \quad f'(p) = A.$$

**Lemma 19.** *The function  $f$  in (23.5) is one-to-one.*

*Proof.* Suppose there exist distinct  $z_1, z_2 \in \Omega$  such that  $f(z_1) = f(z_2) = w \in D$  and let  $\bar{U} \subset \Omega$  be a smoothly bounded region such that  $z_1, z_2 \in U$  and  $f(\gamma)$  is disjoint from  $w$ , where  $\gamma = \partial U$ .

By the argument principle,  $f(\gamma)$  winds twice around  $w$ . However, each  $g_\nu(\gamma)$  winds only once around  $w$ , so since  $g_\nu \rightarrow f$  uniformly on  $\gamma$ , this is a contradiction.  $\square$

**Lemma 20.** *The function  $f$  in (23.5) is onto.*

*Proof.* Let  $\mathcal{O} = f(\Omega)$  and suppose that  $\mathcal{O}$  is not all of  $D$ . In other words, for some  $q \in D$ ,  $\mathcal{O} \subset D$  is open and simply connected,  $0 \in \mathcal{O}$ ,  $q \in D$ , but  $q \notin \mathcal{O}$ .

Now define the Koebe transformation

$$(23.7) \quad K = K_{\mathcal{O},q} : \mathcal{O} \rightarrow D.$$

First define  $F : \mathcal{O} \rightarrow D$  by

$$(23.8) \quad F(z) = \sqrt{\frac{z-q}{1-\bar{q}z}} = \sqrt{\varphi_q(z)}.$$

For any  $|b| < 1$ ,

$$(23.9) \quad \varphi_b(0) = \frac{-b}{1} = -b, \quad \frac{e^{i\theta} - b}{1 - \bar{b}e^{i\theta}} = e^{-i\theta} \frac{e^{i\theta} - b}{e^{-i\theta} - \bar{b}},$$

so therefore  $\varphi_b : D \rightarrow D$ . Also note that  $\varphi_b(b) = 0$ . Since  $\varphi_b$  is a fractional linear transformation,  $\varphi_b$  is one-to-one and onto.

Furthermore,  $\varphi_q(z) \neq 0$  for  $z \in \mathcal{O}$ , so since  $\mathcal{O}$  is simply connected, there is a holomorphic function  $F$  on  $\mathcal{O}$  that satisfies (23.8), once one specifies  $F(0) = (-q)^{1/2}$ .

Now define  $K$  in (23.7) by

$$(23.10) \quad K(z) = \frac{|F'(0)|}{F'(0)} \varphi_{F(0)}(F(z)).$$

Observe that  $\varphi_b(b) = 0$  for any  $b \in D$ , so  $K(0) = 0$ .

**Proposition 70.** *The Koebe transformation is one-to-one and satisfies  $K(0) = 0$  and  $K'(0) > 1$ .*

Given Proposition 70, we have the following proof of Lemma 20. If  $f$  is not onto in  $D$ , then  $\mathcal{O} = f(\Omega) \subset D$  satisfies  $\mathcal{O} \subset D$  is open and simply connected,  $0 \in \mathcal{O}$ , and  $q \in D$  but  $q \notin \mathcal{O}$  for some  $q \in D$ . Then

$$(23.11) \quad g = K \circ f : \Omega \rightarrow D,$$

is one-to-one,  $g(p) = 0$ , since  $f(p) = 0$  and  $K(0) = 0$ , and by the chain rule,

$$(23.12) \quad g'(p) = K'(0)f'(p) = K'(0)A > A.$$

This contradicts (23.4), which implies Lemma 20.  $\square$

*Proof of Proposition 70.* It is straightforward that  $K$  is one-to-one. Next,

$$(23.13) \quad K'(z) = \frac{|F'(0)|}{F'(0)} \varphi'_{F(0)}(F(z))F'(z),$$

and therefore,

$$(23.14) \quad K'(0) = |F'(0)|\varphi'_{F(0)}(F(0)).$$

Therefore, for  $b \in D$ ,

$$(23.15) \quad \varphi'_b(z) = \frac{1 - |b|^2}{(1 - \bar{b}z)^2}, \quad \text{so} \quad \varphi'_b(b) = \frac{1}{1 - |b|^2}, \quad \varphi'_b(0) = 1 - |b|^2,$$

so

$$(23.16) \quad K'(0) = \frac{|F'(0)|}{1 - |F(0)|^2}.$$

Next, by (23.8),

$$(23.17) \quad F'(z) = \frac{1}{2} \varphi_q(z)^{-1/2} \varphi'_q(z) = \frac{\varphi'_q(z)}{2F(z)},$$

so

$$(23.18) \quad F'(0) = \frac{1 - |q|^2}{2F(0)} \Rightarrow |F'(0)| = \frac{1 - |q|^2}{2|q|^{1/2}},$$

since  $|F(0)| = |q|^{1/2}$ . Therefore,

$$(23.19) \quad K'(0) = \frac{1 - |q|^2}{2|q|^{1/2}} \cdot \frac{1}{1 - |q|} = \frac{1 + |q|}{2|q|^{1/2}}.$$

Therefore,

$$(23.20) \quad K'(0) = 1 + \frac{(1 - |q|^{1/2})^2}{2|q|^{1/2}} > 1.$$

□

**Remark 15.** *Read the remarks at the end of section 5.4 of [Tay19].*

Let the  $f$  in the above theorem be denoted a Riemann map. This Riemann map is unique once we require  $f(z_0) = 0$  and  $f'(z_0) > 0$  for some fixed  $z_0 \in \Omega$ . Now consider the boundary behavior of a Riemann map.

**Definition 24.** *We say that  $z_0 \in \partial\Omega$  is regular provided there exists  $r_0(z_0)$  such that, for all  $0 < r < r_0(z_0)$ ,*

$$(23.21) \quad \Omega \cap \{z \in \mathbb{C} : |z - z_0| = r\} = \{z_0 + re^{i\theta} : \theta_1(r) < \theta < \theta_2(r)\},$$

for some  $\theta_1(r) < \theta_2(r)$ , which are continuous in  $r$ . That is,  $\partial D(z_0, r)$  is an arc for  $r > 0$  small. We say that  $\Omega$  is regular provided all points of  $\partial\Omega$  are regular.

**Theorem 25.** *Suppose  $\Omega$  is bounded, simply-connected, and regular. Then any conformal homeomorphism as in Theorem 24 extends to a homeomorphism  $\bar{\Omega} \rightarrow \bar{\mathbb{D}}$ .*

*Proof.* Let  $f : \Omega \rightarrow \mathbb{D}$  be a Riemann map.

**Claim 3.**  $\lim_{z \rightarrow z_0} f(z)$  exists for all  $z_0 \in \partial\Omega$ , the limit being taken from within  $\Omega$ .

*Proof of Claim.* If this fails for some  $z_0 \in \partial\Omega$ , there exist sequences  $\{z_n\}_{n=1}^{\infty}$  and  $\{\zeta_n\}_{n=1}^{\infty}$  in  $\Omega$  converging to  $z_0$  and such that

$$(23.22) \quad f(z_n) \rightarrow w_1, \quad f(\zeta_n) \rightarrow w_2, \quad w_1 \neq w_2 \in \partial\mathbb{D}.$$

Let  $\gamma_1$  be a continuous curve that connects the points  $\{f(z_n)\}_{n=1}^\infty$  and let  $\gamma_2$  be a continuous curve that connects the points  $\{f(\zeta_n)\}_{n=1}^\infty$ . Now let  $\eta_j = f^{-1} \circ \gamma_j$ . The  $\eta_j$  are continuous curves both converging to  $z_0$ . Let

$$(23.23) \quad z_r \in \partial D(z_0, r) \cap \eta_1, \quad \zeta_r \in \partial D(z_0, r) \cap \eta_2.$$

By regularity of  $z_0$ , there exists an arc  $c_r \subset \Omega \cap \partial D(z_0, r)$  with

$$(23.24) \quad f(z_r) - f(\zeta_r) = \int_{c_r} f'(z) dz.$$

Thus,

$$(23.25) \quad |f(z_r) - f(\zeta_r)|^2 \leq \left| \int_{c_r} f'(z) dz \right|^2 \leq \left( \int_{\theta_1(r)}^{\theta_2(r)} |f'(re^{i\theta})| r d\theta \right)^2 \leq 2\pi r \int_{\theta_1(r)}^{\theta_2(r)} |f'(re^{i\theta})|^2 r d\theta.$$

Dividing by  $r$  and integrating over  $0 < r < r_0(z_0)$ ,

$$(23.26) \quad \int_0^{r_0(z_0)} |f(z_r) - f(\zeta_r)|^2 \frac{dr}{r} \leq 2\pi \iint_{\Omega} |f'(z)|^2 dx dy = 2\pi \text{Area}(\mathbb{D}) < \infty.$$

This contradicts the fact that  $f(z_r) \rightarrow w_1$  and  $f(\zeta_r) \rightarrow w_2$ , where  $w_1 \neq w_2$ . Therefore,  $\lim_{z \rightarrow z_0} f(z)$  does exist.  $\square$

Letting  $F(z_0) = \lim_{z \rightarrow z_0} f(z)$ ,  $F : \bar{\Omega} \rightarrow \bar{\mathbb{D}}$  defines a continuous extension of  $f$ . To see why  $F$  is continuous, if it were not, there would have to exist some  $z_0 \in \partial\Omega$  and a sequence  $z_n \in \partial\Omega$  so that  $F(z_n)$  does not converge to  $F(z_0)$  as  $n \rightarrow \infty$ . However, this would imply that there exist  $z'_n \in \Omega$ ,  $z'_n \rightarrow z_0$ , but  $f(z'_n)$  does not converge to  $F(z_0)$ . However, this contradicts the fact that  $\lim_{z \rightarrow z_0} f(z)$  exists.

