A COURSE IN COMPLEX ANALYSIS

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Contents

These notes come from [SS10] and [Tay19].

1. Complex Numbers and the complex plane

The complex numbers arise naturally in the study of polynomials. For example, for the quadratic equation

$$x^2 - 2x + 5 = 0, \tag{1.1}$$

The quadratic formula yields two solutions:

$$z = \frac{2 \pm \sqrt{4-20}}{2} = \frac{2 \pm \sqrt{-16}}{2} = 1 \pm 2i, \tag{1.2}$$

where i is an imaginary number that satisfies $i^2 = -1$.

In general, a complex number z has the form $z = x + iy$, where $x = Re(z)$ and $y = Im(z)$ are the real and imaginary parts. The complex numbers can be visualized as isomorphic to the Euclidean plane $\mathbb{R}^2$, where $x + iy$ is identified with the point $(x, y) \in \mathbb{R}^2$.

Two complex numbers may either be added or multiplied. Suppose $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$. Addition of complex numbers is the same as addition of two vectors in $\mathbb{R}^2$:

$$z_1 + z_2 = (x_1 + x_2) + i(y_1 + y_2). \tag{1.3}$$

Multiplication corresponds to multiplication of 2×2 matrices:

$$z_1 z_2 = (x_1 + iy_1)(x_2 + iy_2) = (x_1 x_2 - y_1 y_2) + i(x_1 y_2 + x_2 y_1). \tag{1.4}$$

We have already discussed how $z \in \mathbb{C}$ may be graphed as a vector in $\mathbb{R}^2$. A complex number $a + ib$ may also be seen as a matrix. Indeed,

$$\begin{pmatrix} x_1 & -y_1 \\ y_1 & x_1 \end{pmatrix} \begin{pmatrix} x_2 \\ y_2 \end{pmatrix} = \begin{pmatrix} x_1 x_2 - y_1 y_2 \\ x_1 y_2 + x_2 y_1 \end{pmatrix}. \tag{1.5}$$

The operations of addition and multiplication are

(1) Commutative
(2) Associative
(3) Distributive

Returning to visualizing complex numbers as vectors in $\mathbb{R}^2$, the absolute value of a complex number z will be defined as the length of the vector corresponding to z, namely,

$$|z| = (x^2 + y^2)^{1/2}. \tag{1.6}$$

Thus, $|z|$ is precisely the distance from the origin to the point (x, y). In particular, the triangle inequality holds,

$$|z + w| \leq |z| + |w|, \qquad \text{for all} \qquad z, w \in \mathbb{C}. \tag{1.7}$$

We also have $|Re(z)| \leq |z|$, $|Im(z)| \leq |z|$, and for all $z, w \in \mathbb{C}$,

$$||z| - |w|| \leq |z - w|, \tag{1.8}$$

since

$$|z| \leq |z - w| + |w|, \qquad \text{and} \qquad |w| \leq |z - w| + |z|. \tag{1.9}$$

The absolute value $|z|$ also corresponds to the square root of the determinant of the matrix corresponding to z, $r^2 = x^2 + y^2$. Then

$$\begin{pmatrix} x & -y \\ y & x \end{pmatrix} = \begin{pmatrix} r & 0 \\ 0 & r \end{pmatrix} \begin{pmatrix} \frac{x}{r} & -\frac{y}{r} \\ \frac{y}{r} & \frac{x}{r} \end{pmatrix}. \tag{1.10}$$

Doing some trigonometry, there exists $0 \leq \theta < 2\pi$ such that

$$\begin{pmatrix} \frac{a}{r} & -\frac{b}{r} \\ \frac{b}{r} & \frac{a}{r} \end{pmatrix} = \begin{pmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{pmatrix}, \tag{1.11}$$

which is a rotation matrix.

Lemma 1. *For any $t \in \mathbb{R}$,*

$$e^{it} = \cos(t) + i \sin t. \tag{1.12}$$

Proof. It is clear that (1.12) holds when $t = 0$. Furthermore,

$$\frac{d}{dt}(e^{it}) = ie^{it}, \tag{1.13}$$

so the velocity of e^{it} is at a right angle to the position e^{It}. Thus, $|e^{it}| = 1$ for all $t \in \mathbb{R}$. Furthermore, (1.13) implies that e^{it} travels at speed one in a counterclockwise direction around the unit circle. This proves (1.12). □

Therefore, for any $z \neq 0$ there exists some $0 < r < \infty$ and $0 \leq \theta < 2\pi$ such that

$$z = re^{i\theta}. \tag{1.14}$$

Definition 1 (Argument)**.** *θ is called the argument of z.*

Moreover, if $z = re^{i\theta}$ and $w = se^{i\varphi}$,

$$zw = rse^{i(\theta+\varphi)}. \tag{1.15}$$

The complex conjugate of z is

$$\bar{z} = x - iy, \tag{1.16}$$

and is obtained by reflecting across the real axis. Therefore,

$$Re(z) = \frac{z+\bar{z}}{2}, \qquad Im(z) = \frac{z-\bar{z}}{2i}, \qquad |z|^2 = z\bar{z}. \tag{1.17}$$

Then if $z \neq 0$,

$$\frac{1}{z} = \frac{\bar{z}}{|z|^2}. \tag{1.18}$$

It is also possible to see (1.18) from a matrix point of view. Applying the usual formula for the inverse of a 2×2 matrix,

$$\begin{pmatrix} x & -y \\ y & x \end{pmatrix}^{-1} = \frac{1}{x^2+y^2}\begin{pmatrix} x & y \\ -y & x \end{pmatrix}. \tag{1.19}$$

Thus $\mathbb{C}$ is a field, every element of $\mathbb{C}$ has an additive inverse, and every nonzero element of $\mathbb{C}$ has a multiplicative inverse. $1 + i0$ is the multiplicative identity.

Theorem 1. *The complex plane $\mathbb{C}$ is a complete, algebraically closed field.*

Proof. The complex plane $\mathbb{C}$ inherits the topology of $\mathbb{R}^2$.

Definition 2 (Convergence)**.** *A sequence $z_n \in \mathbb{C}$ converges to $w \in \mathbb{C}$, that is $w = \lim_{n\to\infty} z_n$ if*

$$|z_n - w| \to 0. \tag{1.20}$$

This is equivalent to $Re(z_n) \to Re(w)$ and $Im(z_n) \to Im(w)$.

It is straightforward to show that $\mathbb{C}$ is complete, that is, every Cauchy sequence converges. Suppose z_n is such a sequence. Then since $|Re(z_n) - Re(z_m)| \leq |z_n - z_m|$, If $|z_n - z_m| \to 0$ as $n, m \to \infty$, then $Re(z_n)$ is a Cauchy sequence in $\mathbb{R}$, and thus $Re(z_n)$ converges. Apply the same argument to $Im(z_n)$.

We can use this topology to define continuity.

Definition 3 (Continuity)**.** *Suppose f is defined on $\Omega \subset \mathbb{C}$. f is continuous at $z_0 \in \Omega$ if for all $\epsilon > 0$ there exists $\delta(\epsilon) > 0$ such that if $|z - z_0| < \delta$ and $z \in \Omega$, then*

$$|f(z) - f(z_0)| < \epsilon. \tag{1.21}$$

This is true if and only if f is continuous as a function of x and y in $\mathbb{R}$. Equivalently, if $z_n \in \Omega$ and $z_n \to z_0$, then $f(z_n) \to f(z_0)$.

As in real analysis, compositions of continuous functions are themselves continuous. Thus, if f is continuous on a set Ω, then $|f(z)|$ is also continuous on a set Ω.

Recall from calculus that a continuous function on a closed interval attains a maximum and a minimum. Here we wish to generalize this fact to functions on the complex plane. Closed and open disks generalize closed and open intervals.

Definition 4. *An open disk centered at $z_0 \in \mathbb{C}$ of radius r is given by*

$$D_r(z_0) = \{z \in \mathbb{C} : |z - z_0| < r\}. \tag{1.22}$$

An open disk centered at z_0 with radius r is given by

$$\bar{D}_r(z_0) = \{z \in \mathbb{C} : |z - z_0| \leq r\}. \tag{1.23}$$

The unit disk is defined

$$\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}. \tag{1.24}$$

Definition 5 (Open set)**.** *$z_0 \in \Omega \subset \mathbb{C}$ is an interior point of Ω if there exists $r > 0$ such that $D_r(z_0) \subset \Omega$. The interior of Ω is the set of all interior points of Ω. A set Ω is open if every point is an interior point.*

Definition 6 (Closed set)**.** *A set $\Omega \subset \mathbb{C}$ is called closed if $\Omega^c = \mathbb{C} - \Omega$ is an open set.*

A point $z \in \mathbb{C}$ is called a limit point of Ω if there exists a sequence $z_n \in \Omega$, $z_n \neq z$ such that $z_n \to z$. The closure of Ω, labeled $\bar{\Omega}$ is equal to the union of Ω and its limit points.

If $\Omega = \bar{\Omega}$ then Ω is a closed set.

The boundary of a set is its closure minus its interior, and is labeled $\partial\Omega$. For example,

$$\partial D_r(z_0) = C_r(z_0) = \{z \in \mathbb{C} : |z - z_0| = r\}. \tag{1.25}$$

Definition 7 (Compact set)**.** *A set is bounded if there exists $M > 0$ such that $\Omega \subset \{z : |z| \leq M\}$.*

If Ω is bounded, we may define the diameter of Ω,

$$diam(\Omega) = \sup_{z,w \in \Omega} |z - w|. \tag{1.26}$$

A set is compact if it is closed and bounded.

Theorem 2. *If f is a continuous function on a compact set Ω, then $|f(z)|$ attains a maximum and a minimum on Ω.*

Here f attains a maximum at $z_0 \in \Omega$ if $|f(z)| \leq f(z_0)|$ for every $z \in \Omega$. f attains a minimum at $z_0 \in \Omega$ if $|f(z)| \geq |f(z_0)|$ for every $z \in \Omega$.

Proof. As in the topology on $\mathbb{R}^2$,

Theorem 3. *A set $\Omega \subset \mathbb{C}$ is compact if and only if every sequence $z_n \in \Omega$ has a subsequence that converges to a point in Ω.*

We can use this fact to show that the image of a compact set under a continuous function is compact. □

The fact that $\mathbb{C}$ is algebraically closed follows from the fundamental theorem of algebra.

Theorem 4 (Fundamental theorem of algebra)**.** *If $p(z)$ is a non-constant polynomial with complex coefficients, then $p(z)$ must have a complex root.*

Proof. Suppose that for some $n \geq 1$,

$$p(z) = a_n z^n + ... + a_1 z + a_0, \qquad a_n \neq 0, \qquad a_j \in \mathbb{C} \qquad \forall 0 \leq j \leq n. \tag{1.27}$$

Therefore, as $|z| \to \infty$,

$$p(z) = a_n z^n (1 + O(z^{-1})), \tag{1.28}$$

which implies that

$$\lim_{|z| \to \infty} |p(z)| = \infty, \tag{1.29}$$

so there exists $0 < R < \infty$ such that

$$\inf_{|z| > R} |p(z)| > |p(0)|, \tag{1.30}$$

and therefore,

$$\inf_{|z| \leq R} |p(z)| = \inf_{z \in \mathbb{C}} |p(z)|. \tag{1.31}$$

Since p is continuous, there exists $z_0 \in D_R$ which satisfies

$$|p(z_0)| = \inf_{z \in \mathbb{C}} |p(z)|, \tag{1.32}$$

where D_R refers to the disk of radius R, $D_R = \{z \in \mathbb{C} : |z| \leq R\}$.

Lemma 2. *If $p(z)$ is a non-constant polynomial and* (1.16) *holds, then $p(z_0) = 0$.*

Proof. Suppose by contradiction that $p(z_0) = a \neq 0$. Since a polynomial in z can easily be rewritten as a polynomial of the same degree in $(z - z_0)$ for any $z_0 \in \mathbb{C}$,

$$p(z_0 + \zeta) = a + q(\zeta), \qquad \zeta = z - z_0, \tag{1.33}$$

where q is a non-constant polynomial of order n. Therefore, for some $k \geq 1$, $b \neq 0$,

$$q(\zeta) = b\zeta^k + ... + b_n \zeta^n. \tag{1.34}$$

The term $b\zeta^k$ dominates the behavior of $q(\zeta)$ for $|\zeta|$ small,

$$q(\zeta) = b\zeta^k + O(\zeta^{k+1}), \qquad \text{as} \qquad \zeta \to 0. \tag{1.35}$$

Therefore, take $S^1 = \{\omega : |\omega| = 1\}$. For any fixed $\omega \in S^1$,

$$p(z_0 + \epsilon\omega) = a + b\omega^k \epsilon^k + O(\epsilon^{k+1}), \qquad \text{as} \qquad \epsilon \searrow 0. \tag{1.36}$$

Since $a \neq 0$ and $b \neq 0$, choose $\omega \in S^1$ such that

$$\frac{b}{|b|}\omega^k = -\frac{a}{|a|}. \tag{1.37}$$

Then,

$$p(z_0 + \epsilon\omega) = a(1 - |\frac{b}{a}|\epsilon^k) + O(\epsilon^{k+1}), \tag{1.38}$$

which contradicts the minimality of $p(z_0)$ when $\epsilon > 0$ is sufficiently small. □

Therefore, $p(z)$ has the root $p(z_0) = 0$. □

Thus, the field $\mathbb{C}$ is algebraically complete.

□

Proposition 1 (Nested sets property). *If $\Omega_1 \supset \Omega_2 \supset ...$ is a sequence of nonempty compact sets such that $diam(\Omega_n) \to 0$, then there exists a unique $w \in \mathbf{C}$ such that $w \in \Omega_n$ for all n.*

Proof. Make a diagonal argument. □

Definition 8 (Connected set). *An open set $\Omega \subset \mathbb{C}$ is connected if we cannot find two nonempty open sets such that $\Omega_1 \cup \Omega_2 = \Omega$ and $\Omega_1 \cap \Omega_2 = \emptyset$.*

The definition is similar for closed sets.

An open set is connected if and only if any two points in Ω can be joined by a curve entirely in Ω.

Definition 9 (Region). *A connected open set is a region.*

Definition 10 (Open covering). *An open covering of a set Ω is a family of open sets $\{U_\alpha\}$ such that $\Omega \subset \cup U_\alpha$.*

The family of open sets need not be countable.

Theorem 5. *A set Ω is compact if and only if every open covering of Ω has a finite sub covering.*

2. Holomorphic functions

Let Ω be an open set in $\mathbb{C}$ and f be a complex valued function on Ω.

Definition 11 (Holomorphic function). *A function f is holomorphic at $z_0 \in \Omega$ if*

$$\lim_{h \to 0} \frac{f(z_0 + h) - f(z_0)}{h} \tag{2.1}$$

exists. Then

$$f'(z_0) = \lim_{h \to 0} \frac{f(z_0 + h) - f(z_0)}{h}. \tag{2.2}$$

This may be rewritten as

$$f(z_0 + h) = f(z_0) + f'(z_0)h + o(h). \tag{2.3}$$

The function f is holomorphic on Ω if f is holomorphic at every point in Ω. If C is closed, f is holomorphic on C if f is holomorphic in a neighborhood of C. If f is holomorphic on all of $\mathbb{C}$, then f is an entire function.

It is possible to apply all the usual limit definition of a derivative rules to functions.

Proposition 2. *The usual rules from calculus hold for holomorphic functions.*

1. *Quotient rule.*
2. *Product rule.*
3. *Chain rule.*
4. *Multiplication by a constant.*
5. *Sum rule.*

Also observe that if f is holomorphic then f is continuous.

Corollary 1. *Polynomials in z are holomorphic. So are rational functions of the form $\frac{f(z)}{g(z)}$, where f and g are polynomials, when $g(z) \neq 0$.*

Proof. By Proposition 2, it suffices to show that $f(z) = z$ is holomorphic. However, that is easy since $f(z+h) - f(z) = h$. □

The above computations in (2.1)–(2.3) are deceptively simple, since holomorphic in one complex variable is a much stronger fact than being differentiable in real variables. To see this, observe that $\bar{z}$ is not holomorphic. Indeed, for any h,

$$\frac{f(z_0 + h) - f(z_0)}{h} = \frac{\bar{h}}{h}. \tag{2.4}$$

Observe that as a mapping from $\mathbb{R}^2$ to $\mathbb{R}^2$, $F(x, y) = (x, -y)$ is differentiable with derivative

$$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{2.5}$$

The function $\bar{z}$ fails to be holomorphic because (2.5) is not the matrix of a complex number.

Theorem 6 (Cauchy–Riemann equations)**.** *Let $f(z) = u(z) + iv(z)$ be a holomorphic function, let $F(x, y) = (u(x, y), v(x, y))$ and let J be the Jacobian of $F(x, y)$,*

$$J_F(x, y) = \begin{pmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{pmatrix}. \tag{2.6}$$

Then the Jacobian $J_F(x, y)$ must be a complex number matrix, that is of the form (1.10)*. That is,*

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \qquad \text{and} \qquad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}. \tag{2.7}$$

Proof. To see this, restrict h to the real line. Then,

$$\lim_{h \to 0} \frac{f(z+h) - f(z)}{h} = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}, \tag{2.8}$$

while

$$\lim_{h \to 0} \frac{f(z+ih) - f(z)}{ih} = -i\frac{\partial u}{\partial y} + \frac{\partial v}{\partial y}. \tag{2.9}$$

Since u and v are real valued functions and (2.8) must equal (2.9), (2.7) must hold. □

Now define two differential operators

$$\begin{aligned} \frac{\partial}{\partial z} &= \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right), \\ \frac{\partial}{\partial \bar{z}} &= \frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right). \end{aligned} \tag{2.10}$$

Corollary 2. *If f is holomorphic at $z_0 = x_0 + iy_0$ then*

$$\frac{\partial f}{\partial \bar{z}}(z_0) = 0, \tag{2.11}$$

$$\frac{\partial f}{\partial x} = \frac{1}{i}\frac{\partial f}{\partial y} = \frac{\partial f}{\partial z}(z_0) = 2\frac{\partial u}{\partial z}(z_0) = f'(z_0), \tag{2.12}$$

and

$$\det J_F(x_0, y_0) = |f'(z_0)|^2. \tag{2.13}$$

The converse is nearly, but not quite true.

Theorem 7. *Suppose $f(z) = u(z) + iv(z)$ is a complex valued function defined on an open set Ω. If u and v are continuously differentiable and satisfy the Cauchy - Riemann equations on Ω, then f is holomorphic on Ω and*

$$f'(z) = \frac{\partial f}{\partial z}. \tag{2.14}$$

Proof. If $h = h_1 + ih_2$ then

$$u(x + h_1, y + h_2) - u(x, y) = \frac{\partial u}{\partial x}h_1 + \frac{\partial u}{\partial y}h_2 + o(h), \tag{2.15}$$

and

$$v(x + h_1, y + h_2) - v(x, y) = \frac{\partial v}{\partial x}h_1 + \frac{\partial v}{\partial y}h_2 + o(h). \tag{2.16}$$

Then

$$f(z + h) - f(z) = (\frac{\partial u}{\partial x} - i\frac{\partial u}{\partial y})(z)(h_1 + ih_2) + o(h). \tag{2.17}$$

Therefore, f is holomorphic and

$$f'(z) = 2\frac{\partial u}{\partial z} = \frac{\partial f}{\partial z}. \tag{2.18}$$

□

Since the polynomials are holomorphic, it is natural to consider power series next.

Definition 12 (Power series)**.** *A power series is a series of the form*

$$\sum_{n=0}^{\infty} a_n z^n. \tag{2.19}$$

Theorem 8 (Radius of convergence)**.** *Given a power series* (2.19) *there exists $0 \leq R \leq \infty$ such that*

$$\frac{1}{R} = \limsup_{n\to\infty} |a_n|^{1/n}, \tag{2.20}$$

where we follow the convention $\frac{1}{0} = \infty$ and $\frac{1}{\infty} = 0$. Equation (2.20) *is called Hadamard's formula. Furthermore,*

1. *If $|z| < R$ the series converges absolutely.*
2. *If $|z| > R$ the series diverges.*

$|z| < R$ is called the disk of convergence.

Proof. Let $L = \frac{1}{R}$. If $|z| < R$ there exists $\epsilon > 0$ such that $(L+\epsilon)|z| = r < 1$. Then (2.20) implies that for $n \geq N_0$ for some N_0 sufficiently large,

$$|a_n|^{1/n} \leq L + \epsilon \Rightarrow |a_n||z|^n \leq r^n. \tag{2.21}$$

Thus (2.19) is absolutely convergent since

$$\sum_{n=N_0}^{\infty} |a_n||z|^n \leq \sum_{n=N_0}^{\infty} r^n = \frac{r^{N_0}}{1-r}. \tag{2.22}$$

Meanwhile, if $|z| > R$ there exists $\epsilon > 0$ such that $(L-\epsilon)|z| = r > 1$. Then (2.20) implies there exists a sequence n_k such that

$$|a_{n_k}||z|^{n_k} \geq r^{n_k} > 1. \tag{2.23}$$

Clearly this sequence does not converge to zero. □

Remark 1. *We could have convergence or divergence on the boundary.*

There are several well - known examples of power series. For example the exponential function

$$e^z = \sum_{n=0}^{\infty} \frac{z^n}{n!}. \tag{2.24}$$

Here the radius of convergence is $R = \infty$. Since $e^z = \cos(z) + i\sin(z)$,

$$\cos(z) = \frac{e^{iz} + e^{-iz}}{2} = \sum_{n=0}^{\infty} (-1)^n \frac{z^{2n}}{(2n)!}, \tag{2.25}$$

and

$$\sin(z) = \frac{e^{iz} - e^{-iz}}{2i} = \sum_{n=0}^{\infty} (-1)^n \frac{z^{2n+1}}{(2n+1)!}. \tag{2.26}$$

Next,

$$\sum_{n=0}^{\infty} z^n = \frac{1}{1-z}, \tag{2.27}$$

and the radius of convergence is $R = 1$. Another important example comes from substituting $-z^2$ for z in (2.25),

$$\sum_{n=0}^{\infty} (-1)^n z^{2n} = \frac{1}{1+z^2}. \tag{2.28}$$

This series has radius of convergence $R = 1$, and singularities at $z = \pm i$. This is an interesting example because a calculus two student could calculate the radius of convergence, but would not see why the solution could not be extended along the real line past $|x| < 1$.

Theorem 9. *The power series*

$$f(z) = \sum_{n=0}^{\infty} a_n z^n \tag{2.29}$$

defines a holomorphic function in its disk of convergence. Also,

$$f'(z) = \sum_{n=0}^{\infty} n a_n z^{n-1}, \tag{2.30}$$

and $f'(z)$ has the same radius of convergence as f.

Proof. Since $\lim_{n\to\infty} n^{\frac{1}{n}} = 1$,

$$\limsup_{n\to\infty} |na_n|^{\frac{1}{n-1}} = (\limsup_{n\to\infty} |a_n|^{\frac{1}{n-1}}) \cdot (\lim_{n\to\infty} n^{\frac{1}{n-1}}) = \limsup_{n\to\infty} |a_n|^{\frac{1}{n}\cdot\frac{n}{n-1}} = \frac{1}{R}. \tag{2.31}$$

Let

$$g(z) = \sum_{n=0}^{\infty} n a_n z^{n-1}, \tag{2.32}$$

and let

$$S_N(z) = \sum_{n=0}^{N} a_n z^n, \qquad f(z) = S_N(z) + E_N(z). \tag{2.33}$$

By direct computation,

$$\begin{aligned} \frac{f(z_0+h) - f(z_0)}{h} - g(z_0) = (&\frac{S_N(z_0+h) - S_N(z_0)}{h} - S_N'(z_0)) \\ &+ (S_N'(z_0) - g(z_0)) + (\frac{E_N(z_0+h) - E_N(z_0)}{h}). \end{aligned} \tag{2.34}$$

Since

$$S_N'(z_0) - g(z_0) = \sum_{n=N+1}^{\infty} n a_n z^n, \tag{2.35}$$

for $N(\epsilon, z_0)$ sufficiently large,

$$|S_N'(z_0) - g(z_0)| \leq \epsilon. \tag{2.36}$$

Meanwhile, by the difference of powers

$$a^n - b^n = (a-b)(a^{n-1} + ... + b^{n-1}), \tag{2.37}$$

$$\sum_{n=N_0+1}^{\infty} a_n \frac{(z_0+h)^n - z_0^n}{h} = \sum_{n=N_0+1}^{\infty} a_n((z_0+h)^n + ... + z_0^n) \leq \sum_{n=N_0+1}^{\infty} n|a_n|(|z_0+h|^n + |z_0|^n), \tag{2.38}$$

and so when $|z_0|$, $|z_0 + h| < R$, for $N_0(\epsilon, z_0)$ sufficiently large,

$$\frac{E_N(z_0+h) - E_N(z_0)}{h} \leq \epsilon. \tag{2.39}$$

Finally, since $\epsilon > 0$ is arbitrary and $S_N(z)$ is a polynomial,

$$\lim_{h\to 0}(\frac{S_N(z_0+h) - S_N(z_0)}{h} - S_N'(z_0)) = 0, \tag{2.40}$$

which proves Theorem 9. □

Corollary 3. *A power series is infinitely complex differentiable in its disk of convergence. Higher derivatives are power series obtained by term wise differentiation.*

It is also possible to have a power series centered at $z_0 \in \mathbb{C}$,

$$\sum_{n=0}^{\infty} a_n (z - z_0)^n. \tag{2.41}$$

Definition 13 (Analytic function)**.** *A function f is said to be analytic at $z_0 \in \Omega$ if there exists $r > 0$ such that*

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n \qquad \forall z \in D_r(z_0), \tag{2.42}$$

and $\sum_{n=0}^{\infty} a_n (z - z_0)^n$ has a positive radius of convergence.