Finally, since  $\mathbb{D}$  is regular, and any sequence  $z_n \in \Omega$  converging to  $z_0 \in \partial\Omega$  can be connected by a continuous curve inside  $\Omega$  (using the continuity of  $\theta_1(r)$  and  $\theta_2(r)$ ), we can apply the same argument to  $f^{-1}$ . Therefore,  $f^{-1}$  extends to a continuous map  $G : \bar{\mathbb{D}} \rightarrow \bar{\Omega}$ . Since  $F \circ G = id_{\bar{\mathbb{D}}}$  and  $G \circ F = id_{\bar{\Omega}}$ , the proof is complete.  $\square$

The same statement applies to unbounded  $\Omega$ . In that case, we can regard  $\Omega$  as a region in  $\mathbb{C}_\infty$ , and call  $\infty$  regular if 0 is regular for  $\Omega^{-1} = \{z^{-1} : z \in \Omega\}$ .

**Lemma 21.** *Any region  $\Omega \subset \mathbb{C}$  so that  $\bar{\Omega}$  is a  $C^1$ -manifold with boundary and corners is regular. This means that for every  $z_0 \in \partial\Omega$  there exists a  $C^1$ -diffeomorphism  $\phi$  of a neighborhood  $U$  of  $z_0$  onto a disk  $D(0, r_1)$  for some  $r_1 = r_1(z_0) > 0$  and such that*

$$(23.27) \quad \phi(\Omega \cap U) = \{re^{i\theta} : 0 < r < r_1, 0 < \theta < \theta_1 < 2\pi\}.$$

One example of such regions are polygons. In that case the conformal maps are given by the Schwarz–Christoffel formulas. For example, the map

$$(23.28) \quad z \mapsto \frac{d\zeta}{\zeta^{1/2}(\zeta-1)^{1/2}} d\zeta, \quad \text{Im}(z) > 0,$$

takes the upper half plane onto the half-strip

$$(23.29) \quad \{z \in \mathbb{C} : \text{Re}(z) \geq 0, 0 < \text{Im}(z) < \pi\},$$

where 0 and 1 are mapped to the two finite vertices of the half-strip. Similarly,

$$(23.30) \quad z \mapsto \int_0^z \frac{d\zeta}{\zeta^{1/2}(\zeta-1)^{1/2}(\zeta-2)^{1/2}} d\zeta, \quad \text{Im}(z) > 0,$$

maps the upper half plane onto a rectangle with 0, 1, and 2 mapped onto three of the vertices and the fourth vertex being the image of  $\infty$ . The square roots  $(z - a)^{1/2}$  here are defined to be positive when  $z > a$  and to take the upper half plane to itself.

#### 24. INFINITE PRODUCTS

In addition to represented holomorphic functions as an infinite sum of functions, it is also useful to represent a function as an infinite product of functions.

First, consider the product of numbers

$$(24.1) \quad \prod_{k=1}^{\infty} (1 + a_k).$$

Disregarding the case when  $a_k = -1$  for some  $k$ , convergence of  $\prod_{k=1}^M (1 + a_k)$  as  $M \rightarrow \infty$  amounts to the convergence of

$$(24.2) \quad \lim_{M \rightarrow \infty} \prod_{k=M}^N (1 + a_k) = 1, \quad \text{uniformly in } N > M.$$

In particular, we require  $a_k \rightarrow 0$  as  $k \rightarrow \infty$ .

Writing out the product

$$(24.3) \quad \prod_{k=M}^N (1 + a_k) = 1 + \sum_{j=M}^N a_j + \sum_{M \leq j_1 < j_2 \leq N} a_{j_1} a_{j_2} + \dots + a_M \cdots a_N,$$

$$(24.4) \quad \left| \prod_{k=M}^N (1 + a_k) - 1 \right| \leq \prod_{k=M}^N (1 + |a_k|) - 1.$$

Now then,

$$(24.5) \quad \log \prod_{k=M}^N (1 + |a_k|) = \sum_{k=M}^N \log(1 + |a_k|).$$

Since  $x \geq 0$  implies  $\log(1 + x) \leq x$ , and  $0 \leq x \leq 1$  implies  $\log(1 + x) \geq \frac{x}{2}$ ,

$$(24.6) \quad \frac{1}{2} \sum_{k=M}^N |a_k| \leq \log \prod_{k=M}^N (1 + |a_k|) \leq \sum_{k=M}^N |a_k|.$$

Therefore,  $\lim_{M \rightarrow \infty} \prod_{k=M}^N (1 + |a_k|) = 1$  uniformly for  $N > M$ , if and only if  $\sum_{k=M}^N |a_k| \rightarrow 0$  uniformly in  $M \rightarrow \infty$ .

Another consequence is the following,

$$(24.7) \quad \text{If } 1 + a_k \neq 0, \quad \text{for all } k, \quad \text{then } \sum_k |a_k| < \infty, \Rightarrow \prod_{k=1}^{\infty} (1 + a_k) \neq 0.$$

Now replace the sequence  $(a_k)$  of complex numbers by a sequence  $(f_k)$  of holomorphic functions.

**Proposition 71.** *Let  $f_k : \Omega \rightarrow \mathbb{C}$  be holomorphic. Assume that for each compact set  $K \subset \Omega$  there exist  $M_k(K)$  such that*

$$(24.8) \quad \sup_{z \in K} |f_k(z)| \leq M_k(K), \quad \text{and} \quad \sum_k M_k(K) < \infty.$$

Then we have a convergent infinite product

$$(24.9) \quad \prod_{k=1}^{\infty} (1 + f_k(z)) = F(z).$$

In fact,

$$(24.10) \quad \prod_{k=1}^n (1 + f_k(z)) \rightarrow F(z), \quad \text{as } n \rightarrow \infty,$$

uniformly on compact subsets of  $\Omega$ . Therefore,  $F$  is holomorphic on  $\Omega$ . If  $z_0 \in \Omega$  and  $1 + f_k(z_0) \neq 0$  for all  $k$ , then  $F(z_0) \neq 0$ .

Now assume that  $f_k, g_k : \Omega \rightarrow \mathbb{C}$  are holomorphic, and assume in addition that  $\sup_K |g_k| \leq M_k(K)$ . Then one has the convergent infinite product

$$(24.11) \quad \prod_{k=1}^{\infty} (1 + g_k(z)) = G(z),$$

with  $G$  holomorphic on  $\Omega$ . Now then,

$$(24.12) \quad (1 + f_k(z))(1 + g_k(z)) = 1 + f_k(z) + g_k(z) + f_k(z)g_k(z),$$

so

$$(24.13) \quad |f_k(z) + g_k(z) + f_k(z)g_k(z)| \leq 2M_k(K) + M_k(K)^2.$$

Therefore,

$$(24.14) \quad \prod_{k=1}^{\infty} (1 + h_k(z)) = H(z),$$

is a convergent infinite product, with  $H(z)$  holomorphic on  $\Omega$ . Moreover, for any  $n$ ,

$$(24.15) \quad \prod_{k=1}^n (1 + f_k(z))(1 + g_k(z)) = \prod_{k=1}^n (1 + f_k(z)) \cdot \prod_{k=1}^n (1 + g_k(z)),$$

so therefore,

$$(24.16) \quad F(z)G(z) = H(z).$$

Consider the infinite product

$$(24.17) \quad S(z) = z \prod_{k=1}^{\infty} \left(1 - \frac{z^2}{k^2}\right).$$

If  $K$  is contained in the set  $\{z : |z| \leq R\}$ ,  $M_k(K) \leq \frac{R^2}{k^2}$ , so  $S(z)$  is holomorphic on all of  $\mathbb{C}$ . Furthermore,  $S(z) = 0$  if and only if  $z \in \mathbb{Z}$ . Also, all zeros of  $S(z)$  are simple.

A familiar function which has the same zeros as  $S(z)$  is  $\sin(\pi z)$ . Since

$$(24.18) \quad \lim_{z \rightarrow 0} \frac{1}{z} S(z) = 1,$$

compare  $S(z)$  to

$$(24.19) \quad s(z) = \frac{1}{\pi} \sin(\pi z).$$

**Lemma 22.** For  $S(z)$  as in (24.17),

$$(24.20) \quad S(z-1) = -S(z).$$

*Proof.* Since  $S(z) = \lim_{n \rightarrow \infty} S_n(z)$ , where

$$(24.21) \quad \begin{aligned} S_n(z) &= z \prod_{k=1}^n \left(1 - \frac{z^2}{k^2}\right) = z \prod_{k=1}^n \left(1 - \frac{z}{k}\right) \left(1 + \frac{z}{k}\right) \\ &= z \prod_{k=1}^n \left(\frac{k-z}{k}\right) \cdot \left(\frac{k+z}{k}\right) = \frac{(-1)^n}{(n!)^2} (z-n)(z-n+1) \cdots (z+n-1)(z+n). \end{aligned}$$

Plugging in  $z-1$ ,

$$(24.22) \quad S_n(z-1) = \frac{(-1)^n}{(n!)^2} (z-1-n)(z-n) \cdots (z+n-2)(z+n-1) = \frac{z-n-1}{z+n} S_n(z).$$

Taking  $n \rightarrow \infty$ ,

$$(24.23) \quad S(z-1) = -S(z).$$

□

Since  $\sin(\pi(z-1)) = -\sin(\pi z)$ ,  $s(z-1) = -s(z)$ . Now take

$$(24.24) \quad f(z) = \frac{1}{S(z)} - \frac{1}{s(z)}.$$

This function is holomorphic on  $\mathbb{C} \setminus \mathbb{Z}$  and satisfies  $f(z-1) = -f(z)$ . Furthermore,

$$(24.25) \quad S(z) = zH(z), \quad s(z) = zh(z),$$

with  $H$  and  $h$  holomorphic on  $\mathbb{C}$ , with  $H(0) = h(0) = 1$ . Therefore, on some neighborhood  $\mathcal{O}$  of 0,

$$(24.26) \quad \frac{1}{H(z)} = 1 + zA(z), \quad \frac{1}{h(z)} = 1 + za(z),$$

Consequently, on  $\mathcal{O} \setminus 0$ ,

$$(24.27) \quad \frac{1}{S(z)} - \frac{1}{s(z)} = \frac{1}{z}(1 + zA(z)) - \frac{1}{z}(1 + za(z)) = A(z) - a(z).$$

Thus,  $f(z)$  has a removable singularity at  $z = 0$ , setting  $f(0) = A(0) - a(0)$ . Setting  $f(-k) = (-1)^k[A(0) - a(0)]$  for each  $k \in \mathbb{Z}$ ,

$$(24.28) \quad f : \mathbb{C} \rightarrow \mathbb{C}, \quad \text{holomorphic.}$$

**Lemma 23.** We have  $f(z) \rightarrow 0$  as  $|z| \rightarrow \infty$ , uniformly on the set

$$(24.29) \quad \{z \in \mathbb{C} : 0 \leq \operatorname{Re}(z) \leq 1\}.$$

*Proof.* Since

$$(24.30) \quad \sin(x+iy) = \frac{1}{2i}(e^{-y+ix} - e^{y-ix}),$$

$|\sin(x+iy)| \rightarrow \infty$  as  $|y| \rightarrow \infty$ . Meanwhile, for  $S(z)$ ,

$$(24.31) \quad \left|1 - \frac{z^2}{k^2}\right| \geq 1 + \frac{y^2 - x^2}{k^2} \geq 1 + \frac{y^2 - 1}{k^2},$$

so  $|\operatorname{Re}(z)| \leq 1$  and  $|\operatorname{Im}(z)| \geq 1$  implies  $|S(z)| \geq |z|$ . Therefore,  $|S(z)| \rightarrow \infty$  as  $|z| \rightarrow \infty$ . □

Therefore,  $f(z)$  is bounded. Since  $f$  is holomorphic on  $\mathbb{C}$ ,  $f$  is constant. Finally, since  $f(z) \rightarrow 0$  as  $|z| \rightarrow \infty$ ,  $0 \leq \operatorname{Re}(z) \leq 1$ ,  $f(z) = 0$ . Therefore,

**Proposition 72.** For  $z \in \mathbb{C}$ ,

$$(24.32) \quad \sin \pi z = \pi z \cdot \prod_{k=1}^{\infty} \left(1 - \frac{z^2}{k^2}\right).$$

## 25. MORE INFINITE PRODUCTS

Now we seek a holomorphic function  $f(z)$  that has zeros that precisely are  $\mathbb{N} = \{1, 2, 3, \dots\}$ , all simple. We cannot take

$$(25.1) \quad \prod_{k \geq 1} \left(1 - \frac{z}{k}\right),$$

because fails to satisfy (24.8) for  $f_k(z) = -\frac{z}{k}$ . This can be fixed by taking

$$(25.2) \quad G(z) = \prod_{k=1}^{\infty} \left(1 - \frac{z}{k}\right) e^{z/k}.$$

Indeed, writing out the Taylor expansion of  $e^{z/k}$ ,

$$(25.3) \quad \left(1 - \frac{z}{k}\right) \left(1 + \frac{z}{k} + \sum_{j=2}^{\infty} \frac{1}{j!} \frac{z^j}{k^j}\right) = 1 - \frac{z^2}{k^2} + \left(1 - \frac{z}{k}\right) \sum_{j=2}^{\infty} \frac{1}{j!} \frac{z^j}{k^j}.$$

Therefore,

$$(25.4) \quad |f_k(z)| \leq C \left|\frac{z}{k}\right|^2,$$

for  $|z| \leq k$ . Therefore,

$$(25.5) \quad G(z) = 0, \Leftrightarrow k \in \mathbb{N},$$

and all zeros of  $G(z)$  are simple.

**Remark 16.**

$$(25.6) \quad S(z) = zG(z)G(-z).$$

Write

$$(25.7) \quad G(z) = \lim_{n \rightarrow \infty} G_n(z), \quad G_n(z) = \prod_{k=1}^n \left(1 - \frac{k}{z}\right) e^{z/k}.$$

Then,

$$(25.8) \quad G_n(z) = (1 - z) \cdots \left(1 - \frac{z}{n}\right) e^{z(1 + \dots + \frac{1}{n})} = n! \frac{1 - z}{1} \cdots \frac{n - z}{n} e^{z(1 + \dots + \frac{1}{n})}.$$

Since  $\frac{k-z}{k} = \frac{k+1-(z-1)}{k}$ ,

$$(25.9) \quad G_n(z) = \frac{z-1}{z-n-1} G_n(z-1) \exp\left(\sum_{k=1}^n \frac{1}{k}\right).$$

Therefore,

$$(25.10) \quad G_n(z) = -\frac{z-1}{n+1-z} G_n(z-1) e^{\gamma_n}.$$

At the limit,  $\gamma_n \rightarrow \gamma$ , where  $\gamma$  is the Euler constant

$$(25.11) \quad \lim_{n \rightarrow \infty} \sum_{k=1}^n \frac{1}{k} - \log(n+1).$$

The constant  $\gamma$  is called Euler's constant.

The factor  $(1-z)e^z$  is the first in a sequence of factors that figure into more general infinite product expansions. Let  $E(z, 0) = 1 - z$ , and for  $p \in \mathbb{N}$ ,

$$(25.12) \quad E(z, p) = (1-z) \exp\left(\sum_{k=1}^p \frac{z^k}{k}\right).$$

Since  $\log \frac{1}{1-z} = \sum_{k=1}^{\infty} \frac{z^k}{k}$ ,

$$(25.13) \quad \log E(z, p) = - \sum_{k=p+1}^{\infty} \frac{z^k}{k},$$

for  $|z| < 1$ . Therefore,

$$(25.14) \quad E(z, p) - 1 = O(z^{p+1}).$$

Indeed, by direct computation,

$$(25.15) \quad \frac{d}{dz}(1 - E(z, p)) = z^p \exp\left(z + \frac{z^2}{2} + \dots + \frac{z^p}{p}\right).$$

This function has a power series whose coefficients are all  $\geq 0$ . Therefore, the same also holds for the holomorphic function,

$$(25.16) \quad \varphi(z) = \frac{1 - E(z, p)}{z^{p+1}}.$$

Therefore,  $\sup_{|z| \leq 1} |\varphi(z)| = \varphi(1) = 1$ .