If f is analytic at every point in Ω, then f is said to be analytic in Ω.

3. Harmonic functions on a planar domain

The notes for this section come from [Tay19]. Suppose Ω is an open domain and $f \in C^{\infty}(\Omega)$ is a holomorphic function. Applying $(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y})$ to the Cauchy–Riemann equations (2.11) implies

$$(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y})(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y})f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = 0, \tag{3.1}$$

on the open set $\Omega \subset \mathbb{C}$.

Such a function is called harmonic. More generally, if $\mathcal{O}$ is an open set in $\mathbb{R}^n$, a function $f \in C^2(\mathcal{O})$ is said to be harmonic on $\mathcal{O}$ if $\Delta f = 0$ on $\mathcal{O}$, where

$$\Delta f = \frac{\partial^2 f}{\partial x_1^2} + ... + \frac{\partial^2 f}{\partial x_n^2} = 0. \tag{3.2}$$

Taking the real and imaginary parts of (3.1), $f = u + iv$,

$$\Delta u = 0, \qquad \Delta v = 0. \tag{3.3}$$

Therefore, if $f \in C^{\infty}(\Omega)$ is a holomorphic function, the real and imaginary parts of f are harmonic functions on Ω.

Many domains $\Omega \subset \mathbb{C}$ have the property that if $u \in C^2(\Omega)$ is a real-valued, harmonic function, then there exists a real-valued harmonic function $v \in C^2(\Omega)$ such that $f = u + iv$ is holomorphic on Ω.

Definition 14. *v is said to be the harmonic conjugate of u.*

To obtain the harmonic conjugate, we define integrals along curves in $\mathbb{C}$. A parameterized curve is a function $z(t)$ that maps a closed interval $[a, b] \subset \mathbb{R}$ to the complex plane. A parameterized curve is smooth if $z'(t)$ exists and is continuous on $[a, b]$, $z'(t) \neq 0$ for $t \in [a, b]$. The quantities $z'(a)$ and $z'(b)$ are defined as the right–handed derivative at a and the left–handed derivative at b,

$$z'(a) = \lim_{h \searrow 0} \frac{z(a + h) - z(a)}{h}, \qquad z'(b) = \lim_{h \nearrow 0} \frac{z(b + h) - z(b)}{h}. \tag{3.4}$$

A curve is called piecewise smooth if z is continuous on $[a, b]$ and there exist $a = a_0 < a_1 < ... < a_n = b$ where $z(t)$ is smooth on $[a_k, a_{k+1}]$. In this case the left hand derivative of $z(t)$ at a_k need not equal the right hand derivative there. For example, a stop sign is a piecewise smooth curve.

Definition 15. *Two parameterizations* $z : [a, b] \to \mathbf{C}$ *and* $\tilde{z} : [c, d] \to \mathbf{C}$ *are equivalent if there exists a continuously differentiable bijection* $s \mapsto t(s)$ *such that* $t'(s) > 0$ *and* $\tilde{z}(s) = z(t(s))$.

The condition $t'(s) > 0$ *says precisely that the orientation is preserved, as* s *travels from* c *to* d, $t(s)$ *travels from* a *to* b. *The family of all parameterizations that are equivalent to* $z(t)$ *determines a smooth curve* $\gamma \subset \mathbb{C}$. *We can define a curve* γ^{-} *obtained from* γ *by reversing its orientation. In particular, take* $z^{-}(t) = z(b + a - t)$, $t \in [a, b]$.

A closed curve is a curve that satisfies $z(a) = z(b)$. *A curve is simple if it is not self - intersecting, that is* $z(t) \neq z(s)$ *unless* $s = t$ *or* $s = a$ *and* $t = b$.

Given a smooth curve γ in $\mathbb{C}$, parameterized by $z : [a, b] \to \mathbb{C}$, and f a continuous function on γ, we define the integral of f along γ by

$$\int_{\gamma} f(z)dz = \int_{a}^{b} f(z(t))z'(t)dt. \tag{3.5}$$

This integral is independent of parameterization. Suppose $\tilde{z}$ is an equivalent parameterization,

$$\int_{a}^{b} f(z(t))z'(t)dt = \int_{c}^{d} f(z(t(s)))z'(t(s))t'(s)ds = \int_{c}^{d} f(\tilde{z}(s))\tilde{z}'(s)ds. \tag{3.6}$$

If γ is piecewise smooth then

$$\int_{\gamma} f(z)dz = \sum_{k=0}^{n-1} \int_{a_k}^{a_{k+1}} f(z(t))z'(t)dt. \tag{3.7}$$

The length of the curve γ is given by

$$Length(\gamma) = \int_{a}^{b} |z'(t)|dt = \int_{c}^{d} |z'(t(s))|t'(s)ds = \int_{c}^{d} |\tilde{z}'(s)|ds. \tag{3.8}$$

Now then, given $\alpha = a + ib$ and $z = x + iy$, let $\gamma_{\alpha z}$ denote a path from $a + ib$ to $a + iy$, and then the horizontal line from $a + iy$ to $x + iy$. Next, let $\sigma_{\alpha z}$ denote the horizontal line segment from $a + ib$ to $x + ib$, and then the vertical line segment from $x + ib$ to $x + iy$. Let $R_{\alpha z}$ denote the rectangle bounded for the four line segments.

Proposition 3. *Let* $\Omega \subset \mathbb{C}$ *be open,* $\alpha = a + ib \in \Omega$, *and assume that the following property holds: If* $z \in \Omega$, *then* $R_{\alpha z} \subset \Omega$. *Let* $u \in C^2(\Omega)$ *be harmonic. Then* u *has a harmonic conjugate* $v \in C^2(\Omega)$.

Proof. For $z \in \Omega$, set

$$v(z) = \int_{\gamma_{\alpha z}} (-\frac{\partial u}{\partial y}dx + \frac{\partial u}{\partial x}dy) = \int_{b}^{y} \frac{\partial u}{\partial x}(a, s)ds - \int_{a}^{x} \frac{\partial u}{\partial y}(t, y)dt. \tag{3.9}$$

Also set

$$\tilde{v}(z) = \int_{\sigma_{\alpha z}} (-\frac{\partial u}{\partial y}dx + \frac{\partial u}{\partial x}dy) = -\int_{a}^{x} \frac{\partial u}{\partial y}(t, b)dt + \int_{b}^{y} \frac{\partial u}{\partial x}(x, s)ds. \tag{3.10}$$

By the fundamental theorem of calculus,

$$\frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}(z), \qquad \frac{\partial \tilde{v}}{\partial y}(z) = \frac{\partial u}{\partial x}(z). \tag{3.11}$$

Furthermore, since $R_{\alpha z} \subset \Omega$, by Green's theorem, since u is a harmonic function,

$$\tilde{v}(z) - v(z) = \int_{\partial R_{\alpha z}} (-\frac{\partial u}{\partial y}dx + \frac{\partial u}{\partial x}dy) = \int\int_{R_{\alpha z}} (\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2})dxdy = 0. \tag{3.12}$$

Therefore, u and v satisfy the Cauchy–Riemann equations. □

It is possible to prove this this proposition without Green's theorem.

Proposition 4. *Let $\Omega \subset \mathbb{C}$ be open, $\alpha = a + ib \in \Omega$, and assume the following property holds: If also $z \in \Omega$ then $\gamma_{\alpha z} \subset \mathbb{C}$.*

Let $u \in C^2(\Omega)$ be harmonic. Then u has a harmonic conjugate $v \in C^2(\Omega)$.

Proof. Define v as in (3.9). Then $\frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}$. Also, by (3.9),

$$\frac{\partial v}{\partial y}(z) = \frac{\partial u}{\partial x}(a, y) - \int_a^x \frac{\partial^2 u}{\partial y^2}(t, y)dt = \frac{\partial u}{\partial x}(a, y) + \int_a^x \frac{\partial^2 u}{\partial x^2}(t, y)dt = \frac{\partial u}{\partial x}(z). \tag{3.13}$$

Therefore, u and v satisfy the Cauchy–Riemann equations. □

Proposition 5 (Mean value theorem for harmonic functions)**.** *If $u \in C^2(\Omega)$ is harmonic, $z_0 \in \Omega$, and $\overline{D_r(z_0)} \subset \Omega$, then*

$$u(z_0) = \frac{1}{2\pi} \int_0^{2\pi} u(z_0 + re^{i\theta})d\theta. \tag{3.14}$$

Proof. Since u is a continuous function,

$$\lim_{r \searrow 0} \frac{1}{2\pi} \int_0^{2\pi} u(z_0 + re^{i\theta})d\theta = u(z_0). \tag{3.15}$$

Taking the derivative with respect to r,

$$\frac{d}{dr}\frac{1}{2\pi} \int_0^{2\pi} u(z_0 + re^{i\theta})d\theta = \frac{1}{2\pi} \int_0^{2\pi} u_r(z_0 + re^{i\theta})d\theta. \tag{3.16}$$

By Green's theorem,

$$\frac{1}{2\pi} \int_0^{2\pi} u_r(z_0 + re^{i\theta})d\theta = \frac{1}{2\pi} \int_{D_r(z_0)} \Delta u dx dy = 0. \tag{3.17}$$

This proves (3.14). □

Writing (3.14) in polar coordinates,

$$u(z_0) = \frac{1}{\pi r^2} \int\int_{D_r(z_0)} u(z) dx dy. \tag{3.18}$$

With this, we can establish a maximum principle for harmonic functions.

Proposition 6. *Let $\Omega \subset \mathbb{C}$ be a connected open set. If $u : \Omega \to \mathbb{R}$ is harmonic on Ω, then given $z_0 \in \Omega$,*

$$u(z_0) = \sup_{z \in \Omega} u(z) \quad \Rightarrow \quad u \quad \textit{is constant on} \quad \Omega. \tag{3.19}$$

If, in addition, Ω is bounded and $u \in C(\bar{\Omega})$, then

$$\sup_{z \in \bar{\Omega}} u(z) = \sup_{z \in \partial\Omega} u(z). \tag{3.20}$$

Proof. Equation (3.20) follows from (3.19) if Ω is bounded, since u must achieve a maximum somewhere on $\bar{\Omega}$. Thus, assume there exists $z_0 \in \Omega$ such that the hypotheses of (3.19) hold. Set

$$\mathcal{O} = \{\zeta \in \Omega : u(\zeta) = u(z_0)\}. \tag{3.21}$$

Since $z_0 \in \mathcal{O}$, $\mathcal{O}$ is not empty. Moreover, by continuity, $\mathcal{O}$ is a closed subset of Ω. Moreover, by (3.18), if there exists a a disk of radius ρ, $\overline{D_\rho(\zeta_0)} \subset \Omega$, since u is the supremum, $u(z) = u(\zeta_0)$ for all $z \in D_\rho(\zeta_0)$. □

Corollary 4. *If $f(z)$ is a holomorphic function, and $f \in C^\infty(\Omega)$, given $z_0 \in \Omega$,*

$$|f(z_0)| = \sup_{z \in \Omega} |f(z)| \qquad \Rightarrow f \qquad \text{is constant on} \qquad \Omega. \tag{3.22}$$

If, in addition, Ω is bounded, and $f \in C(\bar{\Omega})$, then

$$\sup_{z \in \bar{\Omega}} |f(z)| = \sup_{z \in \partial\Omega} |f(z)|. \tag{3.23}$$

Proof. If $f = u + iv$, and u and v are harmonic functions, by the product rule,

$$\Delta(u^2 + v^2) = |\nabla u|^2 + |\nabla v|^2. \tag{3.24}$$

Plugging this fact into the proof of Proposition 5,

$$u(z_0)^2 + v(z_0)^2 \leq \frac{1}{\pi r^2} \int_{D_r(z_0)} (|\nabla u(z)|^2 + |\nabla v(z)|^2) dxdy. \tag{3.25}$$

Moreover, equality holds if and only if $|\nabla u| = 0$ and $|\nabla v| = 0$ on $D_r(z_0)$. □

Next, Liouville's theorem for harmonic functions on $\mathbb{C}$.

Proposition 7. *If $u \in C^2(\Omega)$ is bounded and harmonic on all of $\mathbb{C}$, then u is constant.*

Proof. Choose any two points $p, q \in \mathbb{C}$. For all $r > 0$,

$$u(p) - u(q) = \frac{1}{\pi r^2} [\int \int_{D_r(p)} u(z) dxdy - \int_{D_r(q)} u(z) dxdy]. \tag{3.26}$$

Hence,

$$|u(p) - u(q)| \leq \frac{1}{\pi r^2} \int \int_{\Delta(p,q,r)} |u(z)| dxdy, \tag{3.27}$$

where $\Delta(p, q, r)$ is the set of points contained in $D_r(p)$ or $D_r(q)$, but not both. Therefore, $\Delta(p, q, r) \sim r$ as $r \to \infty$. Taking $r \to \infty$ in (9.6), since $|u|$ is bounded, $u(p) - u(q) = 0$. □

Corollary 5. *If $f : \mathbb{C} \to \mathbb{C}$ is holomorphic and bounded, and $f \in C^\infty(\Omega)$, then f is constant.*

Proof. Since f is holomorphic, $f = u + iv$, where u and v are harmonic functions. Since $|f|$ is uniformly bounded, $|u|$ and $|v|$ are uniformly bounded, and therefore, by Proposition 7, u and v are constant. □

4. Cauchy integral formula (Three lectures)

The Cauchy integral formula is a consequence of the mean value theorem for harmonic functions, Proposition 5. Indeed, if $f \in C^\infty(\Omega)$ is a holomorphic function, then since $f = u + iv$, where u and v are harmonic functions, so

$$f(z_0) = u(z_0) + iv(z_0) = \frac{1}{2\pi}\int_0^{2\pi} u(z_0 + re^{i\theta})d\theta + i\frac{1}{2\pi}\int_0^{2\pi} v(z_0 + re^{i\theta})d\theta = \frac{1}{2\pi}\int_0^{2\pi} f(z_0 + re^{i\theta})d\theta. \tag{4.1}$$

However, it is possible to prove (4.1) without making the a priori assumption that $f \in C^\infty(\Omega)$. To do so, define the primitive of a function.

Definition 16 (Primitive)**.** *A primitive for f on the region Ω is a function $F(z)$ that is holomorphic on Ω and such that $F'(z) = f(z)$ for all $z \in \Omega$.*

Remark 2. *A function may only have a primitive on a subset $\mathbb{C}$ but not all of $\mathbb{C}$. For example, the function $f(z) = \frac{1}{z}$ has a primitive on the region $Re(z) > 0$, for example.*

Theorem 10. *If f is a continuous function with primitive F in Ω, and γ is a curve in Ω that begins at w_1 and ends at w_2, then*

$$\int_\gamma f(z)dz = F(w_2) - F(w_1). \tag{4.2}$$

Proof. Let $z(t)$ be some parameterization of γ with $z(a) = w_1$ and $z(b) = w_2$. Then

$$\int_\gamma f(z)dz = \int_a^b f(z(t))z'(t)dt = \int_a^b \frac{d}{dt}F(z(t))dt = F(z(b)) - F(z(a)). \tag{4.3}$$

A similar argument could be made for a piecewise smooth curve. □

Corollary 6. *If γ is a closed curve in an open set Ω and f is continuous and also has a primitive in Ω, then*

$$\int_\gamma f(z)dz = 0. \tag{4.4}$$

Corollary 7. *If f is holomorphic in a region Ω and $f'(z) = 0$ on this region, then f is constant.*

Proof. For any $w \in \Omega$ take a path connecting w_0 and w. □

Theorem 11 (Cauchy integral formula)**.** *If f is holomorphic on the open set $\Omega \subset \mathbb{C}$, and $\overline{D_r(z_0)} \subset \Omega$, then*

$$f(z_0) = \frac{1}{2\pi}\int_0^{2\pi} f(z_0 + re^{i\theta})d\theta = \frac{1}{2\pi i}\int_{\partial D_r(z_0)} \frac{f(\zeta)}{\zeta - z_0}d\zeta. \tag{4.5}$$

Proof. As in the proof of Proposition 5,

$$\lim_{r \searrow 0} \frac{1}{2\pi}\int_0^{2\pi} f(z_0 + re^{i\theta})d\theta = f(z_0). \tag{4.6}$$

Taking a derivative with respect to r,

$$\frac{d}{dr}\frac{1}{2\pi}\int_0^{2\pi} f(z_0 + re^{i\theta})d\theta = \frac{1}{2\pi}\int_0^{2\pi} f'(z_0 + re^{i\theta})e^{i\theta}d\theta = \frac{1}{2\pi i r}\int_{\partial D_r(z_0)} f'(z_0 + \zeta)d\zeta = 0. \tag{4.7}$$

To see why the last equality in (4.7) is true, observe a parameterization with positive orientation of the circle of radius r centered at z_0,

$$C_r(z_0) = \{z \in \mathbb{C} : |z - z_0| = r\}, \tag{4.8}$$

would be the parameterization

$$z(t) = z_0 + re^{it}, \qquad 0 \leq t \leq 2\pi, \tag{4.9}$$

and a parameterization with a negative orientation would be the parameterization

$$z(t) = z_0 + re^{-it}, \qquad 0 \leq t \leq 2\pi. \tag{4.10}$$

□

Corollary 8. *If f is holomorphic on Ω, then $f \in C^{\infty}(\Omega)$.*

Proof. We can compute the derivative of (4.10) directly. Indeed,

$$\frac{f(z+h) - f(z)}{h} = \frac{1}{h}[\int_{\partial D_r(z)} \frac{f(\zeta)}{(\zeta - (z+h))} d\zeta - \int_{\partial D_r(z+h)} \frac{f(\zeta)}{(\zeta - z)}]. \tag{4.11}$$

Now then, for $|h| \ll r$ sufficiently small,

$$\int_{\partial D_r(z+h)} \frac{f(\zeta)}{(\zeta - (z+h))} d\zeta = \int_{\partial D_r(z)} \frac{f(\zeta)}{(\zeta - (z+h))} d\zeta. \tag{4.12}$$

Taking the limit definition of the derivative,

$$\lim_{h \to 0} \frac{1}{h}[\frac{1}{\zeta - (z+h)} - \frac{1}{(\zeta - z)}] dh = \frac{1}{(\zeta - z)^2}. \tag{4.13}$$

Therefore, arguing by induction,

$$(\frac{d}{dz})^n f(z) = \frac{(-1)^n n!}{2\pi i} \int_{\partial D_r(z)} \frac{f(\zeta)}{(\zeta - z)^{n+1}} d\zeta. \tag{4.14}$$

This integral converges for any n.

Equality (4.12) holds due to the fact that a holomorphic function has a primitive in a disk. □

Theorem 12 (Cauchy's theorem for a disk)**.** *A function that is holomorphic in an open disk has a primitive in that disk.*

Proof. Without loss of generality suppose that the open disk is centered at the origin. Then for any $z = x + iy$ in the disk, let γ_z be the contour traveling from the origin to x, and then on to $x + iy$. Then let

$$F(z) = \int_{\gamma_z} f(w) dw. \tag{4.15}$$

The function $F(z)$ is holomorphic in D and $F'(z) = f(z)$. Indeed, by the fundamental theorem of calculus, since $d\zeta = idy$ on the contour from x to $x + iy$,

$$\frac{1}{i} \frac{\partial F}{\partial y} = f(z). \tag{4.16}$$

Meanwhile, by the Cauchy–Riemann equations, since f is holomorphic,

$$\frac{\partial F}{\partial x} = f(x + i0) + \int_0^y \frac{\partial f}{\partial x}(x + is) i ds = f(x + iy) + \int_0^y \frac{\partial f}{\partial y}(x + is) ds = f(z). \tag{4.17}$$

Therefore, F satisfies the Cauchy–Riemann equations and $F'(z) = f(z)$, so F is holomorphic. $\square$

In particular, Theorem 12 implies that if f is holomorphic in a disk,

$$\int_\gamma f(z)dz = 0 \tag{4.18}$$

It is possible to extend Theorem 11 and Corollary 8 to a general open set Ω. The first step is to use Goursat's theorem to show that the integral of a holomorphic function on a closed curve, where f is holomorphic in the interior, is zero. The proof of Goursat's theorem uses the following elementary facts about the integral,

Proposition 8. *Integration of continuous functions over curves satisfies the following properties:*

(1) *If $\alpha, \beta \in \mathbb{C}$ then*

$$\int_\gamma (\alpha f(z) + \beta g(z))dz = \alpha \int_\gamma f(z)dz + \beta \int_\gamma g(z)dz. \tag{4.19}$$

(2) *If γ^- is γ with the reverse orientation then*

$$\int_\gamma f(z)dz = -\int_{\gamma^-} f(z)dz. \tag{4.20}$$

(3)

$$|\int_\gamma f(z)dz| \leq \sup_{z\in\gamma} |f(z)| \cdot length(\gamma). \tag{4.21}$$

Theorem 13 (Goursat's theorem)**.** *If Ω is an open set in $\mathbb{C}$ and $T \subset \Omega$ is a triangle whose interior that is also contained in Ω, then*

$$\int_T f(z)dz = 0, \tag{4.22}$$

whenever f is holomorphic on Ω.

Proof. Let $T^{(0)}$ be the original triangle with positive orientation. Partition the triangle into four smaller triangles $T_1^{(1)}$, $T_2^{(1)}$, $T_3^{(1)}$, and $T_4^{(4)}$. Then there exists $1 \leq j \leq 4$ such that

$$|\int_{T^{(0)}} f(z)dz| \leq 4|\int_{T_j^{(1)}} f(z)dz|. \tag{4.23}$$

Iterating, there exists a unique

$$z_0 \in \cap_{n=0}^\infty T_{j(n)}^{(n)}. \tag{4.24}$$

For z near z_0,

$$f(z) = f(z_0) + f'(z_0)(z - z_0) + o(|z - z_0|). \tag{4.25}$$

Since 1 and $z - z_0$ have a primitive,

$$|\int_{T_{j(n)}^{(n)}} f(z)dz| = |\int_{T_{j(n)}^{(n)}} o(|z - z_0|)dz| \leq 4^{-n} o_n(1). \tag{4.26}$$

Now by (4.23),

$$|\int_{T^{(0)}} f(z)dz| \leq 4^n |\int_{T_{j(n)}^{(n)}} f(z)dz| \leq o_n(1). \tag{4.27}$$

Taking $n \to \infty$ proves the theorem. ☐

Corollary 9. *If f is holomorphic in an open set Ω that contains a rectangle R in its interior, then*

$$\int_R f(z)dz = 0. \tag{4.28}$$

The Cauchy integral formula actually holds for any z in a disk $D_r(z_0) = D$.

Theorem 14. *Suppose f is holomorphic on an open set that contains $\bar{\Omega}$, where Ω is an open set with compact closure. Let $\partial\Omega$ be the boundary of Ω with positive orientation. Then for any $z \in \Omega$,*

$$f(z) = \frac{1}{2\pi i} \int_C \frac{f(\zeta)}{\zeta - z} d\zeta. \tag{4.29}$$

Proof. For a fixed $z \in \Omega$, there exists some $\epsilon(z) > 0$ sufficiently small such that $D_\epsilon(z) \subset \Omega$. Then by Theorem 11,

$$\int_{\partial D_\epsilon(z)} \frac{f(\zeta)}{\zeta - z} d\zeta = f(z). \tag{4.30}$$

If $\partial D_\epsilon(z)$ and ∂D were constructed out of sides of rectangles, then

$$\int_{\partial D_\epsilon(z)} \frac{f(\zeta)}{\zeta - z} d\zeta - \int_{\partial D} \frac{f(\zeta)}{\zeta - z} d\zeta = 0. \tag{4.31}$$

This is not quite true, however, for a fixed $\epsilon > 0$, if f is continuous on a closed curve γ, the difference between a curve constructed out of the sides of rectangles of length at most h and the curve γ, call it R_γ,

$$\int_{R_{h,\gamma}} f(\zeta)d\zeta - \int_\gamma f(\zeta)d\zeta = o_{h,\gamma,f}(1)|\gamma|. \tag{4.32}$$

Taking $h \to 0$ proves the theorem. ☐

Corollary 10. *If f is holomorphic in an open set Ω then f has infinite many complex derivatives in Ω. Moreover, if $C \subset \Omega$ is a closed contour whose interior is contained in Ω, then*

$$f^{(n)}(z) = \frac{n!}{2\pi i} \int_C \frac{f(\zeta)}{(\zeta - z)^{n+1}} d\zeta, \tag{4.33}$$

for all z in the interior of C.