**Proposition 73.** *Let  $z_k$  be a sequence in  $\mathbb{C} \setminus 0$ . Assume that  $|z_k| \rightarrow \infty$ , and furthermore that*

$$(25.17) \quad \sum_k |z_k|^{-(p+1)} < \infty,$$

for some  $p \in \mathbb{Z}_{\geq 0}$ . Then

$$(25.18) \quad f(z) = \prod_k E\left(\frac{z}{z_k}, p\right),$$

converges locally uniformly to a holomorphic function on  $\mathbb{C}$ . The zeros of  $f$  are precisely the points  $z_k$ . The multiplicity of the zero  $w$  of  $f$  is equal to the number of  $k$ 's such that  $z_k = w$ .

There are some examples for which  $E(z, 2)$  plays a role. Choose  $\alpha, \beta \in \mathbb{C}$  which are linearly independent over  $\mathbb{R}$ , and consider the lattice

$$(25.19) \quad \Gamma = \{m\alpha + n\beta : m, n \in \mathbb{Z}\} \subset \mathbb{C}.$$

Then consider

$$(25.20) \quad H(z) = z \prod_{\omega \in \Gamma \setminus 0} E\left(\frac{z}{\omega}, 2\right).$$

By (25.14),

$$(25.21) \quad E\left(\frac{z}{\omega}, 2\right) = 1 + f_\omega(z), \quad |f_\omega(z)| \leq C \frac{|z|^3}{|\omega|^3},$$

and since

$$(25.22) \quad \sum_{\omega \in \Lambda \setminus \{0\}} |\omega|^{-3} < \infty,$$

the product (25.20) defines a holomorphic function.

**Lemma 24.** *Let  $(z_k)$  be a sequence in  $\mathbb{C} \setminus \{0\}$ . Assume  $|z_k| \rightarrow \infty$  and that  $p_k \in \mathbb{Z}^+$  has the property that*

$$(25.23) \quad \sum_{k \geq 1} \left| \frac{r}{z_k} \right|^{p_k+1} < \infty, \quad \forall r > 0.$$

Then the product

$$(25.24) \quad f(z) = \prod_{k \geq 1} E\left(\frac{z}{z_k}, p_k\right),$$

converges locally uniformly on  $\mathbb{C}$  to a holomorphic function whose zeros are given by  $z_k$ .

*Proof.* By (25.14), if  $|z| \leq |z_k|$ ,

$$(25.25) \quad \left| E\left(\frac{z}{z_k}, p_k\right) - 1 \right| \leq \left| \frac{z}{z_k} \right|^{p_k+1}.$$

Since  $|z_k| \rightarrow \infty$ ,  $R \leq |z_k|$  for all but finitely many  $k$ . □

The product on the right hand side of (25.24) is called a Weierstrass product.

**Proposition 74** (Weierstrass product theorem). *If  $(z_k)$  is a sequence in  $\mathbb{C}$  such that  $|z_k| \rightarrow \infty$ , then there exists an entire function  $f$  with  $\{z_k\}$  as its set of zeros, counted with multiplicity.*

*Proof.* Only finitely many  $z_k$  are zero, let's say  $m$ , so that  $|z_k| > 0$  for  $k \geq m+1$ . For  $R > 0$ , there exists  $K > m$  such that  $|z_k| > 2R$  for each  $k > K$ . Therefore,

$$(25.26) \quad \sum_{k=m+1}^{\infty} \left| \frac{z}{z_k} \right|^k,$$

converges uniformly on  $\{z : |z| \leq R\}$ . Therefore, by Lemma 24,

$$(25.27) \quad z^m \prod_{k \geq m+1} E\left(\frac{z}{z_k}, p_k\right), \quad p_k = k - 1,$$

converges. □

We can also prove the Weierstrass factorization theorem for entire functions.

**Proposition 75.** *Let  $f : \mathbb{C} \rightarrow \mathbb{C}$  be entire and not identically zero. Let  $f$  have a zero of order  $m$  at  $z = 0$ , and let  $(z_k)$  be the sequence of other zeros of  $f$ , counted with multiplicity. Then there exist  $p_k \in \mathbb{N}$  and a holomorphic function  $h : \mathbb{C} \rightarrow \mathbb{C}$  such that*

$$(25.28) \quad f(z) = e^{h(z)} z^m \prod_{k \geq 1} E\left(\frac{z}{z_k}, p_k\right).$$

Here,  $p_k$  can be any sequence for which (25.23) holds.

*Proof.* By Proposition 74, we have a holomorphic function with the required set of zeros, call it  $g(z)$ . This means that  $\frac{f(z)}{g(z)}$  is an entire function that is nowhere vanishing, and thus has the form  $e^{h(z)}$  for some entire function  $h(z)$ .  $\square$

**Remark 17.** Read pages 104–105 of [Tay19] at home.

## 26. FOURIER SERIES AND THE POISSON INTEGRAL

Let  $f : S^1 \rightarrow \mathbb{C}$  be an integrable function. Identifying  $S^1$  with  $\mathbb{T}^1 = \mathbb{R}/(2\pi\mathbb{Z})$ , we desire to write

$$(26.1) \quad f(\theta) = \sum_{k=-\infty}^{\infty} \hat{f}(k) e^{ik\theta}.$$

If  $f$  is written in the form (26.1), where  $\hat{f}(k)$  converges in some sense, using

$$(26.2) \quad \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{il\theta} d\theta = 0, \quad \text{if } l \neq 0, \quad 1, \quad \text{if } l = 0,$$

then

$$(26.3) \quad \hat{f}(k) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(\theta) e^{-ik\theta} d\theta.$$

The series in (26.1) is called the Fourier series of  $f$ . If  $\hat{f}$  is given by (26.3), and (26.1) holds, (26.1) is called the Fourier inversion formula.

For  $0 < r < 1$ , set

$$(26.4) \quad J_r f(\theta) = \sum_{k=-\infty}^{\infty} \hat{f}(k) r^{|k|} e^{ik\theta}.$$

Since

$$(26.5) \quad |\hat{f}(k)| \leq \frac{1}{2\pi} \int_{-\pi}^{\pi} |f(\theta)| d\theta,$$

so the series (26.4) is absolutely convergent for each  $r \in [0, 1)$ . By Fubini's theorem,

$$(26.6) \quad J_r f(\theta) = \sum_{k=-\infty}^{\infty} \hat{f}(k) r^{|k|} e^{ik\theta} = \frac{1}{2\pi} \sum_{k=-\infty}^{\infty} r^{|k|} \int_{-\pi}^{\pi} e^{ik(\theta-\theta')} f(\theta') d\theta' = \int_{S^1} f(\theta') p_r(\theta - \theta') d\theta',$$

where

$$(26.7) \quad p_r(\theta) = \frac{1}{2\pi} \sum_{k=-\infty}^{\infty} r^{|k|} e^{ik\theta} = \frac{1}{2\pi} \left[ 1 + \sum_{k=1}^{\infty} (r^k e^{ik\theta} + r^k e^{-ik\theta}) \right] = \frac{1}{2\pi} \left[ \frac{1}{1 - r e^{i\theta}} + \frac{1}{1 - r e^{-i\theta}} - 1 \right] = \frac{1}{2\pi} \frac{1 - r^2}{1 - 2r \cos \theta + r^2}.$$

Both the numerator and denominator of  $p_r(\theta)$  are positive, so  $p_r(\theta) > 0$  for each  $r \in [0, 1)$ . Also,

$$(26.8) \quad \int_{S^1} p_r(\theta) d\theta = \frac{1}{2\pi} \int_{-\pi}^{\pi} \sum_{k=-\infty}^{\infty} r^{|k|} e^{ik\theta} d\theta = 1,$$

and  $p_r(\theta) \rightarrow 0$  as  $r \nearrow 1$  if  $e^{i\theta} \neq 1$ .

**Proposition 76.** *If  $f \in C(S^1)$ , then*

$$(26.9) \quad J_r f \rightarrow f \quad \text{uniformly on } S^1 \quad \text{as } r \nearrow 1.$$

*Proof.* Since  $f$  is continuous on  $S^1$  and  $S^1$  is compact,  $f$  is uniformly continuous on  $S^1$ . Therefore, there exists some  $\delta > 0$  such that if  $|\theta - \theta'| < \delta$ ,  $|f(\theta) - f(\theta')| < \epsilon$ . Therefore,

$$(26.10) \quad \int_{S^1} f(\theta') p_r(\theta - \theta') d\theta' = \int_{|\theta - \theta'| < \delta} f(\theta) p_r(\theta - \theta') d\theta' + \int_{|\theta - \theta'| < \delta} [f(\theta) - f(\theta')] p_r(\theta - \theta') d\theta' + \int_{|\theta - \theta'| > \delta} f(\theta') p_r(\theta - \theta') d\theta'.$$

The first integral converges to  $f(\theta)$ , and the second two converge to zero, uniformly as  $r \nearrow 1$ .  $\square$

**Proposition 77.** *Assume  $f \in C(S^1)$ . If the Fourier coefficients form a summable series,*

$$(26.11) \quad \sum_{k=-\infty}^{\infty} |\hat{f}(k)| < \infty,$$

*then the identity (26.1) holds for each  $\theta \in S^1$ .*

*Proof.* If  $\sum_k |a_k| < \infty$ , then

$$(26.12) \quad \sum_k a_k = S, \Rightarrow \lim_{r \nearrow 1} \sum_k r^{|k|} a_k = S.$$

To see this, observe that for any fixed  $k$ , (26.12) holds. Therefore, choose  $N(\epsilon)$  sufficiently large so that  $\sum_k |a_k| < \epsilon$ , so the proof is complete.  $\square$

Condition (26.11) holds for piecewise  $C^2$  functions that are continuous on  $S^1$ . Integrating by parts,

$$(26.13) \quad \hat{f}(k) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(\theta) \frac{i}{k} \frac{\partial}{\partial \theta} (e^{-ik\theta}) d\theta.$$

Therefore,

$$(26.14) \quad |\hat{f}(k)| \leq \frac{1}{2\pi|k|} \int_{-\pi}^{\pi} |f'(\theta)| d\theta.$$

Taking a second derivative and integrating by parts,  $|\hat{f}(k)| \leq \frac{1}{2\pi k^2} \int_{-\pi}^{\pi} |f''(\theta)| d\theta$ .

**Proposition 78.** *Let  $f$  be continuous and piecewise  $C^2$  on  $S^1$ . Then  $|\hat{f}(k)| \leq \frac{C}{k^2+1}$ .*

Given  $f \in C(S^1)$ , let us say

$$(26.15) \quad f \in \mathcal{A}(S^1) \Leftrightarrow \sum_k |\hat{f}(k)| < \infty.$$

**Proposition 79.** *Given  $f \in \mathcal{A}(S^1)$ ,*

$$(26.16) \quad \sum_k |\hat{f}(k)|^2 = \frac{1}{2\pi} \int_{S^1} |f(\theta)|^2 d\theta.$$

*More generally, if  $g \in \mathcal{A}(S^1)$ , then*

$$(26.17) \quad \sum_k \hat{f}(k) \overline{\hat{g}(k)} = \frac{1}{2\pi} \int_{S^1} f(\theta) \overline{g(\theta)} d\theta.$$

*Proof.* By Fubini's theorem,

$$(26.18) \quad \frac{1}{2\pi} \int_{S^1} f(\theta) \overline{g(\theta)} d\theta = \frac{1}{2\pi} \int_{S^1} \sum_{j,k} \hat{f}(j) \overline{\hat{g}(k)} e^{-i(j-k)\theta} d\theta = \sum_k \hat{f}(k) \overline{\hat{g}(k)}.$$

□

Define the  $L^2$  norm of  $f$ ,

$$(26.19) \quad \|f\|_{L^2}^2 = \frac{1}{2\pi} \int_{S^1} |f(\theta)|^2 d\theta.$$

This space is an inner product space, where the inner product is given by

$$(26.20) \quad (f, g)_{L^2} = \frac{1}{2\pi} \int_{S^1} f(\theta) \overline{g(\theta)} d\theta.$$

Thus, if  $f \in \mathcal{A}(S^1)$ , (26.16) states that

$$(26.21) \quad \sum |\hat{f}(k)|^2 = \|f\|_{L^2}^2,$$

and (26.17) states that

$$(26.22) \quad \sum \hat{f}(k) \overline{\hat{g}(k)} = (f, g)_{L^2}.$$

The  $l^2$  norm on sequences can be defined by the inner product (26.22). Both the  $L^2$  norm and  $l^2$  norm are Banach norms, and satisfy the triangle inequality.

Now consider a general square integrable function  $f$  on  $S^1$ ,  $f$  need not lie in  $\mathcal{A}(S^1)$ . Let

$$(26.23) \quad S_N f = \sum_{|k| \leq N} \hat{f}(k) e^{ik\theta}.$$

Then split

$$(26.24) \quad f = S_N f + R_N f.$$

By (26.23),

$$(26.25) \quad (f, S_N f)_{L^2} = (S_N f, S_N f)_{L^2},$$

so

$$(26.26) \quad (S_N f, R_N f)_{L^2} = 0.$$

Equivalently,

$$(26.27) \quad \|f\|_{L^2}^2 = \|S_N f\|_{L^2}^2 + \|R_N f\|_{L^2}^2 \geq \|S_N f\|_{L^2}^2.$$

**Proposition 80.** *Let  $f, f_\nu$  be square integrable on  $S^1$ . Assume*

$$(26.28) \quad \lim_{\nu \rightarrow \infty} \|f - f_\nu\|_{L^2} = 0,$$

and for each  $\nu$ ,

$$(26.29) \quad \lim_{N \rightarrow \infty} \|f_\nu - S_N f_\nu\|_{L^2} = 0.$$

Then

$$(26.30) \quad \lim_{N \rightarrow \infty} \|f - S_N f\|_{L^2} = 0.$$

In this case,

$$(26.31) \quad \|f\|_{L^2}^2 = \sum |\hat{f}(k)|^2.$$

*Proof.* The limit (26.30) may be proved by splitting

$$(26.32) \quad \|f - S_N f\|_{L^2} \leq \|f - f_\nu\|_{L^2} + \|f_\nu - S_N f_\nu\|_{L^2} + \|S_N(f - f_\nu)\|_{L^2}.$$

Since  $\|S_N f\|_{L^2}^2 = \sum_{|k| \leq N} |\hat{f}(k)|^2$ , (26.31) also holds.  $\square$

For  $f \in C(S^1)$ , Proposition 76 implies that  $J_r f \rightarrow f$  uniformly as  $r \nearrow 1$ . Since  $f_\nu \in \mathcal{A}(S^1)$ ,

$$(26.33) \quad f \in C(S^1) \Rightarrow S_N f \rightarrow f \quad \text{in} \quad L^2, \quad \sum |\hat{f}(k)|^2 = \|f\|_{L^2}^2.$$

Using Lebesgue integration, one may go further, and prove

**Proposition 81.** *Given  $f \in L^2(S^1)$ , there exists  $f_\nu \in C(S^1)$  such that  $f_\nu \rightarrow f$  in  $L^2$ .*

**Proposition 82.** *If  $(f_\nu)$  is a Cauchy sequence in  $L^2$ , there exists  $f \in L^2$  such that  $f_\nu \rightarrow f$  in the  $L^2$  norm.*

The Fourier series connects with the theory of harmonic functions. Taking  $z = re^{i\theta}$ ,

$$(26.34) \quad J_r f(\theta) = \sum_{k=0}^{\infty} \hat{f}(k) z^k + \sum_{k=1}^{\infty} \hat{f}(-k) \bar{z}^k.$$

Rewriting this as

$$(26.35) \quad (PIf)(z) = (PI_+ f)(z) + (PI_- f)(z),$$

$PI$  is the sum of a holomorphic function and a conjugate holomorphic function. For  $|w| = 1$  and  $|z| < 1$ ,

$$(26.36) \quad 1 - 2r \cos(\theta - \theta') + r^2 = |w - z|^2,$$

so

$$(26.37) \quad PIf(z) = \frac{1 - |z|^2}{2\pi} \int_{S^1} \frac{f(w)}{|w - z|^2} ds(w).$$

Applying this fact to the Dirichlet boundary value problem,

**Proposition 83.** *If  $D \subset \mathbb{C}$  is an open disk and  $f \in C(\partial D)$  is given, there exists a unique  $u \in C(\bar{D}) \cap C^2(D)$  satisfying*

$$(26.38) \quad \Delta u = 0 \quad \text{on} \quad D, \quad u|_{\partial D} = f.$$

**Remark 18.** *Read pages 148, 149, 150 of [Tay19].*

## 27. THE FOURIER TRANSFORM

Suppose  $f$  is a function that is integrable on  $\mathbb{R}$ , that is,

$$(27.1) \quad \|f\|_{L^1(\mathbb{R})} = \int_{-\infty}^{\infty} |f(x)| dx < \infty.$$

Then define the Fourier transform of  $f$  to be

$$(27.2) \quad \hat{f}(\xi) = \mathcal{F}f(\xi) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{-ix\xi} dx, \quad \xi \in \mathbb{R},$$

and set

$$(27.3) \quad \mathcal{F}^* f(\xi) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{ix\xi} dx.$$