Proof. We compute the difference quotient. Let D be the region in the interior of C.

$$\frac{1}{h}[\int_C \frac{f(\zeta)}{\zeta - (z+h)} d\zeta - \int_C \frac{f(\zeta)}{\zeta - z} d\zeta] = \int_C \frac{f(\zeta)}{(\zeta - (z+h))(\zeta - z)} d\zeta. \tag{4.34}$$

Taking $h \searrow 0$, since $\bar{D}$ is a compact set,

$$f'(z) = \int_C \frac{f(\zeta)}{(\zeta - z)^2} d\zeta. \tag{4.35}$$

Now, by induction, suppose

$$f^{(n-1)}(z) = \frac{(n-1)!}{2\pi i} \int_C \frac{f(\zeta)}{(\zeta - z)^n} d\zeta. \tag{4.36}$$

Taking a difference quotient,

$$\begin{aligned}&\frac{1}{h}\frac{(n-1)!}{2\pi i}[\int_C \frac{f(\zeta)}{(\zeta-(z+h))^n}d\zeta - \int_C \frac{f(\zeta)}{(\zeta-z)^n}d\zeta]\\ &= \frac{(n-1)!}{2\pi i}[\int_C \frac{f(\zeta)}{(\zeta-(z+h))^n(\zeta-z)^n}((\zeta-z)^{n-1}+...+(\zeta-(z+h))^{n-1})d\zeta].\end{aligned} \tag{4.37}$$

Taking the limit as $h \searrow 0$,

$$f^{(n)}(z) = \frac{n!}{2\pi i}\int_C \frac{f(\zeta)}{(\zeta-z)^{n+1}}d\zeta. \tag{4.38}$$

This proves (4.33) by induction. □

Remark 3. *This also implies that holomorphic functions are smooth.*

Also, by direct computation,

Corollary 11. *If f is holomorphic in an open set that contains the closure of a disk centered at z_0 and of radius R, then*

$$|f^{(n)}(z_0)| \le \frac{n!\|f\|_C}{R^n}, \tag{4.39}$$

where

$$\|f\|_C = \sup_{z\in C}|f(z)|. \tag{4.40}$$

The Cauchy integral formula gives a new proof of the fundamental theorem of algebra.

Theorem 15 (Fundamental theorem of algebra)**.** *Every non constant polynomial*

$$P(z) = a_n z^n + ... + a_0 \tag{4.41}$$

has a root in $\mathbb{C}$.

Proof. If $P(z)$ has no roots, then $\frac{1}{P(z)}$ is a bounded, holomorphic function. Indeed,

$$\frac{P(z)}{z^n} \to a_n \tag{4.42}$$

as $|z| \to \infty$. Therefore, $|P(z)| \ge C|z|^n$ when $|z| > R$. Since $\frac{1}{P(z)}$ is holomorphic, this implies $\frac{1}{P(z)}$ is bounded on $\mathbb{C}$, and therefore constant. □

The integral representation formula gives a power series expansion.

Theorem 16. *Suppose f is holomorphic in an open set Ω. If D is a disk centered at z_0 with $\bar{D} \subset \Omega$, then f has a power series expansion*

$$f(z) = \sum_{n=0}^{\infty} a_n(z-z_0)^n, \qquad \textit{for all} \qquad z \in D, \qquad a_n = \frac{f^{(n)}(z_0)}{n!}. \tag{4.43}$$

Thus, if f is holomorphic on an open set $\Omega \subset \mathbb{C}$, then f is analytic on $\Omega \subset \mathbb{C}$, that is, f has a power series representation that converges. By Theorem 9, the converse also holds.

Proof. Take the integral representation

$$f(z) = \frac{1}{2\pi i} \int_C \frac{f(\zeta)}{(\zeta - z)} d\zeta. \tag{4.44}$$

Computing,

$$\frac{1}{\zeta - z} = \frac{1}{\zeta - z_0 - (z - z_0)} = \frac{1}{(\zeta - z_0)(1 - \frac{(z-z_0)}{(\zeta - z_0)})}. \tag{4.45}$$

There exists some $0 < r < 1$ such that $|\frac{z-z_0}{\zeta-z_0}| < r$ for any $\zeta \in C$, so the series

$$\frac{1}{1 - \frac{z-z_0}{\zeta-z_0}} = \sum_{n=0}^{\infty} (\frac{z - z_0}{\zeta - z_0})^n \tag{4.46}$$

converges uniformly for $\zeta \in C$. Therefore,

$$f(z) = \sum_{n=0}^{\infty} (\frac{1}{2\pi i} \int_C \frac{f(\zeta)}{(\zeta - z_0)^{n+1}} d\zeta) \cdot (z - z_0)^n = \sum_{n=0}^{\infty} (\frac{n!}{2\pi i} \int_C \frac{f(\zeta)}{(\zeta - z_0)^{n+1}} d\zeta) \cdot \frac{(z - z_0)^n}{n!} = \sum_{n=0}^{\infty} \frac{f^{(n)}(z_0)}{n!} (z - z_0)^n. \tag{4.47}$$

□

Thus if f is entire then the power series expansion converges on $\mathbb{C}$.

Recall that in general, smooth functions need not have a convergent power series expansion. For example, consider the classical example

$$f(x) = \begin{cases} 0 & \text{if } x \leq 0 \\ e^{-\frac{1}{x^2}} & \text{if } x > 0. \end{cases} \tag{4.48}$$

This function is smooth, but $f^{(n)}(0) = 0$ for every n, so the power series fails to converge for any $x > 0$. Theorem 16 prevents this for holomorphic functions.

Theorem 17. *Suppose f is a holomorphic function in a region Ω that vanishes on a sequence of distinct points with a limit point in Ω. Then $f \equiv 0$.*

Proof. Proof: Suppose $z_0 \in \Omega$ is a limit point for $\{w_k\}_{k=1}^{\infty}$. Make a power series expansion of f in a disk around z_0. If f is not identically zero there exists a smallest m such that $a_m \neq 0$, and $a_n = 0$ for $0 \leq n < m$. Then

$$f(z) = a_m (z - z_0)^m (1 + g(z - z_0)), \tag{4.49}$$

where $g(z - z_0) \to 0$ as $z \to z_0$. But then for all k, $a_m(w_k - z_0)^m \neq 0$ and $1 + g(w_k - z_0) \neq 0$ but $f(w_k) = 0$.

Now let U be the interior of the set of points such that $f(z) = 0$. U is both open and closed. If $z_n \to z$, then $z \in U$. Then f vanishes in a neighborhood of z. So let $V = \Omega - u$. U and V are open and disjoint, $\Omega = U \cup V$. Then Ω connected implies that either U or V is empty. □

Corollary 12. *Suppose f and g are holomorphic in a region Ω and $f(z) = g(z)$ for all z in a non - empty, open subset of Ω, or for z in some sequence of disjoint points with a limit point in Ω. Then $f(z) = g(z)$ throughout Ω.*

Definition 17 (Analytic continuation)**.** *Suppose f and F are analytic in regions Ω and Ω', $\Omega \subset \Omega'$. If $f = F$ on Ω we say that F is an analytic continuation of f into Ω'. By corollary 12, the continuation of F is uniquely determined by f.*

5. Applications of the Cauchy integral formula (Two lectures)

Now we will discuss some more applications of the Cauchy integral formula.

Theorem 18 (Morera's theorem)**.** *Suppose f is a continuous function in an open disk D such that for any triangle T contained in D,*

$$\int_T f(z)dz = 0. \tag{5.1}$$

Then f is holomorphic.

Proof. Let

$$F(z) = \int_{\gamma_z} f(s)ds. \tag{5.2}$$

Then since f is continuous,

$$F(z+h) - F(z) = \int_0^1 f(z + \frac{h}{|h|}ds)hds = f(z)h + o(h). \tag{5.3}$$

Then $F'(z) = f(z)$, so F is holomorphic, and therefore f is holomorphic. □

Morera's theorem may be used to prove the Schwarz reflection principle. The Schwarz reflection principle enables extending holomorphic functions on certain regions to holomorphic functions in larger regions. Suppose $\Omega \subset \mathbb{C}$ is an open set that is symmetric across the real line, that is,

$$z \in \Omega \Leftrightarrow \bar{z} \in \Omega. \tag{5.4}$$

Let $I = \Omega \subset \mathbb{R}$. I is the interior of the part of the boundary of Ω^+ and Ω^- that lies on the real axis.

Theorem 19 (Symmetry principle)**.** *If $f^+(z)$ and $f^-(z)$ are holomorphic functions in Ω^+ and Ω^- that extend continuously to I, and $f^+(x) = f^-(x)$ for all $x \in I$, then*

$$f(z) = \begin{cases} f^+(z) & \text{if } z \in \Omega^+, \\ f^-(z) & \text{if } z \in \Omega^-, \\ f^+(z) = f^-(z) & \text{if } z \in I \end{cases} \tag{5.5}$$

is holomorphic on all of Ω.

Proof. By Morera's theorem, it suffices to show that

$$\int_T f(z)dz = 0 \tag{5.6}$$

for any triangle $T \subset \Omega$. If a triangle lies above or below the real axis, then (5.6) holds since $f^+(z)$ and $f^-(z)$ are holomorphic. If T has a side or a vertex on the real axis, then (5.6) holds because by continuity, (5.6) may be approximated by integrals on triangles slightly above or below the real axis. If T overlaps the real axis, then T may be split into three triangles whose edges lie along the real axis. □

Theorem 20 (Schwarz reflection principle)**.** *Suppose that f is a holomorphic function on Ω^+ that extends continuously to I, and that f is real valued on I. Then there exists $F(z)$ that is holomorphic on Ω such that $F(z) = f(z)$ on Ω^+.*

Proof. For $z \in \Omega^-$ let

$$F^-(z) = \overline{f(\bar{z})}. \tag{5.7}$$

Since $f(z)$ is real valued on I, $\overline{f(x)} = f(x)$ for all $x \in I$. To prove $F^-(z)$ is holomorphic on Ω^-, suppose $z_0 \in \Omega^+$. Then there exists $r > 0$ such that if $|z - z_0| < r$,

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n, \tag{5.8}$$

so for $|\bar{z} - \bar{z}_0| < r$,

$$F^-(z) = \overline{f(\bar{z})} = \sum_{n=0}^{\infty} \bar{a}_n (z - \bar{z}_0)^n, \tag{5.9}$$

which proves $F^-(z)$ is holomorphic for $|z - \bar{z}_0| < r$. Then by the symmetry principle, the proof is complete. □

Morera's theorem implies holomorphicity for uniform limits of holomorphic functions.

Theorem 21 (Sequence of holomorphic functions). *If $\{f_n\}_{n=1}^{\infty}$ is a sequence of holomorphic functions that converges uniformly to a function f on every compact subset of Ω, then f is holomorphic on Ω.*

Proof. If $T \subset \Omega$ is a triangle then for any n,

$$\int_T f_n(z) dz = 0. \tag{5.10}$$

Since $f_n \to f$ uniformly on any compact set,

$$\int_T f(z) dz = 0. \tag{5.11}$$

□

Theorem 22. *Under the hypotheses of Theorem 21, the sequence of derivatives $\{f_n'\}_{n=1}^{\infty}$ converges uniformly to f' on every compact subset of Ω.*

Proof. Suppose $f_n(z)$ converges uniformly on every compact subset of Ω. Let Ω_δ be the set of points that are distance $\delta > 0$ from the boundary of Ω,

$$\Omega_\delta = \{z \in \Omega : \bar{D}_\delta(z) \subset \Omega\}. \tag{5.12}$$

For every $z \in \Omega_\delta$, if $F(z)$ is holomorphic and $\overline{D_\delta(z)} \subset \Omega$, then

$$F'(z) = \frac{1}{2\pi i} \int_{C_\delta(z)} \frac{F(\zeta)}{(\zeta - z)^2} d\zeta \leq \frac{1}{2\pi} \int_{C_\delta(z)} \frac{|F(\zeta)|}{|\zeta - z|^2} |d\zeta| \leq \frac{1}{2\pi} \sup_{\zeta \in C_\delta(z)} |F(z)| \frac{1}{\delta^2} 2\pi\delta. \tag{5.13}$$

Therefore,

$$\sup_{\zeta \in \Omega_\delta} |F'(z)| \leq \frac{1}{\delta} \sup_{\zeta \in \Omega} |F(\zeta)|. \tag{5.14}$$

In particular, if $f_n(z) \to f(z)$ uniformly on any compact subset of Ω, then

$$\lim_{n \to \infty} |f_n'(z) - f'(z)| = 0. \tag{5.15}$$

This same analysis may be applied to the other derivatives of $f_n(z)$. □

One particular example of this is the power series expansion of functions. Showing that a power series converges on some disk implies that the power series is a holomorphic function. More generally, one may construct a holomorphic function

$$F(z) = \sum_{n=1}^{\infty} f_n(z). \tag{5.16}$$

A function may also be defined as an integral. Suppose

$$f(z) = \int_a^b F(z,s)ds, \tag{5.17}$$

where for any s, $F(z,s)$ is holomorphic in z and continuous in s. Making a change of variables, suppose $a = 0$ and $b = 1$.

Theorem 23. *Let $F(z,s)$ be defined for $(z,s) \in \Omega \times [0,1]$, where Ω is an open set in $\mathbb{C}$. Suppose*

1. *$F(z,s)$ is holomorphic in z for each s.*
2. *F is continuous on $\Omega \times [0,1]$.*

Then the function

$$f(z) = \int_0^1 F(z,s)ds, \tag{5.18}$$

is holomorphic.

Proof. Suppose T is a triangle in $D \subset \Omega$, where D is a disk. It suffices to show that for any such triangle,

$$\int_T \int_0^1 F(z,s)dsdz = 0. \tag{5.19}$$

Approximating an integral with a Riemann sum, let

$$f_n(z) = \frac{1}{n} \sum_{k=1}^{n} F(z, \frac{k}{n}). \tag{5.20}$$

Then for any n, $f_n(z)$ is holomorphic. Now since $D \times [0,1]$ is compact, for any $\epsilon > 0$ there exists $\delta > 0$ such that if $|(z,s) - (z, \frac{k}{n})| < \delta$, then

$$|F(z,s) - F(z, \frac{k}{n})| < \epsilon. \tag{5.21}$$

Then,

$$|f_n(z) - f(z)| = |\sum_{k=1}^{n} \int_{\frac{k-1}{n}}^{\frac{k}{n}} F(z, \frac{k}{n}) - F(z,s)ds| \leq \sum_{k=1}^{n} \int_{\frac{k-1}{n}}^{\frac{k}{n}} |F(z, \frac{k}{n}) - F(z,s)|ds < \epsilon, \tag{5.22}$$

so $|\int_T f(z)dz| < \epsilon \cdot length(T)$. Therefore $f_n(z) \to f(z)$ uniformly on D. □

Now recall the Weierstrass approximation theorem that any continuous function on a compact interval may be approximated uniformly by polynomials. One means of doing this is by the power series approximation. However, Morera's theorem implies that this is not enough on the complex plane. For example, consider the function $f(z) = \frac{1}{z}$, where K is the unit circle. Then

$$\int_K f(z)dz = 2\pi i, \tag{5.23}$$

but for any polynomial $p(z)$,

$$\int_K p(z)dz = 0. \tag{5.24}$$

Therefore, Runge's approximation theorem includes the rational functions,

$$f(z) = \frac{1}{(z - z_0)^n}. \tag{5.25}$$

Theorem 24 (Runge's approximation theorem)**.** *Any function holomorphic in a neighborhood of a compact set K can be approximated uniformly on K by rational functions whose singularities are in K^c.*

If K^c is connected, any function holomorphic in a neighborhood of K can be approximated uniformly on K by polynomials.

The proof of Theorem 24 uses an integral representation formula.

Lemma 3. *Suppose f is holomorphic on an open set Ω and $K \subset \Omega$ is compact. Then there exists finitely many segments γ_1, ..., γ_N in $\Omega - K$ such that*

$$f(z) = \sum_{n=1}^{N} \frac{1}{2\pi i} \int_{\gamma_n} \frac{f(\zeta)}{\zeta - z} d\zeta. \tag{5.26}$$

Proof. Make a grid of side length $d = c \cdot dist(K, \Omega^c)$ with $c < \frac{1}{\sqrt{2}}$ with sides parallel to the axes. Let $\mathcal{Q} = \{Q_1, ..., Q_M\}$ be the collection of squares that intersect K, and give the boundaries positive orientations. Let γ_n be the boundaries of the squares.

Each γ_n does not intersect K. So if $z \in Q_j$ and is not on the boundary, then

$$\frac{1}{2\pi i} \int_{\partial Q_j} \frac{f(\zeta)}{\zeta - z} d\zeta = f(z), \tag{5.27}$$

and for all $m \neq j$,

$$\frac{1}{2\pi i} \int_{\partial Q_m} \frac{f(\zeta)}{\zeta - z} d\zeta = 0, \tag{5.28}$$

and therefore (5.26) holds. By continuity (5.26) also holds for z on the boundary. □

Lemma 4. *For any line segment γ contained entirely in $\Omega - K$, there exists a sequence of rational functions with singularities on γ that approximate the integral*

$$\int_\gamma \frac{f(\zeta)}{\zeta - z} d\zeta, \tag{5.29}$$

uniformly on K.

Proof. If $\gamma(t) : [0, 1] \to \mathbb{C}$ is a parameterization for γ, then

$$\int_\gamma \frac{f(\zeta)}{\zeta - z} d\zeta = \int_0^1 \frac{f(\gamma(t))}{\gamma(t) - z} \gamma'(t) dt. \tag{5.30}$$

Now take

$$F(z, t) = \frac{f(\gamma(t))}{\gamma(t) - z}. \tag{5.31}$$

Because $\gamma(t)$ does not intersect K, $F(z,t)$ is jointly continuous on $K \times [0,1]$. Also since K is compact, for any $\epsilon > 0$ there exists $\delta > 0$ such that

$$|t_1 - t_2| < \delta \Rightarrow |F(z,t_1) - F(z,t_2)| < \epsilon. \tag{5.32}$$

Each of the Riemann sums is a rational function with singularities on γ, which implies Lemma 4. □

Observe that Lemmas 3 and 4 imply the first part of Theorem 24. To prove the second part of the theorem,

Lemma 5. *If K^c is connected and $z_0 \notin K$, then the function $\frac{1}{z-z_0}$ can be approximated uniformly on K by polynomials.*

Proof. Choose z_1 to be outside a disk containing K and let $z \in K$. Then,

$$\frac{1}{z - z_1} = -\frac{1}{z_1}\frac{1}{1-\frac{z}{z_1}} = -\sum_{n=1}^{\infty}\frac{z^n}{z_1^{n+1}}. \tag{5.33}$$

It suffices to prove that $\frac{1}{z-z_0}$ may be approximated on K uniformly by a polynomial in $\frac{1}{z-z_1}$. Indeed, if

$$\frac{1}{(z-z_0)} = \sum_{m=0}^{\infty} b_m \frac{1}{(z-z_1)^m}, \tag{5.34}$$

then

$$\frac{a_n}{(z-z_0)^n} = a_n(\sum_{m=0}^{\infty}\frac{b_m}{(z-z_1)^m})^n, \tag{5.35}$$

and truncating to $\sum_{m=0}^{M}\frac{b_m}{(z-z_1)^m}$ for M large gives the estimate.

Now suppose $\gamma(t)$ is a curve in K^c that connects z_0 to z_1, with $\gamma(0) = z_0$ and $\gamma(1) = z_1$. Let $\rho = \frac{1}{2}dist(K,\gamma) > 0$, since γ and K are compact.

Now choose a sequence of points $0 \le j \le l$ such that $w_0 = z_0$, $w_l = z_1$, and for $0 \le j \le l-1$, $|w_j - w_{j+1}| < \rho$. Now if $z \in K$ and $|w - w'| < \rho$, then $\frac{1}{z-w}$ can be approximated uniformly on K by polynomials in $\frac{1}{z-w'}$. Note that

$$\frac{1}{z-w} = \frac{1}{z-w'} \cdot \frac{1}{1-\frac{w-w'}{z-w'}} = \sum_{n=0}^{\infty}\frac{(w-w')^n}{(z-w')^{n+1}}. \tag{5.36}$$

This sum converges uniformly for $z \in K$, which proves the claim. Iterating this argument, one may walk back from z_0 to z_1 in finitely many steps. This proves the lemma. □

6. Zeros and singularities

Definition 18 (Point singularity)**.** *A point singularity is $z_0 \in \mathbb{C}$ such that f is defined in a neighborhood of z_0, but not at the point z_0. Such a singularity is also called an isolated singularity.*

All the point singularities are divided into three parts: the removable singularities, the poles, and the essential singularities.

Definition 19 (Removable singularity)**.** *If there exists $c \in \mathbb{C}$ such that $g(z) = f(z)$ for $z \neq z_0$ and $g(z) = c$ when $z = z_0$ is holomorphic at z_0, then f has a removable singularity at z_0.*

For example, take the function $f(z) = z$ when $z \neq 0$, but f is not defined at 0. If p is a removable singularity, then f is bounded near p. The converse is also true.

Theorem 25 (Riemann's theorem on removable singularities)**.** *If $z_0 \in \Omega$ and f is holomorphic on $\Omega \setminus \{z_0\}$ and bounded, then z_0 is a removable singularity.*

Proof. Consider the function $g : \Omega \to \mathbb{C}$ defined by

$$g(z) = (z - z_0)^2 f(z), \qquad z \in \Omega \setminus \{z_0\}, \qquad g(z_0) = 0. \tag{6.1}$$

Since f is bounded, g is continuous on Ω. Also, g is complex differentiable at each point of Ω, since

$$g'(z) = 2(z - z_0)f(z) + (z - z_0)^2 f'(z), \qquad z \in \Omega \setminus \{z_0\}, \qquad g'(z_0) = 0. \tag{6.2}$$

Therefore, by Goursat's theorem, g is holomorphic on Ω, so on a neighborhood U of z_0, g has the convergent power series

$$g(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n, \qquad z \in U. \tag{6.3}$$

Since $g(z_0) = g'(z_0) = 0$, $a_0 = a_1 = 0$, so

$$g(z) = (z - z_0)^2 h(z), \qquad h(z) = \sum_{n=0}^{\infty} a_{n+2}(z - z_0)^n, \qquad z \in U. \tag{6.4}$$

Comparing (6.4) to (6.1), $h(z) = f(z)$ on $U \setminus \{z_0\}$, so set

$$\tilde{f}(z) = f(z), \qquad z \in \Omega \setminus \{z_0\}, \qquad \tilde{f}(z_0) = h(z_0) = a_2. \tag{6.5}$$

□

Definition 20 (Pole)**.** *If the function*

$$g(z) = \begin{cases} \frac{1}{f(z)} & \text{if } 0 < |z - z_0| < r \\ 0 & \text{if } z = z_0 \end{cases} \tag{6.6}$$

is holomorphic in a neighborhood of z_0, then f has a pole at z_0.

For example take $f(z) = \frac{1}{z}$. Observe that a necessary condition for (6.6) to be true is that $f(z) \to \infty$ as $z \to 0$. This is also a sufficient condition, because if $f(z) \to \infty$ as $z \to z_0$, then there exists a neighborhood near z_0 for which $|f(z)| \geq 1$, and therefore $\frac{1}{f(z)}$ is holomorphic. Also, $\frac{1}{f(z)} \to 0$ if and only if $|f(z)| \to \infty$.

Examining (6.6), it is apparent that the study of poles and the study of zeros of holomorphic functions is very closely intertwined.