**Proposition 84.** *If  $f$  is integrable on  $\mathbb{R}$ , then  $\hat{f}$  is bounded and continuous on  $\mathbb{R}$ .*

*Proof.* Since  $|e^{ix\xi}| = 1$ ,

$$(27.4) \quad |\hat{f}(\xi)| \leq \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} |f(x)| dx.$$

Then for  $\epsilon > 0$ , there exists  $N(\epsilon) < \infty$  such that

$$(27.5) \quad \int_{|x| \geq N} |f(x)| dx \leq \epsilon.$$

Therefore, for  $\xi, \eta \in \mathbb{R}$ ,

$$(27.6) \quad |\hat{f}(\xi) - \hat{f}(\eta)| \leq 2\epsilon + \frac{N}{\sqrt{2\pi}} \|f\|_{L^1(\mathbb{R})} |\xi - \eta|,$$

which proves continuity of  $\hat{f}(\xi)$ , since  $\epsilon > 0$  is arbitrary. □

The Fourier inversion formula asserts that

$$(27.7) \quad f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \hat{f}(\xi) e^{ix\xi} d\xi,$$

depending on the nature of  $f$ . For  $\epsilon > 0$ , set

$$(27.8) \quad J_\epsilon f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \hat{f}(\xi) e^{-\epsilon\xi^2} e^{ix\xi} d\xi.$$

Then for  $f \in L^1(\mathbb{R})$ , by Fubini's theorem,

$$(27.9) \quad J_\epsilon f(x) = \frac{1}{2\pi} \int \int f(y) e^{i(x-y)\xi} e^{-\epsilon\xi^2} dy d\xi = \int_{-\infty}^{\infty} f(y) H_\epsilon(x-y) dy,$$

where

$$(27.10) \quad H_\epsilon(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-\epsilon\xi^2 + ix\xi} d\xi = \frac{1}{\sqrt{4\pi\epsilon}} e^{-x^2/4\epsilon}.$$

**Proposition 85.** *Assume  $f$  is integrable on  $\mathbb{R}$ . Then*

$$(27.11) \quad J_\epsilon f(x) \rightarrow f(x),$$

*whenever  $f$  is continuous at  $x$ . If, in addition,  $f$  is continuous on  $\mathbb{R}$  and  $f(x) \rightarrow 0$  as  $|x| \rightarrow \infty$ , then  $J_\epsilon f(x) \rightarrow f(x)$  uniformly on  $\mathbb{R}$ .*

**Corollary 11.** *Assume that  $f$  is bounded and continuous on  $\mathbb{R}$ , and that  $f$  and  $\hat{f}$  are integrable on  $\mathbb{R}$ . Then (27.7) holds for all  $x \in \mathbb{R}$ .*

*Proof.* If  $f \in L^1(\mathbb{R})$ , then  $\hat{f}$  is bounded and continuous. If  $\hat{f} \in L^1$  also holds, then  $\mathcal{F}^* \hat{f}$  is also continuous. Furthermore, since  $\mathcal{F}^* \hat{f} \in L^1$ ,

$$(27.12) \quad J_\epsilon f(x) \rightarrow \mathcal{F}^* \hat{f}(x), \quad \text{as } \epsilon \rightarrow 0.$$

Therefore,  $f(x) = \mathcal{F}^* \hat{f}(x)$ . □

Now set

$$(27.13) \quad \mathcal{A}(\mathbb{R}) = \{f \in L^1(\mathbb{R}) : f \text{ is bounded and continuous, } \hat{f} \in L^1(\mathbb{R})\}.$$

As in the case of the Fourier series, Integrating by parts,

$$(27.14) \quad \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f^{(j)}(x) e^{-ix\xi} dx = (i\xi)^j \hat{f}(\xi),$$

so

$$(27.15) \quad C_0^2(\mathbb{R}) \subset \mathcal{A}(\mathbb{R}).$$

## 28. $L^2$ FUNCTIONS AND INNER PRODUCT SPACES

We now attempt to show that the Fourier transform on  $\mathcal{A}(\mathbb{R})$  has a continuous extension to  $L^2(\mathbb{R})$ . Consider functions in  $L^p(\mathbb{R})$ . For example,  $L^1(\mathbb{R})$  functions are the general class of such functions is the class of Lebesgue measurable functions for which (27.1) holds. We say that  $f_1 \sim f_2$  if  $f_1 = f_2$  almost everywhere. The space of  $L^1$  functions satisfies the triangle inequality,

$$(28.1) \quad \|f + g\|_{L^1} \leq \|f\|_{L^1} + \|g\|_{L^1}.$$

We say that  $f_\nu \rightarrow f$  in  $L^1$  if  $\|f_\nu - f\|_{L^1} \rightarrow 0$ . The  $L^1$  functions are a complete metric space.

**Proposition 86.** (1) Given  $f \in L^1(\mathbb{R})$  and  $k \in \mathbb{N}$ , there exist  $f_\nu \in C_0^k(\mathbb{R})$  such that  $f_\nu \rightarrow f$  in  $L^1$ .

(2) If  $(f_\nu)$  is a Cauchy sequence in  $L^1(\mathbb{R})$ , there exists  $f \in L^1(\mathbb{R})$  such that  $f_\nu \rightarrow f$  in  $L^1$ .

Next, consider the  $L^2$  norm

$$(28.2) \quad \|f\|_{L^2}^2 = \int_{-\infty}^{\infty} |f(x)|^2 dx < \infty.$$

The  $L^2$  norm satisfies the triangle inequality

$$(28.3) \quad \|f + g\|_{L^2} \leq \|f\|_{L^2} + \|g\|_{L^2}.$$

This follows from the fact that

$$(28.4) \quad \|f\|_{L^2}^2 = (f, f)_{L^2},$$

where

$$(28.5) \quad (f, g)_{L^2} = \int_{-\infty}^{\infty} f(x) \overline{g(x)} dx.$$

**Proposition 87.** (1) Given  $f \in L^2(\mathbb{R})$  and  $k \in \mathbb{N}$ , there exist  $f_\nu \in C_0^k(\mathbb{R})$  such that  $f_\nu \rightarrow f$  in  $L^2$ .

(2) If  $(f_\nu)$  is a Cauchy sequence in  $L^2(\mathbb{R})$ , there exists  $f \in L^2(\mathbb{R})$  such that  $f_\nu \rightarrow f$  in  $L^2$ .

Since  $\hat{f} \in L^1$ , then the Fourier inversion formula holds for all  $f \in \mathcal{A}(\mathbb{R})$ . It follows from this that  $f \in \mathcal{A}(\mathbb{R}) \Rightarrow \hat{f} \in \mathcal{A}(\mathbb{R})$ . Also, note that

$$(28.6) \quad \mathcal{A}(\mathbb{R}) \subset L^2(\mathbb{R}),$$

since

$$(28.7) \quad \|f\|_{L^2}^2 = \int |f(x)|^2 \leq \sup |f(x)| \cdot \int |f(x)| dx \leq \frac{1}{\sqrt{2\pi}} \|\hat{f}\|_{L^1} \|f\|_{L^1}.$$

Checking (27.2) and (27.3), if  $f, g \in \mathcal{A}(\mathbb{R})$ ,

$$(28.8) \quad (\mathcal{F}f, g) = (f, \mathcal{F}^*g).$$

Then by Corollary 11,

$$(28.9) \quad f, g \in \mathcal{A}(\mathbb{R}) \Rightarrow (\mathcal{F}f, \mathcal{F}g) = (\mathcal{F}^* \mathcal{F}f, g) = (f, g).$$

The same holds with  $\mathcal{F}$  replaced by  $\mathcal{F}^*$ . Therefore, for  $f \in \mathcal{A}(\mathbb{R})$ ,

$$(28.10) \quad \|\mathcal{F}f\|_{L^2}^2 = \|\mathcal{F}^* f\|_{L^2}^2 = \|f\|_{L^2}^2.$$

**Proposition 88.** *The maps  $\mathcal{F}$  and  $\mathcal{F}^*$  have unique continuous linear extensions from*

$$(28.11) \quad \mathcal{F}, \mathcal{F}^* : \mathcal{A}(\mathbb{R}) \rightarrow \mathcal{A}(\mathbb{R}),$$

to

$$(28.12) \quad \mathcal{F}, \mathcal{F}^* : L^2(\mathbb{R}) \rightarrow L^2(\mathbb{R}),$$

with the identities

$$(28.13) \quad \mathcal{F}^* \mathcal{F}f = f, \quad \mathcal{F} \mathcal{F}^* f = f.$$

The identities (28.10) also hold for any  $f \in L^2(\mathbb{R})$ .