Theorem 26. *Suppose that f is holomorphic in a connected set, has a zero at a point $z_0 \in \Omega$, and is not identically zero. Then there exists an open neighborhood of $z_0 \in U \subset \Omega$ such that $g(z)$ is a non - vanishing holomorphic function on U, and there exists a positive integer n such that*

$$f(z) = (z - z_0)^n g(z) \qquad \forall z \in U. \tag{6.7}$$

Proof. By analytic continuation, the points where $f(z) = 0$ are isolated unless $f \equiv 0$. If $f(z_0) = 0$ then either $f \equiv 0$ or there exists an open set $z_0 \in U$ such that $f(z) \neq 0$ for all $z \in \Omega$, $z \neq z_0$. Therefore, Make a power series expansion about z_0,

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n. \tag{6.8}$$

This power series converges absolutely on some open neighborhood $z_0 \in U$. Since f is not identically zero there exists a smallest n such that $a_n \neq 0$, and

$$f(z) = (z - z_0)^n [a_n + a_{n+1}(z - z_0) + ...] = (z - z_0)^n g(z). \tag{6.9}$$

This n is unique. Indeed, suppose there exists $m \neq n$ such that

$$f(z) = (z - z_0)^n g(z) = (z - z_0)^m h(z), \qquad g(z_0) \neq 0, \qquad h(z_0) \neq 0. \tag{6.10}$$

If $m > n$ then

$$g(z) = (z - z_0)^{m-n} h(z). \tag{6.11}$$

Letting $z \to z_0$, we have $g(z_0) = 0$, which gives a contradiction. Thus $m = n$ and $g(z) = h(z)$. n is called the order of the zero at z_0. □

Definition 21 (Multiplicity)**.** *If f has a zero of order n at z_0 then n is called the multiplicity of the zero at z_0. If f has a zero of order one at z_0 then it is a simple zero.*

We can apply this analysis to poles at z_0 in a deleted neighborhood of z_0.

Definition 22 (Deleted neighborhood of z_0)**.** *The deleted neighborhood of z_0 is the open disk centered at z_0 minus the point z_0,*

$$\{z : 0 < |z - z_0| < r\}. \tag{6.12}$$

Theorem 27. *If f has a pole at $z_0 \in \Omega$, then in a neighborhood of the point, there exists a non–vanishing holomorphic function h and a unique positive integer n such that in a deleted neighborhood of z_0,*

$$f(z) = (z - z_0)^{-n} h(z). \tag{6.13}$$

The number n is called the order of multiplicity of the pole. If $n = 1$ the pole is called a simple pole.

The order of the pole describes the rate the function grows as $z \to z_0$.

Theorem 28. *If f has a pole of order n at z_0, then*

$$f(z) = \frac{a_{-n}}{(z - z_0)^n} + \frac{a_{-n+1}}{(z - z_0)^{n-1}} + ... + \frac{a_{-1}}{(z - z_0)} + G(z), \tag{6.14}$$

where $G(z)$ is holomorphic in a neighborhood of z_0.

Proof. By Theorem 27,

$$f(z) = (z - z_0)^{-n} h(z). \tag{6.15}$$

Making a power series expansion of $h(z)$ near z_0,

$$h(z) = A_0 + A_1(z - z_0) + ..., \tag{6.16}$$

which proves the theorem. □

Definition 23 (Principal part)**.** *The sum*

$$\frac{a_{-n}}{(z - z_0)^n} + ... + \frac{a_{-1}}{(z - z_0)}, \tag{6.17}$$

is called the principal part of f at the pole z_0.

Definition 24 (Residue)**.** *The constant a_{-1} is the residue of f at the pole,*

$$res_{z_0} f = a_{-1}. \tag{6.18}$$

If $n > 1$ then $\frac{a_{-n}}{(z-z_0)^n}$ has a primitive in a deleted neighborhood of z_0. Therefore, if $P(z)$ is the principal part of f, and C is a circle centered at z_0, then

$$\frac{1}{2\pi i} \int_C P(z) dz = a_{-1}. \tag{6.19}$$

If f has a simple pole at z_0 then

$$res_{z_0} f = \lim_{z \to z_0} (z - z_0) f(z). \tag{6.20}$$

More generally,

Theorem 29. *If f has a pole of order n at z_0, then*

$$res_{z_0} f = \lim_{z \to z_0} \frac{1}{(n-1)!} (\frac{d}{dz})^{n-1} (z - z_0)^n f(z). \tag{6.21}$$

Proof. By (6.14),

$$(z - z_0)^n f(z) = a_{-n} + a_{-n+1}(z - z_0) + ... + a_{-1}(z - z_0)^{n-1} + G(z)(z - z_0)^n. \tag{6.22}$$

□

Proposition 9. *If $f : \mathbb{C} \to \mathbb{C}$ is holomorphic and $|f(z)| \to \infty$ as $|z| \to \infty$, then $f(z)$ is a polynomial.*

Proof. Define the function $g : \mathbb{C} \setminus \{0\} \to \mathbb{C}$ defined by

$$g(z) = f(\frac{1}{z}). \tag{6.23}$$

Since $|g(z)| \to \infty$ as $z \to \infty$, g has a pole at 0. Then, by Theorem 27,

$$g(z) = z^{-k} G(z), \tag{6.24}$$

on $\mathbb{C} \setminus \{0\}$ for some $k \in \mathbb{Z}^+$, with G holomorphic on $\mathbb{C}$ and $G(0) \neq 0$. Then,

$$G(z) = \sum_{j=0}^{k-1} g_j z^j + z^k h(z), \tag{6.25}$$

and therefore,

$$g(z) = \sum_{j=0}^{k-1} g_j z^{j-k} + h(z). \tag{6.26}$$

Therefore,

$$f(z) = \sum_{j=0}^{k-1} g_j z^{k-j} + h(\frac{1}{z}), \tag{6.27}$$

so

$$f(z) - \sum_{j=0}^{k-1} g_j z^{k-j}, \tag{6.28}$$

is holomorphic on $\mathbb{C}$, and approaches $h(0)$ as $|z| \to \infty$. Therefore, by Liouville's theorem, the difference is constant, so $f(z)$ is a polynomial. □

If an isolated singularity is not a removable singularity or a pole, then it is an essential singularity. An essential singularity will have $a_{-n} \neq 0$ for infinitely many $n > 0$. One example of this is $h(z) = e^{1/z}$. Observe that if $z \to 0$ along the positive real axis then $h(z) \to \infty$. If $z \to 0$ along the negative real axis then $h(z) \to 0$.

Proposition 10 (Casorati-Weierstrass theorem)**.** *Suppose $f : \Omega \setminus \{z_0\} \to \mathbb{C}$ has an essential singularity at z_0. Then for any neighborhood U of z_0, the image of $U \setminus \{z_0\}$ is dense in $\mathbb{C}$.*

Proof. Suppose there exists a neighborhood U of z_0 such that the image of $U \setminus \{z_0\}$ omits a neighborhood of $w_0 \in \mathbb{C}$. Replacing $f(z)$ by $f(z) - w_0$, suppose without loss of generality $w_0 = 0$. Then

$$g(z) = \frac{1}{f(z)} \tag{6.29}$$

is holomorphic and bounded on $U \setminus \{z_0\}$, so $g(z)$ has a removable singularity at z_0, so $\tilde{g}(z)$ has a holomorphic extension on U. If $\tilde{g}(z_0) \neq 0$, then z_0 is a removable singularity for f. If $\tilde{g}(z_0) = 0$, then z_0 is a pole of f. □

Remark 4. *In fact, with at most one exception f takes on every complex value infinitely often.*

7. The residue formula (Two lectures)

Theorem 30. *Suppose that f is holomorphic in an open set containing a circle C and its interior, except for a pole at z_0 inside C. Then*

$$\int_C f(z)dz = 2\pi i \cdot res_{z_0} f(z). \tag{7.1}$$

Proof. By (6.14), for some $0 < n < \infty$,

$$f(z) = \frac{a_{-n}}{(z-z_0)^n} + \frac{a_{-n+1}}{(z-z_0)^{n-1}} + ... + \frac{a_{-1}}{(z-z_0)} + G(z). \tag{7.2}$$

Since $G(z)$ is holomorphic, G has a primitive, so $\int_C G(z)dz = 0$. Also, for $n > 1$, $n \in \mathbb{Z}$, $(z-z_0)^{-n}$ has the primitive $-\frac{1}{(n-1)(z-z_0)^{n-1}}$. Therefore,

$$\frac{1}{2\pi i} \int_C \frac{a_{-1}}{z - z_0} dz = a_{-1} = \frac{1}{2\pi i} \int_C f(z)dz, \tag{7.3}$$

which proves the Theorem. □

Corollary 13. *Suppose that f is holomorphic in an open set containing a circle C and its interior, except for z_1, ..., z_N inside C. Then*

$$\frac{1}{2\pi i} \int_C f(z)dz = \sum_{k=1}^{N} res_{z_k} f(z). \tag{7.4}$$

Proof. We can prove this by deforming the contour to small circles around $z_1, ..., z_N$. □

Corollary 14. *Suppose that f is holomorphic in an open set containing a toy contour γ and its interior, except for poles at the points z_1, ..., z_N inside γ. Then*

$$\frac{1}{2\pi i}\int_{\gamma} f(z)dz = \sum_{k=1}^{N} res_{z_k} f. \tag{7.5}$$

Definition 25 (Residue formula)**.** *Equation* (7.5) *is called the residue formula.*

The classic application of the residue formula is the computation of the integral of $\frac{1}{1+x^2}$. Of course, by trig substitution we can show that this integral has the anti–derivative $\tan^{-1}(x)$.

Lemma 6.

$$\int_{-\infty}^{\infty} \frac{1}{1+x^2} dx = \pi. \tag{7.6}$$

Proof. Since the integral is absolutely convergent,

$$\int_{-\infty}^{\infty} \frac{1}{1+x^2} dx = \lim_{R\to\infty} \int_{-R}^{R} \frac{1}{1+x^2} dx. \tag{7.7}$$

Now define a contour γ_R that travels along the real line from $-R$ to R, and then is a semicircle of radius R. Then,

$$\int_{-R}^{R} \frac{1}{1+x^2} dx = \int_{\gamma_R} \frac{1}{1+z^2} dz - \int_{0}^{\pi} \frac{1}{1+R^2 e^{2i\theta}} Rie^{i\theta} d\theta. \tag{7.8}$$

Then for large R,

$$\int_{0}^{\pi} \frac{1}{1+R^2 e^{2i\theta}} Rie^{i\theta} d\theta = O(\frac{1}{R}). \tag{7.9}$$

By the residue formula,

$$\int_{-R}^{R} \frac{1}{1+x^2} dx = 2\pi i \cdot res_{z=i} \frac{1}{1+z^2} + O(\frac{1}{R}) = 2\pi i \cdot \frac{1}{2i} + O(\frac{1}{R}) = \pi + O(\frac{1}{R}). \tag{7.10}$$

Taking the limit as $R \to \infty$ proves the theorem. □

Lemma 7. *For $0 < a < 1$,*

$$\int_{-\infty}^{\infty} \frac{e^{ax}}{1+e^x} dx = \frac{\pi}{\sin(\pi a)}. \tag{7.11}$$

Proof. For $0 < a < 1$, integral (7.11) converges absolutely, so

$$\int_{-\infty}^{\infty} \frac{e^{ax}}{1+e^x} dx = \lim_{R\to\infty} \int_{-R}^{R} \frac{e^{ax}}{1+e^x} dx. \tag{7.12}$$

Now let γ_R be a contour traveling along the real axis from $-R$ to R, then vertically to $R+2\pi i$, then traveling along a line from $R + 2\pi i$ to $-R + 2\pi i$, and then down to $-R$. By the residue formula

$$\int_{\gamma_R} \frac{e^{az}}{1+e^z} dz = res_{z=\pi i} \frac{e^{az}}{1+e^z} = \frac{e^{\pi i a}}{e^{\pi i}} = -e^{i\pi a}. \tag{7.13}$$

Computing the contour integral,

$$\int_{0}^{2\pi} \frac{e^{a(R+is)}}{1+e^{(R+is)}} i ds \leq \frac{2\pi}{e^{R(1-a)}} = O(e^{R(a-1)}), \tag{7.14}$$

$$\int_{2\pi}^{0} \frac{e^{a(-R+is)}}{1+e^{(-R+is)}} i ds \leq \frac{2\pi}{e^{aR}} = O(e^{-aR}), \tag{7.15}$$

and

$$\int_{R}^{-R} \frac{e^{a(2\pi i+s)}}{1+e^{(2\pi i+s)}} ds = -e^{2\pi i a} \int_{-R}^{R} \frac{e^{as}}{1+e^{s}} ds. \tag{7.16}$$

Therefore, by (7.13)–(7.16),

$$-e^{i\pi a} = \frac{1}{2\pi i} \int_{\gamma_R} \frac{e^{az}}{1+e^{z}} dz = \frac{1-e^{2\pi i a}}{2\pi i} \int_{-R}^{R} \frac{e^{ax}}{1+e^{x}} dx + O(e^{R(a-1)}) + O(e^{-aR}). \tag{7.17}$$

Doing some algebra and taking the limit as $R \to \infty$,

$$\int_{-\infty}^{\infty} \frac{e^{ax}}{1+e^{x}} dx = -2\pi i \cdot \frac{e^{i\pi a}}{1-e^{2\pi i a}} = \pi \frac{-2i}{e^{-i\pi a} - e^{i\pi a}} = \frac{\pi}{\sin(\pi a)}. \tag{7.18}$$

This proves the lemma. □

Lemma 8.

$$\int_{-\infty}^{\infty} \frac{e^{-2\pi i x\xi}}{\cosh \pi x} dx = \frac{1}{\cosh(\pi\xi)}. \tag{7.19}$$

Here $\cosh(x) = \frac{e^{x}+e^{-x}}{2}$.

Proof. Again by absolute convergence,

$$\int_{-\infty}^{\infty} \frac{e^{-2\pi i x\xi}}{\cosh(\pi x)} dx = \lim_{R\to\infty} \int_{-R}^{R} \frac{e^{-2\pi i x\xi}}{\cosh(\pi x)} dx. \tag{7.20}$$

Now let γ_R be a contour traveling along the real line from $-R$ to R, then up to $R+2i$, then over to $-R+2i$, and then down to $-R$. Now by some algebra, $|e^{\pi z}| = |e^{-\pi z}|$ implies $Re(z) = 0$. Then $e^{\pi i s} + e^{-\pi i s} = \cos(\pi s)$, so the poles lie on $z = \frac{i}{2}$ and $z = \frac{3i}{2}$. Therefore,

$$\begin{aligned} \int_{\gamma_R} \frac{e^{-2\pi i z\xi}}{\cosh(\pi z)} dz &= 2\pi i \cdot res_{z=\frac{i}{2}} \frac{e^{-2\pi i z\xi}}{\cosh(\pi z)} + 2\pi i \cdot res_{z=\frac{3i}{2}} \frac{e^{-2\pi i z\xi}}{\cosh(\pi z)} \\ &= 2\pi i \cdot \frac{e^{\pi\xi}}{\pi i} - 2\pi i \frac{e^{3\pi\xi}}{\pi i} = 2(e^{\pi\xi} - e^{3\pi\xi}) = -4e^{2\pi\xi}\sinh(\pi\xi). \end{aligned} \tag{7.21}$$

Computing (7.21) explicitly,

$$\int_{0}^{2\pi} \frac{e^{-2\pi i(R+is)\xi}}{\cosh(\pi(R+is))} i ds \leq \frac{4\pi}{e^{\pi R}} = O(\frac{1}{e^{\pi R}}), \tag{7.22}$$

$$\int_{2\pi}^{0} \frac{e^{-2\pi i(R+is)\xi}}{\cosh(\pi(R+is))} i ds \leq \frac{4\pi}{e^{\pi R}} = O(\frac{1}{e^{\pi R}}), \tag{7.23}$$

and

$$\int_{-R}^{R} \frac{e^{-2\pi i(2i+s)\xi}}{\frac{1}{2}(e^{\pi(s+2i)} + e^{-\pi(s+2i)}} ds = -e^{4\pi\xi} \int_{-R}^{R} \frac{e^{-2\pi i s\xi}}{\cosh(\pi s)} ds. \tag{7.24}$$

Therefore,

$$(1-e^{4\pi\xi})(\int_{-R}^{R} \frac{e^{-2\pi i x\xi}}{\cos(\pi x)} dx) = -2e^{2\pi\xi}(e^{\pi\xi} - e^{-\pi\xi}) + O(e^{-\pi R}). \tag{7.25}$$

Doing some algebra, using the difference of squares, and letting $R \to \infty$,

$$\int_{-\infty}^{\infty} \frac{e^{-2\pi i x\xi}}{\cos(\pi x)} dx = \frac{-2(e^{\pi\xi} - e^{-\pi\xi})}{e^{-2\pi\xi} - e^{2\pi\xi}} = \frac{2}{e^{\pi\xi} + e^{-\pi\xi}} = \frac{1}{\cosh(\pi\xi)}. \tag{7.26}$$

This proves the lemma. □

Theorem 31.

$$\int_0^{\infty} \frac{1 - \cos x}{x^2} dx = \frac{\pi}{2}. \tag{7.27}$$

Proof. Since $1 - \cos x = O(x^2)$ when x is near zero, this integral is well - defined, and because the integrand is symmetric across the origin,

$$\int_0^{\infty} \frac{1 - \cos x}{x^2} dx = \frac{1}{2} \lim_{R \to \infty} [\int_{-R}^{-\frac{1}{R}} \frac{1 - \cos x}{x^2} dx + \int_{\frac{1}{R}}^{R} \frac{1 - \cos x}{x^2} dx]. \tag{7.28}$$

Now then, because $\sin x$ is an odd function,

$$(7.28) = \frac{1}{2} \lim_{R \to \infty} [\int_{-R}^{-\frac{1}{R}} \frac{1 - e^{ix}}{x^2} dx + \int_{\frac{1}{R}}^{R} \frac{1 - e^{ix}}{x^2} dx]. \tag{7.29}$$

Now for any R let A_R be the contour traveling along the real axis from $-R$ to $-\frac{1}{R}$, then traveling along the semicircle $\frac{1}{R} e^{i\theta}$, with θ traveling from $-\pi$ to 0, then traveling along the real axis from $\frac{1}{R}$ to R, and finally traveling along the semicircle $Re^{i\theta}$, with θ traveling from 0 to π.

Because $\frac{1-e^{iz}}{z^2}$ is holomorphic when $z \neq 0$, for any R,

$$\int_{A_R} \frac{1 - e^{iz}}{z^2} dz = 0. \tag{7.30}$$

Now then, for z near the origin,

$$\frac{1 - e^{iz}}{z^2} = \frac{-iz}{z^2} + o(\frac{1}{|z|}). \tag{7.31}$$

To simplify notation let $r = \frac{1}{R}$.

$$\frac{1}{2} \int_{\pi}^{0} \frac{re^{i\theta}}{re^{i\theta}} d\theta = -\frac{\pi}{2}. \tag{7.32}$$

Also,

$$\int_{\pi}^{0} \frac{o(r)}{r} d\theta = o_r(1). \tag{7.33}$$

Also, since $\sin(\theta) \geq 0$,

$$\int_0^{\pi} \frac{1 - e^{iRe^{i\theta}}}{R^2 e^{2i\theta}} Rie^{i\theta} d\theta \leq \frac{C}{R} \int_0^{\pi} d\theta = O(\frac{1}{R}). \tag{7.34}$$

Therefore, combining (7.30)–(7.34),

$$0 = -\frac{\pi}{2} + o_r(1) + O(\frac{1}{R}) + \frac{1}{2} \int_{-R}^{-\frac{1}{R}} \frac{1 - \cos x}{x^2} dx + \frac{1}{2} \int_{\frac{1}{R}}^{R} \frac{1 - \cos x}{x^2} dx. \tag{7.35}$$

Taking $R \to \infty$ proves the theorem. □

8. Meromorphic functions

Definition 26. *A function f on an open set Ω is meromorphic if there exists a sequence of points $\{z_0, z_1, z_2, ...\}$ that has no limit points in Ω such that*

(1) *f is holomorphic on $\Omega \setminus \{z_0, z_1, z_2, ...\}$,*
(2) *f has poles at $\{z_0, z_1, z_2, ...\}$.*

Suppose f is holomorphic for large values of z. That is, $F(z) = f(\frac{1}{z})$ which is holomorphic in a deleted neighborhood of the origin.

Definition 27. *We say that f has a pole at ∞ if $F(z)$ has a pole at the origin. The same for an essential singularity or a removable singularity. A function f that is meromorphic in $\mathbb{C}$ that is holomorphic at ∞ or has a pole at ∞ is said to be meromorphic in the extended complex plane.*

Theorem 32. *The functions that are meromorphic in the extended complex plane are the rational functions.*

Proof. Suppose f is meromorphic in the extended complex plane. Then $f(\frac{1}{z})$ has either a pole or a removable singularity at zero. In either case $f(\frac{1}{z})$ is holomorphic in $D_r(0) \setminus \{0\}$ which proves that f has only finitely many poles in $\mathbb{C}$, say $\{z_1, ..., z_n\}$.

Near each pole z_k split $f(z)$ into a principal part f_k and the holomorphic part,

$$f(z) = f_k(z) + g_k(z). \tag{8.1}$$

Since f has a pole at z_k, f_k is a polynomial in $\frac{1}{(z-z_k)}$. Similarly, for the singularity at zero,

$$f(\frac{1}{z}) = \widetilde{f}_\infty(z) + g_\infty(z). \tag{8.2}$$

Then $\tilde{g}_\infty(z)$ is holomorphic in a neighborhood of the origin and $\tilde{f}_\infty(z)$ is a polynomial in $\frac{1}{z}$,

$$f_\infty(z) = \widetilde{f_\infty}(\frac{1}{z}), \tag{8.3}$$

where $\widetilde{f_\infty}$ is a polynomial.

Claim 1. *The function,*

$$H(z) = f(z) - f_\infty(z) - \sum_{k=1}^{n} f_k(z), \tag{8.4}$$

is entire and bounded. Thus, H is constant.

Proof of claim. Near each pole z_k we have subtracted the principal part, and have a removable singularity. The same is true for $H(\frac{1}{z})$ near zero. Then by Liouville's theorem H is constant, which implies that f is a rational function. □

Therefore, $f(z)$ is equal to a polynomial plus a sum of rational functions. Thus a rational function is uniquely determined up to multiplicative constant by prescribing the locations and multiplicities of its zeros and poles. □

The Riemann sphere is a helpful representation of the extended complex plane. For any point (X, Y, Z) on the sphere, the stenographic projection is

$$x = \frac{X}{1-Z}, \qquad y = \frac{Y}{1-Z}, \tag{8.5}$$

and

$$X = \frac{x}{x^2 + y^2 + 1}, \qquad Y = \frac{y}{x^2 + y^2 + 1}, \qquad Z = \frac{x^2 + y^2}{x^2 + y^2 + 1}. \tag{8.6}$$

This is also called the one point compactification. Under this representation the image of a pole is $\mathcal{N}$. A function which is meromorphic in the extended complex plane maps the Riemann sphere to itself.

For a meromorphic function, it is possible to count the number of zeros minus the number of poles using the argument principle. If $z = re^{i\theta}$ then

$$\ln(z) = \ln(r) + i\theta. \tag{8.7}$$

In general,

$$\ln f(z) = \ln |f(z)| + i \arg f(z). \tag{8.8}$$

The function $\arg f(z)$ is only determined up to $2\pi n$, so it is convenient to take the derivative of $\ln f(z)$. Then,

$$\int_{\gamma} \frac{f'(z)}{f(z)} dz \tag{8.9}$$

may be interpreted as the change in the argument of f along the curve γ. Moreover, in general, while

$$\ln(f_1 f_2) \neq \ln(f_1) + \ln(f_2), \tag{8.10}$$

by the product rule,

$$\frac{(f_1 f_2)'}{f_1 f_2} = \frac{f_1' f_2 + f_1 f_2'}{f_1 f_2} = \frac{f_1'}{f_1} + \frac{f_2'}{f_2}. \tag{8.11}$$

More generally,

$$\frac{(\prod_{k=1}^N f_k)'}{\prod_{k=1}^N f_k} = \sum_{k=1}^N \frac{f_k'}{f_k}. \tag{8.12}$$

Now if f is holomorphic and has a zero of order n at z_0, that is

$$f(z) = (z - z_0)^n g(z), \tag{8.13}$$

then,

$$\frac{f'(z)}{f(z)} = \frac{n}{z - z_0} + G(z), \qquad G(z) = \frac{g'(z)}{g(z)}. \tag{8.14}$$

Thus if f has a zero of order n at z_0, then $\frac{f'}{f}$ has a simple pole with residue n at z_0. Meanwhile, if

$$f(z) = (z - z_0)^{-n} h(z), \qquad \frac{f'(z)}{f(z)} = -\frac{n}{z - z_0} + H(z), \qquad H(z) = \frac{h'(z)}{h(z)}. \tag{8.15}$$

Theorem 33 (Argument principle)**.** *Suppose f is meromorphic in an open set containing a circle C and its interior. If f has no poles and never vanishes on C, then*

$$\frac{1}{2\pi i} \int_C \frac{f'(z)}{f(z)} dz = \sharp\{ \text{ zeros of } f \text{ inside } C\} - \sharp\{ \text{ poles of } f \text{ inside } C\}, \tag{8.16}$$

where zeros and poles are counted with their multiplicities.

Corollary 15. *This result also holds for toy contours.*

Theorem 34 (Rouché's theorem)**.** *Suppose that f and g are holomorphic in an open set containing a circle C and its interior. If $|f(z)| > |g(z)|$ for all $z \in C$, then f and $f+g$ have the same number of zeros inside the circle C.*

Proof. For any $t \in [0,1]$ define

$$f_t(z) = f(z) + tg(z), \qquad f_0(z) = f(z), \qquad f_1(z) = f(z) + g(z). \tag{8.17}$$

Then since $|f(z)| > |g(z)|$, for all $z \in C$, $t \in [0,1]$, then f_t has no zeros on the circle C. Now let n_t be the number of zeros inside the circle C.

$$n_t = \frac{1}{2\pi i}\int_C \frac{f_t'(z)}{f_t(z)}dz. \tag{8.18}$$

To prove the theorem it suffices to prove that n_t is constant. This follows from the fact that n_t is an integer and n_t is continuous in t. Since $\frac{f_t'(z)}{f_t(z)}$ is jointly continuous for $t \in [0,1]$ and $z \in C$, this proves the theorem. Therefore $n_0 = n_1$. □

Definition 28 (Open mapping)**.** *A mapping is open if it maps open sets to open sets.*

Theorem 35 (Open mapping theorem)**.** *If f is holomorphic and non–constant in a region Ω, then f is open.*

Proof. Suppose w_0 belongs to the image of f, that is, $w_0 = f(z_0)$. Now suppose w is close to w_0 and define

$$g(z) = f(z) - w = (f(z) - w_0) + (w_0 - w) = F(z) + G(z). \tag{8.19}$$

Now choose $\delta > 0$ so that $\{z : |z - z_0| \le \delta\} \subset \Omega$ and $f(z) \ne w_0$ on $|z - z_0| = \delta$. Then because $\{z : |z - z_0| = \delta\}$ is compact, there is some $\epsilon > 0$ such that $|f(z) - w_0| \ge \epsilon$ on $|z - z_0| = \delta$. Then if $|w - w_0| < \epsilon$, $|F(z)| > |G(z)|$ on $\{z : |z - z_0| = \delta\}$. Since F has a zero inside C, $F + G$ has a zero as well, and therefore there exists $z \in \Omega$ such that $g(z) = w$. □

Definition 29 (Maximum)**.** *The maximum of f is the maximum of the absolute value of f, $|f|$, on Ω.*

Theorem 36 (Maximum modulus principle)**.** *If f is a non–constant holomorphic function in a region Ω then f cannot attain a maximum in Ω.*

Proof. This follows by the open mapping theorem. □

Corollary 16. *Suppose Ω is a region with compact closure $\bar{\Omega}$. If f is holomorphic on Ω and continuous on $\bar{\Omega}$, then*

$$\sup_{z\in\Omega} |f(z)| \le \sup_{z\in\bar{\Omega}\setminus\Omega} |f(z)|. \tag{8.20}$$

Remark 5. *This result does not hold if Ω is not compact. For example consider the function $F(z) = e^{-iz^2}$ on the open first quadrant.*

9. Homotopies and simply connected domains

The study of primitives is closely related to the study of homotopy. Suppose that γ_0 and γ_1 are two curves, $\gamma_0, \gamma_1 : [a,b] \to \mathbf{C}$ with $\gamma_0(a) = \gamma_1(a) = \alpha$ and $\gamma_0(b) = \gamma_1(b) = \beta$.

Definition 30 (Homotopic)**.** *Suppose $\Omega \subset \mathbf{C}$ is a domain. Two curves in Ω are said to be homotopic in Ω if for each $0 \leq s \leq 1$ there exists $\gamma_s \subset \Omega$ such that for all s, $\gamma_s(a) = \alpha$, $\gamma_s(b) = \beta$,*

$$\gamma_s(t)|_{s=0} = \gamma_0(t), \qquad \gamma_s(t)|_{s=1} = \gamma_1(t), \tag{9.1}$$

and $\gamma_s(t)$ is continuous in s and t.

Theorem 37. *If f is holomorphic in Ω and γ_0 and γ_1 are homotopic in Ω, then*

$$\int_{\gamma_0} f(z)dz = \int_{\gamma_1} f(z)dz. \tag{9.2}$$

Proof. For any $0 \leq s \leq 1$, let $F(s,t) = \gamma_s(t)$ be continuous on $[0,1] \times [a,b]$, $F(0,t) = \gamma_0(t)$ and $F(1,t) = \gamma_1(t)$. Thus, the image by F of $[0,1] \times [a,b]$ is compact. Therefore there exists $\epsilon > 0$ such that every disk of radius 3ϵ centered at a point in the image of F is contained in Ω.

Now choose $\delta(\epsilon) > 0$ such that

$$\sup_{t \in [a,b]} |\gamma_{s_1}(t) - \gamma_{s_2}(t)| < \epsilon, \qquad \forall |s_1 - s_2| < \delta. \tag{9.3}$$

Cover the curves γ_{s_1} and γ_{s_2} with disks of radius 2ϵ, $\epsilon > 0$. Furthermore, choose consecutive points $\{z_0, ..., z_{n+1}\}$ on γ_{s_1} and $\{w_0, ..., w_{n+1}\}$ on γ_{s_2} such that the union of these disks covers both curves and $z_i, z_{i+1}, w_i, w_{i+1} \in D_i$. Also let $z_0 = w_0$ be the common beginning end–point and $z_{n+1} = w_{n+1}$ the common ending end–point.

For each disk D_i let $F_i(z)$ be the primitive of f. On the intersection $D_i \cap D_{i+1}$, F_i and F_{i+1} are two primitives of the same function, so they differ by a constant. Thus,

$$F_{i+1}(z_{i+1}) - F_i(z_{i+1}) = F_{i+1}(w_{i+1}) - F_i(w_{i+1}). \tag{9.4}$$

Doing some algebra,

$$F_{i+1}(z_{i+1}) - F_{i+1}(w_{i+1}) = F_i(z_{i+1}) - F_i(w_{i+1}). \tag{9.5}$$

Thus,

$$\begin{aligned} \int_{\gamma_{s_1}} f - \int_{\gamma_{s_2}} f &= \sum_{i=0}^{n} [F_i(z_{i+1}) - F_i(z_i)] - \sum_{i=0}^{n} [F_i(w_{i+1}) - F_i(w_i)] \\ = \sum_{i=0}^{n} F_i(z_{i+1}) - F_i(w_{i+1}) - F_i(z_i) + F_i(w_i) &= F_n(z_{n+1}) - F_n(w_{n+1}) - F_0(z_0) + F_0(w_0). \end{aligned} \tag{9.6}$$

Since $w_{n+1} = z_{n+1}$ and $w_0 = z_0$, we have proved that $\int_{\gamma_{s_1}} f = \int_{\gamma_{s_2}} f$. Subdividing $[0,1]$ into finitely many subintervals $[s_i, s_{i+1}]$ proves the theorem. □

Definition 31 (Simply connected)**.** *A region Ω is simply connected if any two curves in Ω with the same endpoints are homotopic.*

Example: Any open convex set is simply connected. Take

$$\gamma_s(t) = (1-s)\gamma_0(t) + s\gamma_1(t). \tag{9.7}$$

Example: If $\Omega = \mathbf{C} \setminus \{(-\infty, 0]\}$ then

$$\gamma_j(t) = r_j(t)e^{i\theta_j(t)}, \tag{9.8}$$

so take

$$r_s(t) = (1-s)r_0(t) + sr_1(t), \qquad \theta_s(t) = (1-s)\theta_0(t) + s\theta_1(t). \tag{9.9}$$

Then $\gamma_s(t) \subset \Omega$ whenever $0 \leq s \leq 1$.

Example: More generally, the interior of a toy contour is simply connected.

Theorem 38. *Any holomorphic function in a simply connected domain has a primitive.*

Proof. Suppose γ is a curve joining z_0 and z, and let

$$F(z) = \int_\gamma f(w)dw. \tag{9.10}$$

Then,

$$F(z+h) - F(z) = \int_l f(w)dw, \tag{9.11}$$

where l is the line joining z to $z+h$. Then,

$$\lim_{h \to 0} \frac{F(z+h) - F(z)}{h} = f(z). \tag{9.12}$$

□

Corollary 17. *If f is holomorphic in a simply connected region Ω then*

$$\int_\gamma f(z)dz = 0 \tag{9.13}$$

for any closed curve γ in Ω.

It is possible to use homotopy theory to define the complex logarithm. Recalling properties of the logarithm from calculus one, it would be attractive to define the logarithm of $z = re^{i\theta}$,

$$\log(z) = \log(r) + i\theta, \tag{9.14}$$

where $\log(r)$ refers to the standard logarithm of $r > 0$. The problem with this definition is that for any integer n, $e^{2n\pi i} = 1$, so

$$\log(z) = \log(r) + i\theta + 2n\pi i. \tag{9.15}$$

Thus to define the logarithm, we must restrict to a subset of $\mathbb{C}$, called a branch, or sheet.

Theorem 39. *Suppose that Ω is simply connected with $1 \in \Omega$ and $0 \notin \Omega$. Then in Ω there is a branch of the logarithm $F(z) = \log_\Omega(z)$ so that*

(1) *F is holomorphic in Ω,*
(2) *$e^{F(z)} = z$ for all $z \in \Omega$,*

(3) $F(r) = \log(r)$ *whenever r is a real number near* 1.

Proof. Construct F as a primitive of $\frac{1}{z}$. Since $0 \notin \Omega$, $f(z) = \frac{1}{z}$ is holomorphic in Ω. Now define

$$\log_{\Omega}(z) = F(z) = \int_{\gamma} f(w)dw, \tag{9.16}$$

where γ is any curve connecting 1 and z. Since Ω is simply connected, (9.16) is independent of path. Then F is a holomorphic function and $F'(z) = \frac{1}{z}$.

Next,

$$\frac{d}{dz}(ze^{-F(z)}) = e^{-F(z)} - zF'(z)e^{-F(z)} = (1 - zF'(z))e^{-F(z)} = 0. \tag{9.17}$$

Since Ω is connected, $ze^{-F(z)}$ is constant. Then since $F(1) = 0$, $ze^{-F(z)} = 1$. Finally, if r is real and close to 1,

$$F(r) = \int_1^r \frac{dx}{x} = \log(r). \tag{9.18}$$

□

For example, take the set $\Omega = \mathbb{C} \setminus \{(-\infty, 0]\}$. This gives the principal branch of the logarithm,

$$\log(z) = \log(r) + i\theta. \tag{9.19}$$

Indeed,

$$\log(z) = \int_1^r \frac{dx}{x} + \int_0^{\theta} \frac{ire^{it}}{re^{it}} dt = \log(r) + i\theta. \tag{9.20}$$

However, in general,

$$\log(z_1 z_2) \neq \log(z_1) + \log(z_2). \tag{9.21}$$

Take for example $z_1 = z_2 = e^{\frac{2\pi i}{3}}$.

Lemma 9. *For the principal branch of the logarithm,*

$$\log(1+z) = \sum_{n=1}^{\infty} (-1)^n \frac{z^n}{n}, \qquad \textit{for} \qquad |z| < 1. \tag{9.22}$$

Proof. When $|z| < 1$,

$$\frac{d}{dz}(-\sum_{n=1}^{\infty} (-1)^n \frac{z^n}{n}) = -\sum_{n=1}^{\infty} (-1)^n z^{n-1} = \sum_{n=1}^{\infty} (-1)^{n-1} z^{n-1} = \frac{1}{1+z}. \tag{9.23}$$

Thus, the left and right hand sides of (9.22) differ by a constant. Plugging $z = 0$ into both shows that the constant is zero. □

It is possible to use the logarithm to define z^{α} in a simply connected domain for any $\alpha \in \mathbb{C}$. If Ω is simply connected with $1 \in \Omega$, $0 \notin \Omega$, choose the branch of the logarithm with $\log(1) = 0$. Then define

$$z^{\alpha} = e^{\alpha \log(z)}, \tag{9.24}$$

so $1^\alpha = 1$. If $\alpha = \frac{1}{n}$ then

$$(z^{1/n})^n = \prod_{k=1}^{n} e^{\frac{1}{n}\log(z)} = e^{\sum_{k=1}^{n}\frac{1}{n}\log(z)} = e^{\log(z)} = z, \tag{9.25}$$

so $z^{1/n}$ defines the n–th root of z.

Theorem 40. *If f is a nowhere vanishing holomorphic function in a simply connected region Ω, then there exists a holomorphic function g on Ω such that $f(z) = e^{g(z)}$. In this case, $g(z) = \log(f(z))$ determines a branch of the logarithm of f.*

Proof. Fix $z_0 \in \Omega$ and define a function

$$g(z) = \int_\gamma \frac{f'(w)}{f(w)} dw + c_0, \tag{9.26}$$

where γ is a curve connecting z_0 to z, and $c_0 \in \mathbb{C}$ is such that $e^{c_0} = f(z_0)$. This definition is independent of γ. So then, g is holomorphic and $g'(z) = \frac{f'(z)}{f(z)}$. Thus,

$$\frac{d}{dz}(f(z)e^{-g(z)}) = 0, \tag{9.27}$$

so $f(z)e^{-g(z)}$ is constant. Then $f(z_0)e^{-c_0} = 1$ implies the theorem. □

10. Fourier series and the Poisson integral

Many important integrals in complex analysis arise from the study of the Fourier series and Fourier transform. These notes come from [Tay19], although see also [SS09]. We begin with a function that is 2π–periodic.

Let $f : S^1 \to \mathbb{C}$ be an integrable function. Identifying S^1 with $\mathbb{T}^1 = \mathbb{R}/(2\pi\mathbb{Z})$, we desire to write

$$f(\theta) = \sum_{k=-\infty}^{\infty} \hat{f}(k) e^{ik\theta}. \tag{10.1}$$

If f is written in the form (10.1), where $\hat{f}(k)$ converges in some sense, using

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} e^{il\theta} d\theta = 0, \qquad \text{if} \qquad l \neq 0, \qquad 1, \qquad \text{if} \qquad l = 0, \tag{10.2}$$

then

$$\hat{f}(k) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(\theta) e^{-ik\theta} d\theta. \tag{10.3}$$

The series in (10.1) is called the Fourier series of f. If $\hat{f}$ is given by (10.3), and (10.1) holds, (10.1) is called the Fourier inversion formula.

For $0 < r < 1$, set

$$J_r f(\theta) = \sum_{k=-\infty}^{\infty} \hat{f}(k) r^{|k|} e^{ik\theta}. \tag{10.4}$$

Since

$$|\hat{f}(k)| \leq \frac{1}{2\pi}\int_{-\pi}^{\pi} |f(\theta)| d\theta, \tag{10.5}$$

the series (10.4) is absolutely convergent for each $r \in [0,1)$. By Fubini's theorem,

$$J_r f(\theta) = \sum_{k=-\infty}^{\infty} \hat{f}(k) r^{|k|} e^{ik\theta} = \frac{1}{2\pi} \sum_{k=-\infty}^{\infty} r^{|k|} \int_{-\pi}^{\pi} e^{ik(\theta - \theta')} f(\theta') d\theta' = \int_{S^1} f(\theta') p_r(\theta - \theta') d\theta', \tag{10.6}$$

where

$$p_r(\theta) = \frac{1}{2\pi} \sum_{k=-\infty}^{\infty} r^{|k|} e^{ik\theta} = \frac{1}{2\pi}[1 + \sum_{k=1}^{\infty} (r^k e^{ik\theta} + r^k e^{-ik\theta})] = \frac{1}{2\pi}[\frac{1}{1 - re^{i\theta}} + \frac{1}{1 - re^{-i\theta}} - 1] = \frac{1}{2\pi} \frac{1 - r^2}{1 - 2r\cos\theta + r^2}. \tag{10.7}$$

Both the numerator and denominator of $p_r(\theta)$ are positive, so $p_r(\theta) > 0$ for each $r \in [0,1)$. Also,

$$\int_{S^1} p_r(\theta) d\theta = \frac{1}{2\pi} \int_{-\pi}^{\pi} \sum r^{|k|} e^{ik\theta} d\theta = 1, \tag{10.8}$$

and $p_r(\theta) \to 0$ as $r \nearrow 1$ if $e^{i\theta} \neq 1$.

Proposition 11. *If $f \in C(S^1)$, then*

$$J_r f \to f \qquad \textit{uniformly on} \qquad S^1 \qquad \textit{as} \qquad r \nearrow 1. \tag{10.9}$$

Proof. Since f is continuous on S^1 and S^1 is compact, f is uniformly continuous on S^1. Therefore, there exists some $\delta > 0$ such that if $|\theta - \theta'| < \delta$, $|f(\theta) - f(\theta')| < \epsilon$. Therefore,

$$\begin{aligned} \int_{S^1} f(\theta') p_r(\theta - \theta') d\theta' &= \int_{|\theta - \theta'| < \delta} f(\theta) p_r(\theta - \theta') d\theta' \\ &+ \int_{|\theta - \theta'| < \delta} [f(\theta) - f(\theta')] p_r(\theta - \theta') d\theta' + \int_{|\theta - \theta'| > \delta} f(\theta') p_r(\theta - \theta') d\theta'. \end{aligned} \tag{10.10}$$

The first integral converges to $f(\theta)$, and the second two converge to zero, uniformly as $r \nearrow 1$. $\square$

The Fourier series connects with the theory of harmonic functions. Observe that by (3.1), $\bar{z}$, as well as powers of $\bar{z}$, are harmonic functions. Taking $z = re^{i\theta}$, let

$$J_r f(\theta) = \sum_{k=0}^{\infty} \hat{f}(k) z^k + \sum_{k=1}^{\infty} \hat{f}(-k) \bar{z}^k. \tag{10.11}$$

Rewriting this as

$$(PIf)(z) = (PI_+ f)(z) + (PI_- f)(z), \tag{10.12}$$

where PI is the sum of a holomorphic function and a conjugate holomorphic function. For $|w| = 1$ and $|z| < 1$,

$$1 - 2r\cos(\theta - \theta') + r^2 = |w - z|^2, \tag{10.13}$$

so

$$PIf(z) = \frac{1 - |z|^2}{2\pi} \int_{S^1} \frac{f(w)}{|w - z|^2} ds(w). \tag{10.14}$$

Applying this fact to the Dirichlet boundary value problem,

Proposition 12. *If $D \subset \mathbb{C}$ is an open disk and $f \in C(\partial D)$ is given, there exists a unique $u \in C(\bar{D}) \cap C^2(D)$ satisfying*

$$\Delta u = 0 \qquad \textit{on} \qquad D, \qquad u|_{\partial D} = f. \tag{10.15}$$

Proof. Uniqueness follows from the mean value theorem for harmonic functions. Observe that for any $f \in C(\partial D)$, (10.11) defines a smooth, harmonic function with radius of convergence ≤ 1. Furthermore, it follows from (10.10) that $u \in C(D)$ and $u|_{\partial D} = f$. To see this, choose r close to 1 and $\tilde{\theta}$ close to θ. Then, since f is uniformly continuous on $C(\partial D)$, $|f(\theta) - f(\tilde{\theta})| < \epsilon$ for $|\theta - \tilde{\theta}| < \delta(\epsilon)$ sufficiently small, and for $r(\epsilon)$ sufficiently close to 1, $|u(r,\theta) - f(\theta)| < \epsilon$ for any $\theta \in S^1$. □

The Fourier series is extremely useful to many branches of analysis, in large part because a class of functions form a Hilbert space, a generalization of finite dimensional inner product spaces.

Proposition 13. *Assume $f \in C(S^1)$. If the Fourier coefficients form a summable series,*

$$\sum_{k=-\infty}^{\infty} |\hat{f}(k)| < \infty, \tag{10.16}$$

then the identity (10.1) *holds for each $\theta \in S^1$.*

Proof. If $\sum_k |a_k| < \infty$, then

$$\sum_k a_k = S, \qquad \text{implies} \qquad \lim_{r \nearrow 1} \sum_k r^{|k|} a_k = S. \tag{10.17}$$

To see this, observe that for any fixed k, (10.17) holds. Therefore, choose $N(\epsilon)$ sufficiently large so that $\sum_k |a_k| < \epsilon$, so the proof is complete. □

Condition (10.17) holds for piecewise C^2 functions that are continuous on S^1. Integrating by parts,

$$\hat{f}(k) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(\theta) \frac{i}{k} \frac{\partial}{\partial \theta}(e^{-ik\theta}) d\theta. \tag{10.18}$$

Therefore,

$$|\hat{f}(k)| \leq \frac{1}{2\pi |k|} \int_{-\pi}^{\pi} |f'(\theta)| d\theta. \tag{10.19}$$

Taking a second derivative and integrating by parts, $|\hat{f}(k)| \leq \frac{1}{2\pi k^2} \int_{-\pi}^{\pi} |f''(\theta)| d\theta$.

Proposition 14. *Let f be continuous and piecewise C^2 on S^1. Then $|\hat{f}(k)| \leq \frac{C}{k^2+1}$.*

Given $f \in C(S^1)$, let us say

$$f \in \mathcal{A}(S^1) \Leftrightarrow \sum_k |\hat{f}(k)| < \infty. \tag{10.20}$$

Proposition 15. *Given $f \in \mathcal{A}(S^1)$,*

$$\sum_k |\hat{f}(k)|^2 = \frac{1}{2\pi} \int_{S^1} |f(\theta)|^2 d\theta. \tag{10.21}$$

More generally, if $g \in \mathcal{A}(S^1)$, then

$$\sum_k \hat{f}(k) \overline{\hat{g}(k)} = \frac{1}{2\pi} \int_{S^1} f(\theta) \overline{g(\theta)} d\theta. \tag{10.22}$$

Proof. By Fubini's theorem,

$$\frac{1}{2\pi}\int_{S^1} f(\theta)\overline{g(\theta)}d\theta = \frac{1}{2\pi}\int_{S^1}\sum_{j,k}\hat{f}(j)\overline{\hat{g}(k)}e^{-i(j-k)\theta}d\theta = \sum_{k}\hat{f}(k)\overline{\hat{g}(k)}. \tag{10.23}$$

□

Define the L^2 norm of f,

$$\|f\|_{L^2}^2 = \frac{1}{2\pi}\int_{S^1}|f(\theta)|^2 d\theta. \tag{10.24}$$

This space is an inner product space, where the inner product is given by

$$(f,g)_{L^2} = \frac{1}{2\pi}\int_{S^1} f(\theta)\overline{g(\theta)}d\theta. \tag{10.25}$$

Thus, if $f \in \mathcal{A}(S^1)$, (10.23) states that

$$\sum |\hat{f}(k)|^2 = \|f\|_{L^2}^2, \tag{10.26}$$

and

$$\sum \hat{f}(k)\overline{\hat{g}(k)} = (f,g)_{L^2}. \tag{10.27}$$

The l^2 norm on sequences can be defined by the inner product (10.27). Both the L^2 norm and l^2 norm are Banach norms, and satisfy the triangle inequality.

Now consider a general square integrable function f on S^1, f need not lie in $\mathcal{A}(S^1)$. Let

$$S_N f = \sum_{|k|\leq N}\hat{f}(k)e^{ik\theta}. \tag{10.28}$$

Then split

$$f = S_N f + R_N f. \tag{10.29}$$

By (10.28),

$$(f, S_N f)_{L^2} = (S_N f, S_N f)_{L^2}, \tag{10.30}$$

so

$$(S_N f, R_N f)_{L^2} = 0. \tag{10.31}$$

Equivalently,

$$\|f\|_{L^2}^2 = \|S_N f\|_{L^2}^2 + \|R_N f\|_{L^2}^2 \geq \|S_N f\|_{L^2}^2. \tag{10.32}$$

Proposition 16. *Let f, f_ν be square integrable on S^1. Assume*

$$\lim_{\nu\to\infty}\|f - f_\nu\|_{L^2} = 0, \tag{10.33}$$

and for each ν,

$$\lim_{N\to\infty}\|f_\nu - S_N f_\nu\|_{L^2} = 0. \tag{10.34}$$

Then

$$\lim_{N\to\infty}\|f - S_N f\|_{L^2} = 0. \tag{10.35}$$

In this case,

$$\|f\|_{L^2}^2 = \sum |\hat{f}(k)|^2. \tag{10.36}$$

Proof. The limit (10.35) may be proved by splitting

$$\|f - S_N f\|_{L^2} \le \|f - f_\nu\|_{L^2} + \|f_\nu - S_N f_\nu\|_{L^2} + \|S_N(f - f_\nu)\|_{L^2}. \tag{10.37}$$

Since $\|S_N f\|_{L^2}^2 = \sum_{|k|\le N} |\hat{f}(k)|^2$, (10.36) also holds. □

For $f \in C(S^1)$, Proposition 11 implies that $J_r f \to$f uniformly as $r \nearrow 1$. Since $f_\nu \in \mathcal{A}(S^1)$,

$$f \in C(S^1) \Rightarrow S_N f \to f \quad \text{in} \quad L^2, \qquad \sum |\hat{f}(k)|^2 = \|f\|_{L^2}^2. \tag{10.38}$$

Using Lebesgue integration, one may go further, and prove

Proposition 17. *Given $f \in L^2(S^1)$, there exists $f_\nu \in C(S^1)$ such that $f_\nu \to f$ in L^2.*

Proposition 18. *If (f_ν) is a Cauchy sequence in L^2, there exists $f \in L^2$ such that $f_\nu \to f$ in the L^2 norm.*

11. The Fourier transform

If f is a function on $\mathbb{R}$ that satisfies appropriate regularity and decay conditions, then its Fourier transform is defined by

$$\hat{f}(\xi) = \int_{-\infty}^{\infty} f(x) e^{-2\pi i x\xi} dx, \qquad \xi \in \mathbb{R}, \tag{11.1}$$

and its counterpart, the Fourier inversion formula holds,

$$f(x) = \int_{-\infty}^{\infty} \hat{f}(\xi) e^{2\pi i x\xi} d\xi, \qquad x \in \mathbb{R}. \tag{11.2}$$

As a warm–up, we will compute the Fourier transform of the normal distribution.