*Proof.* By Proposition 87, there exists a sequence  $f_\nu \in \mathcal{A}(\mathbb{R})$  such that  $f_\nu \rightarrow f$  in  $L^2$ . Then by (28.10),  $\mathcal{F}f_\nu$  is a Cauchy sequence in  $L^2(\mathbb{R})$ , and therefore,  $\mathcal{F}f_\nu$  converges in  $L^2$ . Label the limit  $\mathcal{F}f$ . This limit is independent of the choice of  $f_\nu$ , since for a different sequence  $f_\mu$ ,  $\|f_\nu - f_\mu\|_{L^2} \rightarrow 0$  as  $\mu, \nu \rightarrow \infty$ . Furthermore,

$$(28.14) \quad \|\mathcal{F}f_\nu - \mathcal{F}f\|_{L^2}, \|\mathcal{F}^* f_\nu - \mathcal{F}^* f\|_{L^2} \rightarrow 0.$$

□

In particular, Proposition 88 implies

$$(28.15) \quad \chi_{[-R,R]} \hat{f} \rightarrow \hat{f}, \quad \text{in } L^2, \quad \text{as } R \rightarrow \infty.$$

**Proposition 89.** *Define  $S_R$  by*

$$(28.16) \quad S_R f(x) = \frac{1}{\sqrt{2\pi}} \int_{-R}^R \hat{f}(\xi) e^{ix\xi} d\xi.$$

Then

$$(28.17) \quad f \in L^2(\mathbb{R}) \Rightarrow S_R f \rightarrow f, \quad \text{in } L^2(\mathbb{R}),$$

as  $R \rightarrow \infty$ .

**Proposition 90.** *There is the inclusion*

$$(28.18) \quad C_0^1(\mathbb{R}) \subset \mathcal{A}(\mathbb{R}).$$

*Proof.* This follows from the fact that if  $g = f'$ , then

$$(28.19) \quad \hat{g}(\xi) = i\xi \hat{f}(\xi).$$

Then use the Cauchy–Schwartz inequality.

□

**Proposition 91.** *Let  $f_\nu \in \mathcal{A}(\mathbb{R})$  and  $f \in C(\mathbb{R}) \cap L^1(\mathbb{R})$ . Assume that*

$$(28.20) \quad f_\nu \rightarrow f \quad \text{in } L^1, \quad \text{and} \quad \|\hat{f}_\nu\|_{L^1} \leq A,$$

for some  $A < \infty$ . Then  $f \in \mathcal{A}(\mathbb{R})$ .

*Proof.* Since  $f_\nu \rightarrow f$  in  $L^1$ ,  $\hat{f}_\nu \rightarrow \hat{f}$  uniformly on  $\mathbb{R}$ . Therefore, for each  $R < \infty$ ,

$$(28.21) \quad \int_{-R}^R |\hat{f}_\nu(\xi) - \hat{f}(\xi)| d\xi \rightarrow 0,$$

so

$$(28.22) \quad \int_{-R}^R |\hat{f}(\xi)| d\xi \leq A,$$

for any  $R < \infty$ . □

Let  $V$  be a complex vector space, that is, a space on which

$$(28.23) \quad f, g \in V \Rightarrow f + g \in V,$$

and

$$(28.24) \quad a \in \mathbb{C}, \quad f \in V \Rightarrow af \in V.$$

Moreover, the commutative, associative, and distributive properties must hold.

$$(28.25) \quad f + g = g + f, \quad (f + g) + h = f + (g + h), \quad f + 0 = f, \quad f + (-f) = 0,$$

for multiplication by scalars,

$$(28.26) \quad a(bf) = (ab)f, \quad 1 \cdot f = f,$$

and

$$(28.27) \quad a(f + g) = af + ag, \quad (a + b)f = af + bf.$$

An inner product on a complex vector space  $V$  assigns to elements  $f, g \in V$  the quantity  $(f, g) \in \mathbb{C}$  in a fashion that obeys three rules:

$$(28.28) \quad \begin{aligned} (a_1 f_1 + a_2 f_2, g) &= a_1 (f_1, g) + a_2 (f_2, g), \\ (f, g) &= \overline{(g, f)}, \\ (f, f) &> 0, \quad \text{unless } f = 0. \end{aligned}$$

For example,

$$(28.29) \quad (f, g) = \frac{1}{2\pi} \int_{S^1} f(\theta) \overline{g(\theta)} d\theta,$$

defines an inner product on  $C(S^1)$ . Similarly,

$$(28.30) \quad (f, g) = \int_{-\infty}^{\infty} f(x) \overline{g(x)} dx$$

defines an inner product on  $\mathcal{A}(\mathbb{R})$ .

Another example is to let  $l^2$  consist of sequences  $(a_k)_{k \in \mathbb{Z}}$  such that

$$(28.31) \quad \sum_{k=-\infty}^{\infty} |a_k|^2 < \infty.$$

There the inner product is given by

$$(28.32) \quad ((a_k), (b_k)) = \sum_{k=-\infty}^{\infty} a_k \overline{b_k}.$$

**Definition 25** (Norm). *A norm on a vector space  $V$  is a function  $f \mapsto \|f\|$  satisfying,*

$$(28.33) \quad \|af\| = |a|\|f\|, \quad a \in \mathbb{C}, \quad f \in V,$$

$$(28.34) \quad \|f\| > 0, \quad \text{unless } f = 0,$$

*and which satisfies the triangle inequality,*

$$(28.35) \quad \|f + g\| \leq \|f\| + \|g\|.$$

A norm on a vector space  $V$  makes  $V$  a metric space. Indeed, for any  $f, g \in V$ , define

$$(28.36) \quad d(f, g) = \|f - g\|.$$

Then clearly,  $d(f, g) = d(g, f)$ , and  $d(f, g) = 0$  if and only if  $f = g$ . Finally,

$$(28.37) \quad d(f, h) = \|f - h\| \leq \|f - g\| + \|g - h\| = d(f, g) + d(g, h).$$

**Theorem 26.** *An inner product space is a metric space.*

*Proof.* Define the norm

$$(28.38) \quad \|f\| = \sqrt{(f, f)}.$$

It is straightforward to verify that (28.28) implies that (28.33) and (28.34) hold. To prove that the triangle inequality holds, observe that

$$(28.39) \quad \|f + g\|^2 = \|f\|^2 + \|g\|^2 + 2\operatorname{Re}(f, g).$$

Therefore, the triangle inequality is equivalent to showing

$$(28.40) \quad \operatorname{Re}(f, g) \leq \|f\|\|g\|.$$

Indeed, since  $0 \leq \|f - g\|^2$ ,

$$(28.41) \quad \|f - g\|^2 = \|f\|^2 - 2\operatorname{Re}(f, g) + \|g\|^2 \geq 0,$$

implies that

$$(28.42) \quad 2\operatorname{Re}(f, g) \leq \|f\|^2 + \|g\|^2.$$

Next, multiplying  $f$  by  $t$  and  $g$  by  $\frac{1}{t}$ ,

$$(28.43) \quad 2\operatorname{Re}(f, g) \leq t^2\|f\|^2 + \frac{1}{t^2}\|g\|^2.$$

Minimizing the right hand side of (28.43) over all  $t > 0$ , take  $t^2 = \frac{\|g\|}{\|f\|}$ . Then,

$$(28.44) \quad \operatorname{Re}(f, g) \leq \|f\|\|g\|.$$

□

In fact,

**Proposition 92.** *For any inner product on a vector space  $V$ ,*

$$(28.45) \quad |(f, g)| \leq \|f\|\|g\|, \quad \forall f, g \in V.$$

*Proof.* Multiplying  $f$  by some  $e^{i\theta}$ ,  $(f, g)$  is real and positive, so (28.44) implies

$$(28.46) \quad |(f, g)| \leq \|f\|\|g\|.$$

□

**Definition 26.** *An inner product space  $V$  is called a Hilbert space if it is a complete metric space. The completion of both  $C(S^1)$  and  $\mathcal{A}(\mathbb{R})$  is the  $L^2$  space.*

## 29. LAPLACE TRANSFORMS AND MELLIN TRANSFORMS

Suppose  $f : \mathbb{R}^+ \rightarrow \mathbb{C}$  that is integrable on  $[0, R]$  for all  $R < \infty$  and satisfies

$$(29.1) \quad \int_0^\infty |f(t)|e^{-at} dt < \infty, \quad \forall a > A,$$

for some  $-\infty < A < \infty$ . Then define the Laplace transform of  $f$  by

$$(29.2) \quad \mathcal{L}f(s) = \int_0^\infty f(t)e^{-st} dt, \quad \operatorname{Re}(s) > A.$$