Theorem 41. *For any $\xi \in \mathbb{R}$,*

$$e^{-\pi\xi^2} = \int_{-\infty}^{\infty} e^{-\pi x^2} e^{-2\pi i x\xi} dx. \tag{11.3}$$

Proof. First, if $\xi = 0$, then

$$\begin{aligned}(\int_{-\infty}^{\infty} e^{-\pi x^2} dx)^2 = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-\pi(x^2+y^2)} dx dy &= \int_0^{\infty}\int_0^{2\pi} e^{-\pi r^2} r d\theta dr \\ &= 2\pi \int_0^{\infty} e^{-\pi r^2} r dr = \int_0^{\infty} e^{-u} du = 1.\end{aligned} \tag{11.4}$$

Since (11.3) is symmetric, suppose $\xi > 0$. Now consider a contour traveling from $-\infty$ to $-R$, then to $-R + i\xi$, then to $R + i\xi$, then down to R, and finally to ∞.

$$\int_{-R}^{R} e^{-\pi(s+i\xi)^2} ds = \int_{-R}^{R} e^{-\pi s^2} e^{-2\pi i s\xi} e^{\pi\xi^2} ds = e^{\pi\xi^2} \int_{-R}^{R} e^{-\pi x^2} e^{-2\pi i x\xi} dx, \tag{11.5}$$

$$\int_0^{\xi} e^{-\pi(R+is)^2} ds = \int_0^{\xi} e^{-\pi R^2} e^{-2\pi i R s} e^{\pi s^2} ds = O(\xi e^{-\pi R^2} e^{\pi\xi^2}), \tag{11.6}$$

$$\int_{\xi}^{0} e^{-\pi(-R+is)^2} ds = -\int_{0}^{\xi} e^{-\pi R^2} e^{2\pi i R s} e^{\pi s^2} ds = O(\xi e^{-\pi R^2} e^{\pi \xi^2}), \tag{11.7}$$

$$\int_{-\infty}^{-R} e^{-\pi x^2} dx, \qquad \int_{R}^{\infty} e^{-\pi x^2} dx = O(\frac{1}{R} e^{-\pi R^2}). \tag{11.8}$$

Therefore, since $e^{-\pi z^2}$ is holomorphic on $\mathbb{C}$,

$$1 = \int_{-\infty}^{\infty} e^{-\pi x^2} dx = e^{\pi \xi^2} \int_{-R}^{R} e^{-\pi x^2} e^{-2\pi i x \xi} dx + O((\xi + \frac{1}{R}) e^{-\pi R^2}). \tag{11.9}$$

Therefore, taking $R \to \infty$,

$$1 = e^{\pi \xi^2} \int_{-\infty}^{\infty} e^{-\pi x^2} e^{-2\pi i x \xi} dx. \tag{11.10}$$

□

Therefore, the Fourier transform of $e^{-\pi x^2}$ is $e^{-\pi \xi^2}$, and the inverse Fourier transform of $e^{-\pi \xi^2}$ is $e^{-\pi x^2}$.

Now then, suppose f and $\hat{f}$ are continuous and satisfy

$$|f(x)| \leq \frac{A}{1 + x^2}, \qquad |\hat{f}(\xi)| \leq \frac{A'}{1 + \xi^2}. \tag{11.11}$$

Proposition 19. *If f satisfies* (11.11)*,*

$$f(x) = \int e^{2\pi i x \xi} \hat{f}(\xi) d\xi. \tag{11.12}$$

Proof. If $\hat{f}$ satisfies (11.11),

$$\int e^{2\pi i x \xi} \hat{f}(\xi) d\xi = \lim_{\epsilon \searrow 0} \int e^{-\epsilon \pi \xi^2} e^{2\pi i x \xi} \hat{f}(\xi) d\xi = \lim_{\epsilon \searrow 0} \int e^{-\epsilon \pi \xi^2} e^{2\pi i x \xi} \int e^{-2\pi i y \xi} f(y) dy d\xi. \tag{11.13}$$

Applying Fubini's theorem and Theorem 41,

$$(11.13) = \lim_{\epsilon \searrow 0} \int f(y) \int e^{-\epsilon \pi \xi^2} e^{2\pi i (x - y) \xi} d\xi dy = \lim_{\epsilon \searrow 0} \epsilon^{-1/2} \int e^{-\frac{\pi}{\epsilon} |x - y|^2} f(y) dy. \tag{11.14}$$

Taking the limit as $\epsilon \searrow 0$, combined with the fact that f is continuous, proves the proposition. □

Now introduce a class of functions. For each $a > 0$, let $\mathcal{F}_a$ be the set of functions f that satisfy the following two conditions:

(1) The function f is holomorphic in the horizontal strip

$$S_a = \{z \in \mathbb{C} : |Im(z)| < a\}. \tag{11.15}$$

(2) There exists a constant $A > 0$ such that

$$|f(x + iy)| \leq \frac{A}{1 + x^2}, \qquad \text{for all} \qquad x \in \mathbb{R} \qquad \text{and} \qquad |y| < a. \tag{11.16}$$

For example, $e^{-\pi x^2}$ belongs to $\mathcal{F}_a$ for all a. Also, the function

$$f(z) = \frac{1}{\pi}\frac{c}{c^2 + z^2}, \tag{11.17}$$

belongs to $\mathcal{F}_a$ for $0 < a < c$, since f has simple poles at $z = \pm ci$.

Another example is provided by $f(z) = \frac{1}{\cosh(\pi z)}$, which belongs $\mathcal{F}_a$ whenever $|a| < \frac{1}{2}$. By the Cauchy integral formula, if $f \in \mathcal{F}_a$, $f^{(n)} \in \mathcal{F}_b$ for any n when $0 < b < a$.

Theorem 42. *If f belongs to the class $\mathcal{F}_a$ for some $a > 0$, then*

$$|\hat{f}(\xi)| \leq Be^{-2\pi b|\xi|}, \tag{11.18}$$

for any $0 \leq b < a$.

Proof. Recall that

$$\hat{f}(\xi) = \int_{-\infty}^{\infty} e^{-2\pi i x\xi} f(x)dx. \tag{11.19}$$

When $b = 0$, (11.16) implies that (11.18) holds when $b = 0$.

Now suppose $0 < b < a$ and suppose that $\xi > 0$. Integrating (11.19) along the contour $x - ib$,

$$\int_{-\infty}^{\infty} f(x - ib)e^{-2\pi i(x-ib)\xi}d\xi = e^{-2\pi b\xi}\int_{-\infty}^{\infty} f(x - ib)e^{-2\pi i x\xi}dx. \tag{11.20}$$

Then (11.16) implies that (11.20) $\leq Be^{-2\pi b\xi} = Be^{-2\pi b|\xi|}$ when $\xi > 0$. Now then, since f is holomorphic on the strip $\{x + iy : |y| < a\}$, let R_b be the contour traveling from R to $R - ib$; then from $R - ib$ to $-R - ib$; then from $-R - ib$ to $-R$. Then

$$\int_{-R}^{R} f(x)e^{-2\pi i x\xi}dx + \int_{R_b} f(z)e^{-2\pi i z\xi} = 0. \tag{11.21}$$

Now then, by (11.16),

$$|\int_0^b f(R + iy)dy|, |\int_0^b f(-R + iy)dy| \leq \frac{A}{R}. \tag{11.22}$$

Therefore, by (11.20) and (11.21), for $\xi > 0$,

$$|\int_{-R}^{R} f(x)e^{-2\pi i x\xi}dx| \leq Be^{-2\pi b\xi} + O(\frac{1}{R}). \tag{11.23}$$

Taking $R \to \infty$ proves (11.18). When $\xi < 0$, the computation is the same, but instead shift the contour to $x + ib$. □

Theorem 43. *If $f \in \mathcal{F}$, then the Fourier inversion formula holds, namely*

$$f(x) = \int_{-\infty}^{\infty} \hat{f}(\xi)e^{2\pi i x\xi}d\xi, \qquad \textit{for all} \qquad x \in \mathbb{R}. \tag{11.24}$$

Proof. The proof of this theorem uses the simple identity

Lemma 10. *If $A > 0$ and B is real, then $\int_0^{\infty} e^{-(A+iB)\xi}d\xi = \frac{1}{A+iB}$.*

Proof. Since $A > 0$ and $B \in \mathbb{R}$, $|e^{-(A+iB)\xi} \leq e^{-A\xi}$, so the integral converges. By definition,

$$\int_0^\infty e^{-(A+iB)\xi} d\xi = \lim_{R\to\infty} \int_0^R e^{-(A+iB)\xi} d\xi = \lim_{R\to\infty} [-\frac{e^{-(A+iB)\xi}}{A+iB}]|_0^R = \frac{1}{A+iB}. \tag{11.25}$$

□

Now split

$$\int_{-\infty}^\infty \hat{f}(\xi) e^{2\pi i x\xi} d\xi = \int_{-\infty}^0 \hat{f}(\xi) e^{2\pi i x\xi} d\xi + \int_0^\infty \hat{f}(\xi) e^{2\pi i x\xi} d\xi. \tag{11.26}$$

Now then, for $\xi > 0$, following Theorem 42,

$$\hat{f}(\xi) = \int_{-\infty}^\infty f(u - ib) e^{-2\pi i(u-ib)\xi} du. \tag{11.27}$$

By Fubini's theorem and the decay properties of f,

$$\begin{aligned} \int_0^\infty \hat{f}(\xi) e^{2\pi i x\xi} d\xi &= \int_0^\infty \int_{-\infty}^\infty f(u - ib) e^{-2\pi i(u-ib)\xi} e^{2\pi i x\xi} du d\xi \\ = \int_{-\infty}^\infty f(u - ib) \int_0^\infty e^{-2\pi i(u-ib-x)\xi} d\xi du &= \int_{-\infty}^\infty f(u - ib) \frac{1}{2\pi b + 2\pi i(u - x)} du \\ &= \frac{1}{2\pi i} \int_{-\infty}^\infty \frac{f(u - ib)}{u - ib - x} du = \frac{1}{2\pi i} \int_{L_1} \frac{f(\zeta)}{\zeta - x} d\zeta, \end{aligned} \tag{11.28}$$

where L_1 denotes the line $\{u - ib : u \in \mathbb{R}\}$ transversed from left to right. Also,

$$\begin{aligned} \int_{-\infty}^0 \hat{f}(\xi) e^{2\pi i x\xi} d\xi &= \int_{-\infty}^0 \int_{-\infty}^\infty f(u + ib) e^{-2\pi i(u+ib)\xi} e^{2\pi i x\xi} du d\xi \\ = \int_{-\infty}^\infty f(u + ib) \int_{-\infty}^0 e^{-2\pi i(u+ib-x)\xi} d\xi du &= -\int_{-\infty}^\infty f(u + ib) \frac{1}{-2\pi b + 2\pi i(u - x)} du \\ &= -\frac{1}{2\pi i} \int_{-\infty}^\infty \frac{f(u + ib)}{u + ib - x} du = -\frac{1}{2\pi i} \int_{L_2} \frac{f(\zeta)}{\zeta - x} d\zeta, \end{aligned} \tag{11.29}$$

where L_2 is the contour that is the real line along the real line shifted up by ib. Furthermore, from (11.16),

$$\int_{L_1} \frac{f(\zeta)}{\zeta - x} d\zeta = \int_{-R}^R \frac{f(y - ib)}{y - ib - x} dy + O_x(\frac{1}{R}), \tag{11.30}$$

$$\int_{L_2} \frac{f(\zeta)}{\zeta - x} d\zeta = \int_{-R}^R \frac{f(y + ib)}{y + ib - x} dy + O_x(\frac{1}{R}), \tag{11.31}$$

$$|\int_{-b}^b \frac{f(R + is)}{(R + is - x)} i ds|, |\int_{-b}^b \frac{f(-R + is)}{-R + is - x} i ds| = O_x(\frac{1}{R}). \tag{11.32}$$

Now let γ_R be the contour from $-R - ib$ to $R - ib$, then from $R - ib$ to $R + ib$, then from $R + ib$ to $-R + ib$, and then from $-R + ib$ to $-R - ib$. Therefore, for R sufficiently large, by the Cauchy integral formula,

$$f(x) = \frac{1}{2\pi i} \int_{\gamma_R} \frac{f(\zeta)}{\zeta - x} d\zeta = \frac{1}{2\pi i} \int_{L_1} \frac{f(\zeta)}{\zeta - x} d\zeta - \frac{1}{2\pi i} \int_{L_2} \frac{f(\zeta)}{\zeta - x} d\zeta + O_x(\frac{1}{R}). \tag{11.33}$$

Taking $R \to \infty$ proves Theorem 43. □

Theorem 44 (Poisson summation formula)**.** *If* $f \in \mathcal{F}$*, then*

$$\sum_{n\in\mathbb{Z}} f(n) = \sum_{n\in\mathbb{Z}} \hat{f}(n). \tag{11.34}$$

Proof. Suppose $f \in \mathcal{F}_a$ and choose some $0 < b < a$. The function $\frac{1}{e^{2\pi iz}-1}$ has simple poles with residue $\frac{1}{2\pi i}$ at the integers. Applying the residue formula, let γ_N be the positively oriented rectangle with vertices at $-N - \frac{1}{2} - ib$, $N + \frac{1}{2} - ib$, $N + \frac{1}{2} + ib$, and $-N - \frac{1}{2} + ib$. Thus,

$$\sum_{|n|\le N} f(n) = \int_{\gamma_N} \frac{f(z)}{e^{2\pi iz}-1} dz. \tag{11.35}$$

Letting $N \to \infty$, (11.16) implies

$$\sum_{n\in\mathbb{Z}} f(n) = \int_{L_1} \frac{f(z)}{e^{2\pi iz}-1} dz - \int_{L_2} \frac{f(z)}{e^{2\pi iz}-1} dz. \tag{11.36}$$

When $z \in L_1$, $|e^{2\pi i(x-ib)}| = |e^{2\pi b+2\pi ix}| > 1$. Using the expansion when $|w| > 1$, $\frac{1}{w-1} = w^{-1}\sum_{n=0}^{\infty} w^{-n}$. Therefore,

$$\frac{1}{e^{2\pi iz}-1} = e^{-2\pi iz}\sum_{n=0}^{\infty} e^{-2\pi inz}. \tag{11.37}$$

Therefore,

$$\int_{L_1} f(z)(e^{-2\pi iz}\sum_{n=0}^{\infty} e^{-2\pi inz})dz = \sum_{n=0}^{\infty}\int_{L_1} f(z)e^{-2\pi i(n+1)z}dz = \sum_{n=0}^{\infty}\int_{-\infty}^{\infty} f(x)e^{-2\pi i(n+1)x}dx = \sum_{n=0}^{\infty}\hat{f}(n+1). \tag{11.38}$$

We use the fact that f is holomorphic, so it is possible to shift the contour from L_1 to the real line.

When $z \in L_2$, $|e^{2\pi iz}| < 1$, so

$$\frac{1}{e^{2\pi iz}-1} = -\sum_{n=0}^{\infty} e^{2\pi inz}. \tag{11.39}$$

Then,

$$-\int_{L_2} f(z)\frac{1}{e^{2\pi iz}-1}dz = \int_{L_2} f(z)(\sum_{n=0}^{\infty} e^{2\pi inz})dz = \sum_{n=0}^{\infty}\int_{L_2} f(z)e^{2\pi inz}dz = \sum_{n=0}^{\infty}\int_{-\infty}^{\infty} f(x)e^{2\pi inx}dz = \sum_{n=0}^{\infty}\hat{f}(-n). \tag{11.40}$$

Adding (11.38) and (11.40) proves the theorem. □

The Poisson summation formula has a number of important applications. Recall that the Fourier transform of $e^{-\pi x^2}$ is $e^{-\pi\xi^2}$. Applying a change of variables formula, the Fourier transform of $e^{-\pi t(x+a)^2}$ is $t^{-1/2}e^{-\pi n^2/t}e^{2\pi ia\xi}$. Applying the Poisson summation formula,

$$\sum_{n=-\infty}^{\infty} e^{-\pi t(n+a)^2} = \sum_{n=-\infty}^{\infty} t^{-1/2}e^{-\pi n^2/t}e^{2\pi ina}. \tag{11.41}$$

Thus, considering the theta function $\theta(t) = \sum_{n=-\infty}^{\infty} e^{-\pi n^2 t}$, (11.41) implies

$$\theta(t) = t^{-1/2}\theta(\frac{1}{t}). \tag{11.42}$$

Likewise, $\frac{1}{\cosh(\pi x)}$ is also its own Fourier transform. Thus, the Fourier transform of $f(x) = \frac{e^{-2\pi i a x}}{\cosh(\frac{\pi x}{t})}$ is $\hat{f}(\xi) = \frac{t}{\cosh(\pi(\xi + a)t)}$. Therefore,

$$\sum_{n=-\infty}^{\infty} \frac{e^{-2\pi i a n}}{\cosh(\frac{\pi n}{t})} = \sum_{n=-\infty}^{\infty} \frac{t}{\cosh(\pi(n+a)t)}. \tag{11.43}$$

12. Paley–Wiener theorem

In this case we do not assume that f is analytic, but we do assume the validity of the Fourier inversion formula,

$$f(x) = \int_{-\infty}^{\infty} \hat{f}(\xi) e^{2\pi i x\xi} d\xi, \qquad \text{if} \qquad \hat{f}(\xi) = \int_{-\infty}^{\infty} f(x) e^{-2\pi i x\xi} dx, \tag{12.1}$$

under the conditions $|f(x)| \leq \frac{A}{1+x^2}$, $|\hat{f}(\xi)| \leq \frac{A'}{1+\xi^2}$.

Theorem 45. *Suppose $\hat{f}$ satisfies the decay condition $|\hat{f}(\xi)| \leq Ae^{-2\pi a|\xi|}$ for some constants $a, A > 0$. Then $f(x)$ is the restriction to $\mathbb{R}$ of a function $f(z)$ that is holomorphic in the strip $S_b = \{z \in \mathbb{C} : |Im(z)| < b\}$, for any $0 < b < a$.*

Proof. Define

$$f_n(z) = \int_{-n}^{n} \hat{f}(\xi) e^{2\pi i \xi z} d\xi, \tag{12.2}$$

and note that f_n is entire, by Theorem 23. Moreover, $f(z)$ may be defined for all z in the strip S_b by

$$f(z) = \int_{-\infty}^{\infty} \hat{f}(\xi) e^{2\pi i x\xi} d\xi, \tag{12.3}$$

since the integral is bounded by $A\int_{-\infty}^{\infty} e^{-2\pi a|\xi|} e^{2\pi b|\xi|} d\xi$, which is finite if $b < a$. Moreover, for $z \in S_b$,

$$|f(z) - f_n(z)| \leq A \int_{|\xi| \geq n} e^{-2\pi a|\xi|} e^{2\pi b|\xi|} d\xi \to 0, \qquad \text{as} \qquad n \to \infty. \tag{12.4}$$

Thus, the sequence $\{f_n\}$ converges to f uniformly in S_b, which proves the theorem. □

Corollary 18. *If $\hat{f}(\xi) = O(e^{-2\pi a|\xi|})$ for some $a > 0$, and f vanishes in a non–empty open interval, then $f = 0$.*

In particular, Corollary 18 implies that it is not possible for both f and $\hat{f}$ to be compactly supported.

Theorem 46 (Paley–Wiener theorem)**.** *Suppose f is continuous and of moderate decrease on $\mathbb{R}$. Then, f has an extension to the complex plane that is entire with $|f(z)| \leq Ae^{2\pi M|z|}$ for some $A > 0$, if and only if $\hat{f}$ is supported in the interval $[-M, M]$.*

Proof. One direction is simple. If $\hat{f}$ is supported on $[-M, M]$, then both f and $\hat{f}$ have moderate decrease. Therefore, applying the inversion formula,

$$f(x) = \int_{-M}^{M} \hat{f}(\xi) e^{2\pi i x \xi} d\xi. \tag{12.5}$$

Since the range of integration is finite, we can replace x by the complex variable in $\mathbb{C}$ by

$$f(z) = \int_{-M}^{M} \hat{f}(\xi) e^{2\pi i z \xi} d\xi. \tag{12.6}$$

By construction, $g(z) = f(z)$ if z is real, and g is holomorphic by Theorem 23. Finally, if $z = x + iy$,

$$|g(z)| \leq \int_{-M}^{M} |\hat{f}(\xi)| e^{-2\pi \xi y} d\xi \leq A e^{2\pi M |z|}. \tag{12.7}$$

Now to prove the converse, that if $|f(z)| \leq A e^{2\pi M |z|}$ for some $A > 0$, then $\hat{f}$ is supported in the interval $[-M, M]$.

Step 1: First suppose that f is holomorphic in the complex plane, and also satisfies the following decay condition,

$$|f(x + iy)| \leq A' \frac{e^{2\pi M |y|}}{1 + x^2}. \tag{12.8}$$

Under this stronger assumption $\hat{f}(\xi) = 0$ if $|\xi| > M$. Observe that if $\xi > M$, since f is holomorphic on $\mathbb{C}$,

$$\hat{f}(\xi) = \int_{-\infty}^{\infty} f(x) e^{-2\pi i \xi x} dx = \int_{-\infty}^{\infty} f(x - iy) e^{-2\pi i \xi (x - iy)} dx. \tag{12.9}$$

Then, by (12.9),

$$|\hat{f}(\xi)| \leq A' \int_{-\infty}^{\infty} \frac{e^{2\pi M y - 2\pi \xi y}}{1 + x^2} dx \leq C e^{-2\pi y (\xi - M)}. \tag{12.10}$$

Recalling that $\xi - M > 0$ and letting $y \to \infty$ proves that $\hat{f}(\xi) = 0$. When $\xi < -M$, making a similar argument and shifting the contour up by $y > 0$ proves $\hat{f}(\xi) = 0$ whenever $\xi < -M$.

Step 2: Now suppose that f only satisfies

$$|f(x + iy)| \leq A e^{2\pi M |y|}. \tag{12.11}$$

This is a relaxation of (12.8), but still falls short of the condition $|f(z)| \leq A e^{2\pi M |z|}$. Suppose $\xi > M$ and for $\epsilon > 0$ let

$$f_\epsilon(z) = \frac{f(z)}{(1 + i\epsilon z)^2}. \tag{12.12}$$

The quantity $\frac{1}{(1+i\epsilon z)^2}$ has absolute value less than or equal to one in the lower half plane, and $\frac{1}{(1+i\epsilon z)^2} \to 1$ as $\epsilon \searrow 0$. Since f satisfies the conditions of moderate decrease, $\hat{f}_\epsilon(\xi) \to \hat{f}(\xi)$ as $\epsilon \searrow 0$, since

$$|\hat{f}_\epsilon(\xi) - \hat{f}(\xi)| \leq \int_{-\infty}^{\infty} |f(x)| [\frac{1}{(1 + i\epsilon x)^2} - 1] dx \to 0. \tag{12.13}$$

Now then, for fixed $\epsilon > 0$, we have

$$|f_\epsilon(x+iy)| \leq A'' \frac{e^{2\pi M|y|}}{1+x^2}. \tag{12.14}$$

Therefore, by Step one, we must have $\hat{f}(\xi) = 0$, and therefore, $\hat{f}(\xi) = 0$ after passing to the limit as $\epsilon \searrow 0$. A similar argument works for $\xi < -M$, but in that case we use the factor $\frac{1}{(1-i\epsilon z)^2}$ instead.

Step 3: To conclude the proof it suffices to show that the conditions in the theorem imply (12.11). To prove this, it suffices to show that if $|f(x)| \leq 1$ for all real x and $|f(z)| \leq e^{2\pi M|z|}$ for all complex z, then

$$|f(x+iy)| \leq e^{2\pi M|y|}. \tag{12.15}$$

Theorem 47 (Phragmen and Lindelöf result)**.** *Suppose F is a holomorphic function in the sector $S = \{z : -\frac{\pi}{4} < arg(z) < \frac{\pi}{4}\}$ that is continuous in the closure of S. Assume $|F(z)| \leq 1$ on the boundary of the sector and that there are constants $C, c > 0$ such that $|F(z)| \leq Ce^{c|z|}$ for all z in the sector. Then, $|F(z)| \leq 1$ for all $z \in S$.*

Remark 6. *Some moderate restriction on the growth of F is needed. For example take $F(z) = e^{z^2}$.*

Proof of Theorem 47. For $\epsilon > 0$, let $F_\epsilon(z) = F(z)e^{-\epsilon z^{3/2}}$. We have chosen the principal branch of the logarithm, so $z^{3/2} = r^{3/2}e^{3i\theta/2}$. Therefore, F_ϵ is holomorphic on S and is continuous up to its boundary. Also,

$$|e^{-\epsilon z^{3/2}}| = e^{-\epsilon r^{3/2}\cos(\frac{3\theta}{2})}. \tag{12.16}$$

Since $-\frac{\pi}{4} < \theta < \frac{\pi}{4}$, $\cos(\frac{3\theta}{2})$ is strictly positive in the sector. Since $|F(z)| \leq Ce^{c|z|}$, $F_\epsilon(z)$ decreases rapidly in the closed sector as $|z| \to \infty$.

Claim 2. $|F_\epsilon(z)| \leq 1$ *for all* $z \in \bar{S}$.

Proof of claim. Define $M = \sup_{z \in \bar{Z}} |F_\epsilon(z)|$. Since F_ϵ decays to zero as $|z|$ becomes large, we can show that $F_\epsilon(z)$ cannot achieve its maximum in the interior of S. Therefore, $F_\epsilon(z)$ must achieve its maximum on the boundary, and by construction $|F_\epsilon(z)| \leq 1$ on the boundary. □

Therefore, letting $\epsilon \searrow 0$ implies that $|F(z)| \leq 1$ on $\bar{S}$. □

Now we can show that if $|f(x)| \leq 1$ and $|f(z)| \leq e^{2\pi M|z|}$, then $|f(z)| \leq e^{2\pi M|y|}$. Then we can rotate the first quadrant to $-\frac{\pi}{4} < \theta < \frac{\pi}{4}$ and the result remains the same. Then consider

$$F(z) = f(z)e^{2\pi i M z}. \tag{12.17}$$

Then $|F(z)| \leq 1$ on both the positive real axis and the positive imaginary axis. Therefore, by the Phragmen Lindelöf result, $|F(z)| \leq 1$ for all $z \in Q$, and therefore, $|f(z)| \leq e^{2\pi M|y|}$ for all $z \in Q$. Applying a similar argument to quadrant two concludes the bounds on the second quadrant. This concludes the proof of the Paley–Wiener theorem. □

Theorem 48. *Suppose f and $\hat{f}$ have moderate decrease. Then $\hat{f}(\xi) = 0$ for all $\xi < 0$ if and only if f can be extended to a continuous and bounded function in the closed upper half plane $\{z = x + iy : y \geq 0\}$ with f holomorphic in the interior.*

Proof. First assume $\hat{f}(\xi) = 0$ for $\xi < 0$. By the Fourier inversion formula,

$$f(x) = \int_0^\infty \hat{f}(\xi) e^{2\pi i x\xi} d\xi, \tag{12.18}$$

which we can extend to the upper half plane by

$$f(z) = \int_0^\infty \hat{f}(\xi) e^{2\pi i z\xi} d\xi. \tag{12.19}$$

Note that the above integral converges and that $|f(z)| \leq A \int_0^\infty \frac{d\xi}{1+\xi^2} d\xi < \infty$, which proves the boundedness of f. The uniform convergence of

$$f_n(z) = \int_0^\infty \hat{f}(\xi) e^{2\pi i z\xi} d\xi, \tag{12.20}$$

to $f(z)$ in the closed upper half plane establishes the continuity of f there and its holomorphicity in the interior.

For the converse let

$$f_{\epsilon,\delta}(z) = \frac{f(z+i\delta)}{(1-i\epsilon z)^2}. \tag{12.21}$$

Then $f_{\epsilon,\delta}$ is holomorphic in a region containing the closed upper half plane. Then we can show that $\hat{f}_{\epsilon,\delta}(\xi) = 0$ for all $\xi < 0$. Indeed, if $\xi < 0$,

$$\int_{-\infty}^\infty f(x) e^{-2\pi i x\xi} dx = \int_{-\infty}^\infty f(x+iM) e^{-2\pi i x\xi} 2^{2\pi M\xi} dx \leq C_{\epsilon,\delta} e^{-2\pi M|\xi|}. \tag{12.22}$$

Taking $M \to \infty$ implies $\hat{f}_{\epsilon,\delta}(\xi) = 0$ for $\epsilon > 0$ and $\delta > 0$. Passing to the limit successively, we have $\hat{f}_{\epsilon,0}(\xi) = 0$ for all $\xi < 0$. Finally, $\hat{f}(\xi) = \hat{f}_{0,0}(\xi) = 0$ for all $\xi < 0$. □

13. The Gamma function

Definition 32 (Gamma function)**.** *For $s > 0$,*

$$\Gamma(s) = \int_0^\infty e^{-t} t^{s-1} dt. \tag{13.1}$$

This integral converges for $s > 0$, because near $t = 0$, the integral of t^{s-1} converges when $s > 0$.

Proposition 20. *The gamma function extends to an analytic function in $Re(s) > 0$ and is given by the integral* (13.1)*.*

Proof. Take the strip $S_{\delta,M} = \{\delta < Re(s) < M\}$. Then the integral can be approximated by

$$\Gamma(s) = \int_0^\infty e^{-t} t^{s-1} dt = \lim_{\epsilon \searrow 0} \int_\epsilon^{1/\epsilon} e^{-t} t^{s-1} dt. \tag{13.2}$$

Indeed,

$$|\Gamma(s) - F_\epsilon(s)| \leq \int_0^\epsilon e^{-t} t^{Re(s)-1} dt + \int_{1/\epsilon}^\infty e^{-t} t^{Re(s)-1} dt \to 0, \tag{13.3}$$

uniformly in $S_{\delta,M}$. □

Now then, it is possible to define $\Gamma(s)$ for other values of s using analytic continuation.

Lemma 11. *If $Re(s) > 0$ then $\Gamma(s+1) = s\Gamma(s)$. Then $\Gamma(n+1) = n!$ for $n = 0, 1, 2, \ldots.$*

Proof. By direct computation,

$$\int_{\epsilon}^{1/\epsilon} \frac{d}{dt}(e^{-t}t^s)dt = -\int_{\epsilon}^{1/\epsilon} e^{-t}t^s dt + s\int_{\epsilon}^{1/\epsilon} e^{-t}t^{s-1}dt \to 0, \tag{13.4}$$

as $\epsilon \searrow 0$. The first term converges to $-\Gamma(s+1)$ and the second term converges to $s\Gamma(s)$. Then,

$$\Gamma(1) = \int_0^\infty e^{-t}dt = 1. \tag{13.5}$$

Therefore, $\Gamma(n+1) = n!$. □

Theorem 49. *The functions $\Gamma(s)$ initially defined for $Re(s) > 0$ has an analytic continuation to a meromorphic function on $\mathbb{C}$, whose only singularities are simple poles at negative integers, $s = 0, -1, -2....$ The residue of $\Gamma(s)$ at $s = -n$ is $\frac{(-1)^n}{n!}$.*

Proof. It suffices to extend to $Re(s) > -1$. Define $F_1(s) = \frac{\Gamma(s+1)}{s}$. $\Gamma(s+1)$ is holomorphic in $Re(s) > -1$ so F_1 is meromorphic with possibly only a simple pole at $s = 0$ with residue 1. Then, for $Re(s) > 0$,

$$F_1(s) = \frac{\Gamma(s+1)}{s} = \Gamma(s). \tag{13.6}$$

Therefore, $F_1(s)$ extends $\Gamma(s)$ to a meromorphic function on $Re(s) > -1$. More generally, for $Re(s) > -m$,

$$F_m(s) = \frac{\Gamma(s+m)}{(s+m-1)\cdots s}. \tag{13.7}$$

Then, $F_m(s)$ is meromorphic in $Re(s) > -m$ and has simple poles at $s = 0, -1, ..., -m+1$. Then,

$$res_{s=-n}F_m(s) = \frac{\Gamma(-n+m)}{(m-1-n)!(-1)(-2)\cdots(-n)} = \frac{(-1)^n}{n!}. \tag{13.8}$$

Also, $F_m(s) = \Gamma(s)$ for $Re(s) > 0$. This gives the desired continuation of $\Gamma(s)$. Furthermore, by construction, $\Gamma(s+1) = s\Gamma(s)$ for $s \neq 0, -1, -2, ...$ Also,

$$res_{s=-n}\Gamma(s+1) = -n res_{s=-n}\Gamma(s). \tag{13.9}$$

When $s = 0$, $\Gamma(1) = \lim_{s\to 0} s\Gamma(s)$.

We may also split

$$\Gamma(s) = \int_0^1 e^{-t}t^{s-1}dt + \int_1^\infty e^{-t}t^{s-1}dt. \tag{13.10}$$

The integral $\int_1^\infty e^{-t}t^{s-1}dt$ is an entire function. Taking a power series,

$$\int_0^1 e^{-t}t^{s-1}dt = \sum_{n=0}^\infty \int_0^1 \frac{(-1)^n}{n!}t^n t^{s-1}dt = \sum_{n=0}^\infty \frac{(-1)^n}{n!(n+s)}. \tag{13.11}$$

Therefore, when $Re(s) > 0$,

$$\Gamma(s) = \sum_{n=0}^\infty \frac{(-1)^n}{n!(n+s)} + \int_1^\infty e^{-t}t^{s-1}dt. \tag{13.12}$$

This gives a meromorphic function with poles at the negative integers with residues $\frac{(-1)^n}{n!}$ at $s = -n$. Next,

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{n!(n+s)} = \sum_{n=0}^{N} \frac{(-1)^n}{n!(n+s)} + \sum_{n=N+1}^{\infty} \frac{(-1)^n}{n!(n+s)}. \tag{13.13}$$

When $N > 2R$, (13.13) is a meromorphic function on $|s| < R$ with poles at the desired points. If $|s| > R$ then $|\frac{(-1)^n}{n!(n+s)}| \leq \frac{1}{n!R}$ so the sum converges. □

Theorem 50. *For all $s \in \mathbb{C}$,*

$$\Gamma(s)\Gamma(1-s) = \frac{\pi}{\sin(\pi s)}. \tag{13.14}$$

Proof. Observe that $\Gamma(1-s)$ has simple poles for $s = 1, 2, 3, ...$ and $\Gamma(s)$ has simple poles at $s = 0, -1, ...$ Therefore, $\Gamma(s)\Gamma(1-s)$ is meromorphic on $\mathbb{C}$ and has simple poles at the integers. Furthermore, observe that $\Gamma(s)$ has a residue of $\frac{(-1)^n}{n!}$ at the integers $n = 0, -1, -2, ...$ Since $\Gamma(1-s) = n!$ at $s = n = 0, -1, -2, ...$, the residues match. Therefore, to prove (13.14), it remains to prove equality for $0 < s < 1$.

Lemma 12. *For $0 < a < 1$,*

$$\int_0^\infty \frac{v^{a-1}}{1+v} dv = \frac{\pi}{\sin(\pi a)}. \tag{13.15}$$

Proof. Make the change of variables $v = e^x$, $dv = e^x dx$. Then,

$$\int_0^\infty \frac{v^{a-1}}{1+v} dv = \int_{-\infty}^\infty \frac{e^{ax}}{1+e^x} dx = \frac{\pi}{\sin(\pi a)}. \tag{13.16}$$

The last equality uses Lemma 7. Now then,

$$\Gamma(1-s) = \int_0^\infty e^{-u} u^{-s} du = t \int_0^\infty e^{-vt} (vt)^{-s} dv. \tag{13.17}$$

Therefore,

$$\begin{aligned} \Gamma(1-s)\Gamma(s) &= \int_0^\infty e^{-t} t^{s-1} \Gamma(1-s) dt = \int_0^\infty e^{-t} t^{s-1} (t \int_0^\infty e^{-vt} (vt)^{-s} dv) dt \\ &= \int_0^\infty \int_0^\infty e^{-t(1+v)} v^{-s} dv dt = \int_0^\infty \frac{v^{-s}}{1+v} dv = \frac{\pi}{\sin(\pi(1-s))} = \frac{\pi}{\sin(\pi s)}. \end{aligned} \tag{13.18}$$

□

Therefore Theorem 50 is true. □

Then if $s = \frac{1}{2}$, since $\Gamma(s) > 0$ when $s > 0$, $\Gamma(\frac{1}{2}) = \sqrt{\pi}$.

Theorem 51. *The function Γ has the following properties.*

(1) *The function $\frac{1}{\Gamma(s)}$ is an entire function of s with simple zeros $s = 0, -1, -2, ...$ and it vanishes nowhere else.*

(2) *The function $\frac{1}{\Gamma(s)}$ has the growth*

$$|\frac{1}{\Gamma(s)}| \leq C_1 e^{C_2 |s| \log |s|}. \tag{13.19}$$

Therefore, $|\frac{1}{\Gamma(s)}|$ *is of order one such that for all* $\epsilon > 0$ *there exists* $C(\epsilon)$ *such that* $|\frac{1}{\Gamma(s)}| \leq C(\epsilon)e^{C_2|s|^{1+\epsilon}}$.

Proof. Recall that

$$\frac{1}{\Gamma(s)} = \Gamma(1-s)\frac{\sin \pi s}{\pi}. \tag{13.20}$$

The simple poles of $\Gamma(1-s)$ at $s = 1, 2, 3, ...$ are cancelled by zeros of $\sin(\pi s)$. Therefore, $\frac{1}{\Gamma(s)}$ is entire with simple zeros at $s = 0, -1, -2, ...$ Now then,

$$\int_1^\infty e^{-t}t^{Re(s)}dt \leq e^{(Re(s)+1)\log(Re(s)+1)}. \tag{13.21}$$

Indeed, by Stirling's approximation, if $Re(s) \leq n \leq Re(s) + 1$,

$$\int_1^\infty e^{-t}t^{Re(s)}dt \leq \int_0^\infty e^{-t}t^n dt = n! \leq e^{(Re(s)+1)\log(Re(s)+1)}. \tag{13.22}$$

Therefore,

$$\frac{1}{\Gamma(s)} = (\sum_{n=0}^\infty \frac{(-1)^n}{n!(n+1-s)})\frac{\sin(\pi s)}{\pi} + (\int_1^\infty e^{-t}t^{-s}dt)\frac{\sin(\pi s)}{\pi}. \tag{13.23}$$

Since $|\int_1^\infty e^{-t}t^{-s}dt| \leq e^{(Re(s)+1)\log(Re(s)+1)}$ and $|\sin(\pi s)| \leq e^{\pi|s|}$, so the second term on the right hand side of (13.23) is bounded by $C_1 e^{C_2|s|\log|s|}$.

Next, consider

$$\sum_{n=0}^\infty \frac{(-1)^n}{n!(n+1-s)}\frac{\sin(\pi s)}{\pi}. \tag{13.24}$$

If $|Im(s)| > 1$, this sum is bounded by $Ce^{\pi|s|}$. If $|Im(s)| \leq 1$ choose k such that $k - \frac{1}{2} \leq Re(s) \leq k + \frac{1}{2}$. Then for $k \geq 1$,

$$\sum_{n=0}^\infty \frac{(-1)^n}{n!(n+1-s)}\frac{\sin(\pi s)}{\pi} = (-1)^{k-1}\frac{\sin(\pi s)}{(k-1)!(k-s)\pi} + \sum_{n\neq k-1}(-1)^n\frac{\sin(\pi s)}{n!(n+1-s)\pi}. \tag{13.25}$$

Both terms on the right hand side are bounded since $\frac{\sin(\pi s)}{(k-s)}$ is a removable singularity. For $k \leq 0$, $Re(s) < \frac{1}{2}$ so

$$\sum_{n=0}^\infty \frac{(-1)^n}{n!(n+1-s)} \leq C\sum \frac{1}{n!}. \tag{13.26}$$

□

By the Hadamard factorization theorem,

Lemma 13. *For all* $s \in \mathbb{C}$,

$$\frac{1}{\Gamma(s)} = e^{\gamma s}\prod_{n=1}^\infty (1 + \frac{s}{n})e^{-s/n}. \tag{13.27}$$

Theorem 52. *Suppose f is entire and has growth order ρ_0, that is, $|f(z)| \le Ae^{B|z|^\rho}$. Let k be the integer so that $k \le \rho_0 < k+1$. If $a_1, a_2, \ldots$ denote the non-zero zeros of f, then*

$$f(z) = e^{P(z)} z^m \prod_{n=1}^{\infty} E_k(\frac{z}{a_n}), \tag{13.28}$$

where P is a polynomial of degree $\le k$ and m is the order of the zero at 0. Also, $E_0(z) = 1 - z$ and $E_k(z) = (1-z)e^{z+z^2/2+\ldots+z^k/k}$, for $k \ge 1$.

Proof of Lemma 13. Since $\frac{1}{\Gamma}$ is an entire function of growth order one, with simple zeros at $s = 0, -1, -2, \ldots,$

$$\frac{1}{\Gamma(s)} = e^{As+B} s \prod_{n=1}^{\infty} (1 + \frac{s}{n}) e^{-s/n}. \tag{13.29}$$

Now then, since $s\Gamma(s) \to 1$ as $s \to 0$, $B = 0$. For $s = 1$,

$$\begin{aligned} 1 = \frac{1}{\Gamma(1)} &= e^A \lim_{N\to\infty} \prod_{n=1}^{N} (1 + \frac{1}{n}) e^{-1/n} = \lim_{N\to\infty} e^A e^{\sum_{n=1}^N [\log(1+\frac{1}{n}) - \frac{1}{n}]} \\ &= \lim_{N\to\infty} e^A e^{-(\sum_{n=1}^N \frac{1}{n}) + \log N + \log(1+\frac{1}{N})} = e^{-\gamma} e^A. \end{aligned} \tag{13.30}$$

□

Therefore, $A = \gamma + 2\pi i k$, $k \in \mathbb{Z}$. Since $\Gamma(s) \in \mathbb{R}$ whenever $s \in \mathbb{R}$, $k = 0$. Therefore, $\frac{1}{\Gamma}$ is essentially characterized as an entire function that satisfies:

(1) Simple zeros at $s = 0, -1, -2, \ldots$ and vanishes nowhere else.
(2) Has order of growth ≤ 1.

Remark 7. *The order of growth of a function f is given by $\rho_f = \inf \rho$, where $|f(z)| \le Ae^{B|z|^\rho}$ for some A, B.*

Definition 33. *The constant $\gamma \in \mathbb{R}$ is Euler's constant,*

$$\gamma = \lim_{N\to\infty} \sum_{n=1}^{N} \frac{1}{n} - \log N. \tag{13.31}$$

This constant is well–defined since

$$\sum_{n=1}^{N} \frac{1}{n} - \log N = \sum_{n=1}^{N} \frac{1}{N} - \int_1^N \frac{1}{x} dx = \sum_{n=1}^{N-1} \int_n^{n+1} [\frac{1}{n} - \frac{1}{x}] dx + \frac{1}{N}. \tag{13.32}$$

By the mean value theorem, $|\frac{1}{n} - \frac{1}{x}| \le \frac{1}{n^2}$ for $n \le x \le n+1$. Therefore, the right hand side of (13.31) is equal to

$$\sum_{n=1}^{N-1} a_n + \frac{1}{N}, \qquad 0 \le a_n \le \frac{1}{n^2}. \tag{13.33}$$

Therefore, the sum $\sum_n a_n$ converges.

14. Zeta function

The zeta function is defined for $Re(s) > 1$ by the convergent series

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}. \tag{14.1}$$

This function can be extended to the half plane in $\mathbb{C}$, $Re(s) > 1$.

Proposition 21. *The series defining $\zeta(s)$ converges for $Re(s) > 1$ and $\zeta(s)$ is holomorphic in $Re(s) > 1$.*

Proof. If $s = \sigma + it$ then $|n^{-s}| = n^{-\sigma}$. Then the derivative of

$$\frac{d}{ds} \sum_{n=1}^{\infty} \frac{1}{n^s} = \sum_{n=1}^{\infty} \frac{\log(n)}{n^s}, \tag{14.2}$$

which converges when $\sigma > 1$. $\square$

The analytic continuation of ζ to a meromorphic function in $\mathbb{C}$ relates ζ and Γ to the theta function. Let

$$\theta(t) = \sum_{n=-\infty}^{\infty} e^{-\pi n^2 t} = t^{-1/2}\theta(\frac{1}{t}). \tag{14.3}$$

Then as $t \searrow 0$, $\theta(t) \leq t^{-1/2}$. Also, for $t \geq 1$, $|\theta(t) - 1| \leq Ce^{-\pi t}$.

Theorem 53. *If $Re(s) > 1$ then*

$$\pi^{-s/2}\Gamma(\frac{s}{2})\zeta(s) = \frac{1}{2}\int_0^{\infty} u^{s/2-1}[\theta(u) - 1]du. \tag{14.4}$$

Proof. For any $n \geq 1$, by a change of variables

$$\int_0^{\infty} e^{-\pi n^2 u} u^{s/2-1} du = \frac{1}{\pi n^2}\int_0^{\infty} e^{-t}(\frac{t}{\pi n^2})^{s/2-1} dt = (\pi n^2)^{-s/2}\Gamma(\frac{s}{2}). \tag{14.5}$$

Since $\frac{\theta(u)-1}{2} = \sum_{n=1}^{\infty} e^{-\pi n^2 u}$,

$$\begin{aligned} \frac{1}{2}\int_0^{\infty} u^{s/2-1}[\theta(u) - 1]du &= \sum_{n=1}^{\infty}\int_0^{\infty} u^{s/2-1} e^{-\pi n^2 u} du \\ &= \pi^{-s/2}\Gamma(\frac{s}{2})\sum_{n=1}^{\infty} n^{-s} = \pi^{-s/2}\Gamma(\frac{s}{2})\zeta(s). \end{aligned} \tag{14.6}$$

We will call this function $\xi(s)$. $\square$

Definition 34 (ξ function)**.** *For $Re(s) > 1$,*

$$\xi(s) = \pi^{-s/2}\Gamma(\frac{s}{2})\zeta(s). \tag{14.7}$$

Theorem 54. *The function $\xi(s)$ may be analytically continued from $Re(s) > 1$ to a meromorphic function with simple poles at $s = 0$ and $s = 1$. Also,*

$$\xi(s) = \xi(1 - s), \qquad \forall s \in \mathbb{C}. \tag{14.8}$$

Proof. Let $\psi(u) = \frac{\theta(u)-1}{2}$. Because $\theta(u) = u^{-1/2}\theta(\frac{1}{u})$,

$$\psi(u) = \frac{\theta(u)-1}{2} = \frac{u^{-1/2}\theta(\frac{1}{u})-1}{2} = u^{-1/2}\psi(\frac{1}{u}) + \frac{1}{2u^{1/2}} - \frac{1}{2}. \tag{14.9}$$

Then whenever $Re(s) > 1$,

$$\begin{aligned}\xi(s) = \pi^{-s/2}\Gamma(\frac{s}{2})\zeta(s) &= \int_0^\infty u^{s/2-1}\psi(u)du = \int_0^1 u^{s/2-1}\psi(u)du + \int_1^\infty u^{s/2-1}\psi(u)du \\ &= \int_0^1 u^{s/2-1}[u^{-1/2}\psi(\frac{1}{u}) + \frac{1}{2u^{1/2}} - \frac{1}{2}]du + \int_1^\infty u^{s/2-1}\psi(u)du,\end{aligned} \tag{14.10}$$

so by a change of variables,

$$= \frac{1}{s-1} - \frac{1}{s} + \int_1^\infty (u^{-s/2-1/2} + u^{s/2-1})\psi(u)du. \tag{14.11}$$

Therefore $\xi(s)$ has an analytic continuation to all of $\mathbb{C}$ with simple poles at $s = 0$ and $s = 1$. Moreover, this integral is unchanged after interchanging s and $1-s$. □

We can use this to meromorphically continue $\zeta(s)$ to $\mathbb{C}$.

Theorem 55. *The zeta function has a meromorphic continuation into the entire complex plane, whose only singularity is a simple pole at $s = 1$.*

Proof. Define

$$\zeta(s) = \pi^{s/2}\frac{\xi(s)}{\Gamma(\frac{s}{2})}. \tag{14.12}$$

Since $\frac{1}{\Gamma(\frac{s}{2})}$ is entire with simple zeros at $0, -2, -4, ...$, the simple pole of $\xi(s)$ is canceled out by the zero of $\frac{1}{\Gamma(\frac{s}{2})}$. Therefore, the only singularity of $\zeta(s)$ is at $s = 1$. □

There is another more elementary proof of extending $\zeta(s)$ to $Re(s) > 0$.