This integral is absolutely convergent for all  $s$  in the half plane  $H_A = \{z \in \mathbb{C} : \operatorname{Re}(z) > A\}$ , and defines a continuous function

$$(29.3) \quad \mathcal{L}f : H_A \rightarrow \mathbb{C}.$$

Also, if  $\gamma$  is a closed curve in  $H_A$ , we can change the order of integration to see that

$$(29.4) \quad \int_\gamma \mathcal{L}f(s) ds = \int_\gamma \int_0^\infty f(t)e^{-st} dt ds = \int_0^\infty \int_\gamma f(t)e^{-st} ds dt = 0.$$

Therefore, by Morera's theorem,  $\mathcal{L}f$  is holomorphic on  $H_A$ . Also,

$$(29.5) \quad \frac{d}{ds} \mathcal{L}f(s) = \mathcal{L}g(s), \quad \text{where} \quad g(t) = -tf(t).$$

Meanwhile, if  $f \in C^1([0, \infty))$  and  $\int_0^\infty |f'(t)|e^{-at} dt < \infty$  for all  $a > A$ , then integrating by parts,

$$(29.6) \quad \mathcal{L}f'(s) = \int_0^\infty e^{-st} f'(t) dt = s\mathcal{L}f(s) - f(0).$$

Making a similar calculation for higher order derivatives,

$$(29.7) \quad \mathcal{L}f^{(k)}(s) = s^k \mathcal{L}f(s) - s^{k-1} f(0) - \dots - f^{(k-1)}(0).$$

Therefore, if  $f$  satisfies an ODE of the form

$$(29.8) \quad c_n f^{(n)}(t) + c_{n-1} f^{(n-1)}(t) + \dots + c_0 f(t) = g(t),$$

for  $t \geq 0$ , with initial data

$$(29.9) \quad f(0) = a_0, \dots, f^{(n-1)}(0) = a_{n-1},$$

then

$$(29.10) \quad p(s)\mathcal{L}f(s) = \mathcal{L}g(s) + q(s),$$

where

$$(29.11) \quad p(s) = c_n s^n + \dots + c_0, \quad q(s) = c_n(a_0 s^{n-1} + \dots + a_{n-1}) + \dots + c_1 a_0.$$

Therefore, if all the roots of  $p(s)$  satisfy  $\operatorname{Re}(s) \leq B$ ,

$$(29.12) \quad \mathcal{L}f(s) = \frac{\mathcal{L}g(s) + q(s)}{p(s)}, \quad s \in H_C, \quad C = \max\{A, B\}.$$

We are therefore motivated to seek an inverse Laplace transform. This can be obtained by relating the Laplace transform to the Fourier transform. Indeed, if (29.1) holds and  $B > A$ , then

$$(29.13) \quad \mathcal{L}f(B + i\xi) = \int e^{-Bt} e^{-it\xi} f(t) dt = \sqrt{2\pi} \hat{\varphi}(\xi),$$

where

$$(29.14) \quad \varphi(x) = f(x)e^{-Bx}, \quad x \geq 0, \quad \varphi(x) = 0, \quad x < 0.$$

Now then, since

$$(29.15) \quad \varphi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \hat{\varphi}(\xi) e^{ix\xi} d\xi,$$

$$(29.16) \quad f(t) = \frac{e^{Bt}}{2\pi} \int_{-\infty}^{\infty} \mathcal{L}f(B + i\xi) e^{i\xi t} d\xi = \frac{1}{2\pi i} \int_{\gamma} \mathcal{L}f(s) e^{st} ds,$$

where  $\gamma$  is the vertical line  $\gamma(\xi) = B + i\xi$ ,  $-\infty < \xi < \infty$ .

**Proposition 93.** *If  $f_1$  and  $f_2$  are integrable on  $[0, R]$  for all  $R < \infty$  and satisfy (29.1), then*

$$(29.17) \quad \mathcal{L}f_1(s) = \mathcal{L}f_2(s), \quad \forall s \in H_A \Rightarrow f_1 = f_2, \quad \text{on } \mathbb{R}^+.$$

*Proof.* By (29.17),  $\mathcal{L}(f_1 - f_2) = 0$  for all  $s \in H_A$ . Since  $\mathcal{L}f(s)$  is analytic on  $H_A$ ,  $\mathcal{L}(f_1 - f_2) = 0$  everywhere.  $\square$

Next introduce the Mellin transform.

$$(29.18) \quad \{z\} = \int_0^{\infty} f(t) t^{z-1} dt.$$

This function is defined and holomorphic on  $A < \operatorname{Re}(z) < B$ , provided  $f(t)t^{x-1}$  is integrable for real  $x \in (A, B)$ . Making a change of variables,

$$(29.19) \quad \mathcal{M}f(z) = \int_{-\infty}^{\infty} f(e^x) e^{zx} dx.$$

For  $0 \in (A, B)$ , for  $z$  on the imaginary axis,

$$(29.20) \quad \mathcal{M}^{\#}f(\xi) = \int_0^{\infty} f(t) t^{i\xi-1} dt = \int_{-\infty}^{\infty} f(e^x) e^{ix\xi} dx.$$

Using the Fourier inversion formula,

$$(29.21) \quad f(e^x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} (\mathcal{M}^{\#}f)(\xi) e^{-ix\xi} d\xi,$$

so

$$(29.22) \quad f(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} (\mathcal{M}^{\#}f)(\xi) t^{-i\xi} d\xi.$$

Also, by Plancherel's theorem,

$$(29.23) \quad \int_{-\infty}^{\infty} |f(e^x)|^2 dx = \frac{1}{2\pi} \int_{-\infty}^{\infty} |\mathcal{M}^{\#}f(\xi)|^2 d\xi,$$

so by a change of variables,

$$(29.24) \quad \int_0^{\infty} |f(t)|^2 \frac{1}{t} dt = \frac{1}{2\pi} \int_{-\infty}^{\infty} |\mathcal{M}^{\#}f(\xi)|^2 d\xi.$$

We can also treat  $n \times n$  first order systems of differential equations of the form

$$(29.25) \quad f'(t) = Kf(t) + g(t), \quad f(0) = a,$$

where  $K$  is an  $n \times n$  matrix. Let  $e^{tK}$  denote the matrix exponential

$$(29.26) \quad e^{tK} = \sum_{j=0}^{\infty} \frac{1}{j!} t^j K^j.$$

Let  $E_K(t) = e^{tK}$ . Then

$$(29.27) \quad \mathcal{E}_K(s) = \int_0^{\infty} e^{tK} e^{-st} dt,$$

which is valid whenever  $\operatorname{Re}(s) > \|K\|$ .

Assuming that  $g$  and  $f$  have Laplace transforms,

$$(29.28) \quad s\mathcal{L}f(s) - a = K\mathcal{L}f(s) + \mathcal{L}g(s),$$

so

$$(29.29) \quad \mathcal{L}f(s) = (sI - K)^{-1}(a + \mathcal{L}g(s)).$$

Therefore, it remains to identify the right hand side of (29.29) as the Laplace transform of a function.

First observe that

$$(29.30) \quad \mathcal{L}E_K(s) = (sI - K)^{-1}.$$

Indeed, by the fundamental theorem of calculus,

$$(29.31) \quad L \int_0^T e^{-tL} dt = I - e^{-TL},$$

so if  $L$  satisfies  $\|e^{-tL}\| \leq ce^{-\delta t}$  for all  $t > 0$ , then  $\int_0^T e^{-tL} dt = L^{-1}$ . Taking  $L = sI - K$ ,

$$(29.32) \quad \|e^{t(K-sI)}\| = e^{-t\operatorname{Re}(s)} \|e^{tK}\|,$$

so therefore,

$$(29.33) \quad (sI - K)^{-1}a = \mathcal{L}(E_K a)(s).$$

Now let  $f * g$  denote the convolution

$$(29.34) \quad f * g(t) = \int_0^t f(\tau)g(t - \tau)d\tau.$$

Then,

$$(29.35) \quad \mathcal{L}(f * g)(s) = \mathcal{L}f(s) \cdot \mathcal{L}g(s),$$

since

$$(29.36) \quad \int_0^{\infty} e^{-ts} \int_0^t f(\tau)g(t - \tau)d\tau dt = \int_0^{\infty} e^{-ts} f(t)dt \cdot \int_0^{\infty} e^{-\tau s} g(\tau)d\tau.$$

Therefore,

$$(29.37) \quad (sI - K)^{-1}\mathcal{L}g(s) = \mathcal{L}(E_K * g)(s),$$

where

$$(29.38) \quad E_K * g(t) = \int_0^t E_K(t - \tau)g(\tau)d\tau = \int_0^t e^{(t-\tau)K}g(\tau)d\tau.$$

This gives Duhamel's formula,

$$(29.39) \quad f(t) = e^{tK} a + \int_0^t e^{(t-\tau)K} g(\tau) d\tau.$$

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