Proposition 22. *There is a sequence of entire functions $\{\delta_n(s)\}_{n=1}^\infty$ satisfying $|\delta_n(s)| \le \frac{|s|}{n^{\sigma+1}}$, where $s = \sigma + it$ and such that for a large integer N,*

$$\sum_{1\le n<N} \frac{1}{n^s} - \int_1^N \frac{dx}{x^s} = \sum_{1\le n<N} \delta_n(s). \tag{14.13}$$

Proof. Let

$$\delta_n(s) = \int_n^{n+1} [\frac{1}{n^s} - \frac{1}{x^s}]dx. \tag{14.14}$$

By the mean value theorem, if $s = \sigma + it$,

$$|\frac{1}{n^s} - \frac{1}{x^s}| \le \frac{|s|}{n^{\sigma+1}}. \tag{14.15}$$

□

Corollary 19. *For $Re(s) > 0$,*

$$\zeta(s) - \frac{1}{s-1} = H(s), \tag{14.16}$$

where $H(s) = \sum_{n=1}^\infty \delta_n(s)$ is holomorphic in $Re(s) > 0$.

Proof. When $Re(s) > 1$, observe that since $|\delta_n(s)| \leq \frac{|s|}{n^{\sigma+1}}$, so the series $\sum \delta_n(s)$ converges uniformly for any $Re(s) > \delta$ for any $\delta > 0$. If $Re(s) > 1$, the series $\sum n^{-s}$ converges to $\zeta(s)$, and $\zeta(s)$ is analytic in $Re(s) > 1$. Uniform convergence also shows that $\sum \delta_n(s)$ is holomorphic when $Re(s) > 0$, and thus $\zeta(s)$ is extendable to this half plane. □

Remark 8. *The idea described above can be used to extend ζ to the entire complex plane.*

Now for $s = 1 + it$, the growth of $\zeta'(s)$ and $\zeta(s)$ is bounded by $C_\epsilon |t|^\epsilon$ for any $\epsilon > 0$.

Proposition 23. *Suppose $s = \sigma + it$ with σ, $t \in \mathbb{R}$. Then for each σ_0, $0 \leq \sigma_0 \leq 1$, and every $\epsilon > 0$, there exists C_ϵ such that*

1. $|\zeta(s)| \leq C_\epsilon |t|^{1-\sigma_0+\epsilon}$ *for* $\sigma_0 \leq \sigma$ *and* $|t| \geq 1$,
2. $|\zeta'(s)| \leq C_\epsilon |t|^\epsilon$ *if* $1 \leq \sigma$ *and* $|t| \geq 1$.

Proof. Since we have the trivial bound $|\delta_n(s)| \leq \frac{2}{n^{\sigma_0}}$, for any $0 < \delta < 1$,

$$|\delta_n(s)| \leq (\frac{|s|}{n^{\sigma_0+1}})^\delta (\frac{2}{n^{\sigma_0}})^{1-\delta}. \tag{14.17}$$

Now choose $\delta = 1 - \sigma_0 + \epsilon$. Then

$$|\zeta(s)| \leq |\frac{1}{s-1}| + 2|s|^{1-\sigma_0+\epsilon} \sum_{n=1}^{\infty} \frac{1}{n^{1+\epsilon}}. \tag{14.18}$$

Then by the Cauchy integral formula, if $\sigma_0 \geq 1$, $s = \sigma_0 + it$,

$$\zeta'(s) = \frac{1}{2\pi r} \int_0^{2\pi} \zeta(s + re^{i\theta}) e^{i\theta} d\theta, \tag{14.19}$$

so if $r = \epsilon$, the integral is in the half plane $Re(s) \geq 1 - \epsilon$, which proves the proposition. □

15. Zeros of the zeta function

Claim 3. *For $Re(s) > 1$,*

$$\zeta(s) = \prod_p \frac{1}{1 - p^{-s}}. \tag{15.1}$$

Proof. If $2^K \geq N$,

$$\sum_{n=1}^{N} \frac{1}{n^s} \leq \prod_{p \leq N} (1 + \frac{1}{p^s} + ... + \frac{1}{p^{Ks}}) \leq \prod_{p \leq N} (\frac{1}{1 - p^{-s}}) \leq \prod_p (\frac{1}{1 - p^{-s}}). \tag{15.2}$$

The second to last inequality follows from the fact that

$$\frac{1}{1 - p^{-s}} = \sum_{m=0}^{\infty} p^{-ms}. \tag{15.3}$$

Also by the fundamental theorem of arithmetic,

$$\prod_{p \leq N} (1 + \frac{1}{p^s} + ... + \frac{1}{p^{Ks}}) \leq \sum_{n=1}^{\infty} \frac{1}{n^s}. \tag{15.4}$$

Therefore,

$$\prod_p \left(\frac{1}{1-p^{-s}}\right) \leq \sum_{n=1}^{\infty} \frac{1}{n^s}. \tag{15.5}$$

This proves the product formula, (15.5), which then proves that $\zeta(s)$ does not vanish for $Re(s) > 1$. □

Corollary 20. *The sum of the reciprocals of the primes diverges,*

$$\sum_p \frac{1}{p} = \infty. \tag{15.6}$$

Proof. Since $\sum_{n=1}^{N} \frac{1}{n} \nearrow \infty$, (15.2) implies that

$$\prod_{p \leq N} \frac{1}{1-p^{-1}} \nearrow \infty, \tag{15.7}$$

as $N \nearrow \infty$. Since

$$\log\left(\prod_{p \leq N} \frac{1}{1-p^{-1}}\right) = \sum_{p \leq N} \frac{1}{p} + \sum_p O\left(\frac{1}{p^2}\right). \tag{15.8}$$

Since the second term on the right hand side of (15.8) is $O(1)$, this proves $\sum_{p \leq N} \frac{1}{p} \nearrow \infty$. □

Theorem 56. *The only zeros of* ζ *outside the strip* $0 \leq Re(s) \leq 1$ *are at the negative even integers* $-2, -4, -6, ...$

Proof. Since $\xi(s) = \xi(1-s)$,

$$\pi^{-s/2}\Gamma\left(\frac{s}{2}\right)\zeta(s) = \pi^{-(1-s)/2}\Gamma\left(\frac{1-s}{2}\right)\zeta(1-s), \tag{15.9}$$

so

$$\zeta(s) = \pi^{s-1/2}\frac{\Gamma(\frac{1-s}{2})}{\Gamma(\frac{s}{2})}\zeta(1-s). \tag{15.10}$$

Therefore, when $Re(s) < 0$, $\zeta(s) = 0$ if and only if $\frac{1}{\Gamma(\frac{s}{2})} = 0$, which is true if and only if $\frac{s}{2}$ is a negative integer. □

Definition 35 (Critical strip)**.** *The region* $0 \leq Re(s) \leq 1$ *is called the critical strip.*

Theorem 57. *There are no zeros on the line* $Re(s) = 1$, $\zeta(1+it) \neq 0$ *for all* $t \in \mathbb{R}$.

Proof. Use the formula

$$\log\left(\frac{1}{1-x}\right) = \sum_{m=1}^{\infty} \frac{x^m}{m}. \tag{15.11}$$

If $Re(s) > 1$ then

$$\log \zeta(s) = \sum_{p_j, m} \frac{p_j^{-ms}}{m} = \sum_{n=1}^{\infty} c_n n^{-s}, \tag{15.12}$$

where $c_n = \frac{1}{m}$ if $n = p_j^m$ for some p_j, $c_n = 0$ otherwise. This formula holds for all $Re(s) > 1$ by analytic continuation.

Remark 9. *The double sum converges absolutely, so we may sum in either order.*

Next observe that for $\theta \in \mathbb{R}$,

$$3 + 4\cos\theta + \cos 2\theta = 2(1 + \cos\theta)^2 \geq 0. \tag{15.13}$$

Thus,

Lemma 14. *If $\sigma > 1$ and t is real, then*

$$\log |\zeta^3(\sigma)\zeta^4(\sigma + it)\zeta(\sigma + 2it)| \geq 0. \tag{15.14}$$

Proof. Let $s = \sigma + it$. Then,

$$Re(n^{-s}) = Re(e^{-(\sigma+it)\log n}) = e^{-\sigma \log n}\cos(t\log n) = n^{-\sigma}\cos(t\log n). \tag{15.15}$$

Therefore, by (15.12),

$$\begin{aligned} \log |\zeta^3(\sigma)\zeta^4(\sigma + it)\zeta(\sigma + 2it)| &= 3\log|\zeta(\sigma)| + 4\log|\zeta(\sigma + it)| + \log|\zeta(\sigma + 2it)| \\ &= 3Re[\log\zeta(\sigma)] + 4Re[\log\zeta(\sigma + it)] + Re[\log\zeta(\sigma + 2it)] \\ &= \sum c_n n^{-\sigma}(3 + 4\cos\theta_n + \cos 2\theta_n), \qquad \text{where} \qquad \theta_n = t\log(n). \end{aligned} \tag{15.16}$$

Since $c_n \geq 0$, this sum is positive. □

If $\zeta(1 + it_0) = 0$, since $\zeta(1 + it_0) = 0$, ζ must vanish to at least order one at $1 + it_0$. Thus,

$$|\zeta(\sigma + it_0)|^4 \leq C(\sigma - 1)^4, \qquad \sigma \to 1. \tag{15.17}$$

Since $s = 1$ is a simple pole for $\zeta(s)$,

$$|\zeta(\sigma)|^3 \leq C'(\sigma - 1)^{-3}, \qquad \sigma \to 1. \tag{15.18}$$

Finally since $\zeta(s)$ is holomorphic near $\sigma + 2it_0$, $|\zeta(\sigma + 2it_0)|$ remains bounded as $\sigma \to 1$, which means that the product on the left hand side of (15.16) goes to zero. However this means that the logarithm goes to negative infinity, which contradicts (15.14). □

Proposition 24. *For every $\epsilon > 0$ there exists a constant C_ϵ such that*

$$\frac{1}{|\zeta(s)|} \leq C_\epsilon |t|^\epsilon, \tag{15.19}$$

when $s = \sigma + it$, $\sigma \geq 1$, and $|t| \geq 1$.

Proof. By Lemma 14,

$$|\zeta^3(\sigma)\zeta^4(\sigma + it)\zeta(\sigma + 2it)| \geq 1. \tag{15.20}$$

This along with $|\zeta(\sigma + 2it)| \leq C_\epsilon|t|^\epsilon$ from Proposition 23 implies that

$$|\zeta^4(\sigma + it)| \geq c|\zeta^{-3}(\sigma)||t|^{-\epsilon} \geq c'(\sigma - 1)^3|t|^{-\epsilon}. \tag{15.21}$$

This implies that for $\sigma \geq 1$, $|t| \geq 1$,

$$|\zeta(\sigma + it)| \geq c'(\sigma - 1)^{3/4}|t|^{-\epsilon/4}. \tag{15.22}$$

There are two cases to consider, depending on whether the inequality $\sigma - 1 \geq A|t|^{-5\epsilon}$ holds for some appropriate constant A.

Case 1: If $\sigma - 1 \geq A|t|^{-5\epsilon}$ then

$$|\zeta(\sigma + it)| \geq A'|t|^{-4\epsilon}. \tag{15.23}$$

Case 2: If $\sigma - 1 < A|t|^{-5\epsilon}$, then select $\sigma' > \sigma$ with $\sigma' - 1 = A|t|^{-5\epsilon}$. Then by the triangle inequality,

$$|\zeta(\sigma + it)| \geq |\zeta(\sigma' + it)| - |\zeta(\sigma' + it) - \zeta(\sigma + it)|. \tag{15.24}$$

An application of the mean value theorem along with $|\zeta'(\sigma + it)| \leq C_\epsilon |t|^\epsilon$ from Proposition 23 gives

$$|\zeta(\sigma' + it) - \zeta(\sigma + it)| \leq c''|\sigma' - \sigma||t|^\epsilon \leq c''|\sigma' - 1||t|^\epsilon. \tag{15.25}$$

Therefore,

$$|\zeta(\sigma + it)| \geq c'(\sigma' - 1)^{3/4}|t|^{-\epsilon/4} - c''(\sigma' - 1)|t|^\epsilon. \tag{15.26}$$

Choose $A = (c'/(2c''))^4$. Then we have $\sigma' - 1 = A|t|^{-5\epsilon}$, and

$$c'(\sigma' - 1)^{3/4}|t|^{-\epsilon/4} = 2c''(\sigma' - 1)|t|^\epsilon. \tag{15.27}$$

Therefore, in this case as well,

$$|\zeta(\sigma + it)| \geq A''|t|^{-4\epsilon}. \tag{15.28}$$

Taking the reciprocal proves (15.19) when $|t| \geq 1$. □

16. Reduction to the functions ψ and ψ_1

Let $\Lambda(n)$ denote the Von Mangoldt function

Definition 36 (Von Mangoldt function)**.**

$$\Lambda(n) = \begin{cases} \log(p) & \textit{if } n = p^m \textit{ for some prime } p, \\ 0 & \textit{if otherwise,} \end{cases} \tag{16.1}$$

and let $\psi(x)$ denote the Tchebychev counting function

Definition 37 (Tchebychev ψ - function)**.**

$$\psi(x) = \sum_{p^m \leq x} \log p = \sum_{n \leq x} \Lambda(n) = \sum_{p \leq x} \lfloor \frac{\log x}{\log p} \rfloor (\log p). \tag{16.2}$$

Now let $\pi(x)$ denote the prime counting function,

$$\pi(x) = \sharp\{p_j \leq x : p_j \text{ prime }\}. \tag{16.3}$$

We use the notation $f(x) \sim g(x)$ if $\frac{f(x)}{g(x)} \to 1$ as $x \to \infty$. Proving the asymptotics of $\pi(x)$ is equivalent to proving the asymptotics of $\psi(x)$.

Theorem 58. *$\psi(x) \sim x$ as $x \to \infty$ if and only if $\pi(x) \sim \frac{x}{\log x}$ as $x \to \infty$.*

Proof. First suppose $\psi(x) \sim x$.

$$\psi(x) = \sum_{p \leq x} [\frac{\log x}{\log p}] \log p \leq \sum_{p \leq x} \frac{\log x}{\log p} \log p = \pi(x) \log(x). \tag{16.4}$$

Therefore,

$$\frac{\psi(x)}{x} \leq \frac{\pi(x) \log x}{x}. \tag{16.5}$$

Therefore, if $\psi(x) \sim x$,

$$1 \leq \liminf_{x \to \infty} \pi(x) \frac{\log x}{x}. \tag{16.6}$$

Now fix $0 < \alpha < 1$.

$$\psi(x) \geq \sum_{p \leq x} \log p \geq \sum_{x^{\alpha} < p \leq x} \log p \geq (\pi(x) - \pi(x^{\alpha})) \log x^{\alpha}, \tag{16.7}$$

so

$$\psi(x) + \alpha \pi(x^{\alpha}) \log x \geq \alpha \pi(x) \log x. \tag{16.8}$$

Divide both sides by x and make the trivial bound$\pi(x^{\alpha}) \leq x^{\alpha}$. Then for any $\alpha < 1$,

$$1 \geq \alpha \limsup_{x \to \infty} \pi(x) \frac{\log x}{x}. \tag{16.9}$$

Combining (16.9) and (16.6) shows that $\psi(x) \sim x$ implies $\pi(x) \sim \frac{x}{\log x}$.

Now suppose $\pi(x) \sim \frac{x}{\log(x)}$. Again from (16.4),

$$\psi(x) \leq \log(x) \pi(x), \tag{16.10}$$

so $\pi(x) \sim \frac{x}{\log x}$ implies

$$\limsup_{x \to \infty} \frac{\psi(x)}{x} \leq 1. \tag{16.11}$$

Meanwhile,

$$\sum_{p \leq x} \log(p) = \sum_{p \leq x^{\alpha}} \log(p) + \sum_{x^{\alpha} < p \leq x} \log(p) \geq \log(x) O(x^{\alpha}) + (\pi(x) - x^{\alpha}) \alpha \log(x). \tag{16.12}$$

Therefore, if $\frac{\pi(x)}{x} \sim 1$, then for any $\alpha < 1$,

$$\liminf_{x \to \infty} \frac{\psi(x)}{x} > \alpha. \tag{16.13}$$

This proves the converse. □

Now let $\psi_1(x)$ denote the integral of $\psi(x)$.

$$\psi_1(x) = \int_1^x \psi(u) du. \tag{16.14}$$

The asymptotics of $\psi(x)$ follow from the asymptotics of $\psi_1(x)$.

Proposition 25. *If $\psi_1(x) \sim x^2/2$ as $x \to \infty$, then $\psi(x) \sim x$ as $x \to \infty$.*

Proof. Since $\psi(x)$ is increasing in x, for $\alpha < 1 < \beta$,

$$\frac{1}{(1-\alpha)x} \int_{\alpha x}^{x} \psi(u) du \leq \psi(x) \leq \frac{1}{(\beta - 1)x} \int_x^{\beta x} \psi(u) du. \tag{16.15}$$

Therefore, for any $\beta > 1$,

$$\psi(x) \leq \frac{1}{(\beta - 1)x} [\psi_1(\beta x) - \psi_1(x)], \tag{16.16}$$

so

$$\frac{\psi(x)}{x} \leq \frac{1}{(\beta - 1)} [\frac{\psi_1(\beta x)}{(\beta x)^2} \beta^2 - \frac{\psi_1(x)}{x^2}]. \tag{16.17}$$

This implies

$$\limsup_{x\to\infty} \frac{\psi(x)}{x} \leq \frac{1}{\beta-1}[\frac{1}{2}\beta^2 - \frac{1}{2}] = \frac{1}{2}(\beta+1). \tag{16.18}$$

This proves $\limsup \psi(x)/x \leq 1$. A similar argument shows $\liminf \psi(x)/x \geq 1$. □

Now we are are ready to relate ψ_1 and ζ. To see how, recall that

$$\log \zeta(s) = \sum_{m,p} \frac{p^{-ms}}{m}, \tag{16.19}$$

so

$$\frac{\zeta'(s)}{\zeta(s)} = -\sum_{m,p} (\log p) p^{-ms} = -\sum_{n=1}^{\infty} \frac{\Lambda(n)}{n^s}. \tag{16.20}$$

It is possible to use (16.20) to write $\psi_1(x)$ as a contour integral.

Proposition 26. *For all $c > 1$,*

$$\psi_1(x) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \frac{x^{s+1}}{s(s+1)} (-\frac{\zeta'(s)}{\zeta(s)}) ds. \tag{16.21}$$

Proof. If $c > 0$,

$$\frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \frac{a^s}{s(s+1)} ds = \begin{cases} 1 - \frac{1}{a} & \text{if } a \geq 1 \\ 0 & \text{if } 0 < a \leq 1 \end{cases} \tag{16.22}$$

Indeed suppose $1 \leq a = e^{\beta}$, $\beta = \log a \geq 0$. Then

$$f(s) = \frac{a^s}{s(s+1)} = \frac{e^{s\beta}}{s(s+1)}, \tag{16.23}$$

is a function with simple poles at $s = 0$ and $s = 1$. The residue at $s = 0$ is 1 and the residue at $s = -1$ is $\frac{-1}{a}$. Let $\Gamma(T)$ be the contour passing from $c - iT$ to $c + iT$ and then making a semicircle on the left hand side of the line $c + it$. The residue formula implies

$$\frac{1}{2\pi i} \int_{\Gamma(T)} f(s) ds = 1 - \frac{1}{a}. \tag{16.24}$$

Moreover, the integral over the half circle goes to zero. If $0 < a \leq 1$ then the contour must go to the right. In this case the contour contains no poles, and so the integral is zero.

Now then,

$$\psi(u) = \sum_{n=1}^{\infty} \Lambda(n) f_n(u), \tag{16.25}$$

where $f_n(u) = 1$ if $n \leq u$ and zero otherwise.

$$\psi_1(x) = \int_0^x \psi(u) du = \sum_{n=1}^{\infty} \int_0^x \Lambda(n) f_n(u) du = \sum_{n \leq x} \Lambda(n) \int_n^x du = \sum_{n \leq x} \Lambda(n)(x - n). \tag{16.26}$$

By direct computation,

(16.27)

$$\frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \frac{x^{s+1}}{s(s+1)} (-\frac{\zeta'(s)}{\zeta(s)}) ds = x \sum_{n=1}^{\infty} \Lambda(n) \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \frac{(x/n)^s}{s(s+1)} ds = x \sum_{n \leq x} \Lambda(n)(1 - \frac{n}{x}) = \psi_1(x).$$

This proves the proposition. □

It remains to be shown that $\psi_1(x) \sim x^2/2$. There are two important facts that will be utilized in this proof. The integral formula,

$$\psi_1(x) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \frac{x^{s+1}}{s(s+1)}(-\frac{\zeta'(s)}{\zeta(s)})ds, \qquad c > 1, \tag{16.28}$$

as well as

$$\zeta(1+it) \neq 0 \qquad \forall t \in \mathbb{R}. \tag{16.29}$$

Fix $c = 2 > 1$ and assume x is fixed, $x \geq 2$. Let $\gamma(T)$ be the path that travels along 1 - it, goes around s = 1 as a three sided rectangle with sides $\{t - iT : 1 \leq t \leq 2\}$, $\{2 + it : -T \leq t \leq T\}$, and $\{t + iT : 1 \leq t \leq 2\}$, and then continues along the path 1 + it. There are no residues in between, and $|\frac{\zeta'(s)}{\zeta(s)}| \leq A|t|^{\eta}$ for any fixed $\eta > 0$. Therefore, $|F(s)| \leq A'|t|^{-2+\eta}$, so

$$\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} F(s)ds = \frac{1}{2\pi i}\int_{\gamma(T)} F(s)ds. \tag{16.30}$$

Now take the contour $\gamma(T, \delta)$. This contour travels along s = 1 + it until it $t = -T$, then it turns left and travels over to $1 - \delta - iT$, then travels up to $1 - \delta + iT$, then turns right and travels back to the line Re(s) = 1, and finally goes to infinity. This contour consists of five lines, γ_1, ..., γ_5. For $Re(s) > 0$,

$$\zeta(s) = \frac{1}{s-1} + H(s), \tag{16.31}$$

where $H(s)$ is a regular holomorphic function. The residue of $F(s)$ at $s = 1$ equals $\frac{x^2}{2}$. Therefore,

$$\frac{1}{2\pi i}\int_{\gamma(T)} F(s)ds = \frac{x^2}{2} + \frac{1}{2\pi i}\int_{\gamma(T,\delta)} \frac{x^{s+1}}{s(s+1)}(-\frac{\zeta'(s)}{\zeta(s)})ds. \tag{16.32}$$

Now decompose the contour and estimate each piece. There exists a T sufficiently large such that $|\int_{\gamma_1} F(s)ds| \leq \frac{\epsilon}{2}x^2$ and $|\int_{\gamma_5} F(s)ds| \leq \frac{\epsilon}{2}x^2$. This follows from $|x^{1+s}| = x^2$ and $|\zeta'(s)/\zeta(s)| \leq A|t|^{1/2}$. Therefore,

$$|\int_{\gamma_1,\gamma_5} F(s)ds| \leq Cx^2 \int_T^{\infty} \frac{|t|^{1/2}}{t^2}dt. \tag{16.33}$$

Choose $\delta > 0$ sufficiently small so that no other zeros are enclosed by the contours $\gamma(T, \delta)$ and $\gamma(T)$. Then,

$$|x^{1+s}| = x^{2-\delta}, \tag{16.34}$$

so there exists a constant C_T such that

$$|\int_{\gamma_3} F(s)ds| \leq C_T x^{2-\delta}. \tag{16.35}$$

Finally,

$$|\int_{\gamma_2,\gamma_4} F(s)ds| \leq C_T' \int_{1-\delta}^{1} x^{1+\sigma}d\sigma \leq C_T' \frac{x^2}{\log x}. \tag{16.36}$$

Therefore asymptotically, the term $x^2/2$ will win against all the other terms. This gives us $\gamma_1(x) \sim x^2/2$. This concludes the proof of the prime number theorem.

17. Jensen's formula

Recall the fundamental theorem of algebra, which says that if $p(z)$ is a polynomial of order n, then $p(z)$ has n zeroes. Here, we obtain a bound on the size of $|f(0)|$ from the location of the zeroes in a disk.

Theorem 59 (Jensen's formula)**.** *Let Ω be an open set that contains the closure of a disk D_R and suppose that f is holomorphic in Ω, $f(0) \neq 0$, and f vanishes nowhere on C_R. If $z_1, ..., z_N$ denote the zeroes of f inside the disk (counted with multiplicities),*

$$\log|f(0)| = \sum_{k=1}^{N} \log(\frac{|z_k|}{R}) + \frac{1}{2\pi}\int_0^{2\pi} \log|f(Re^{i\theta})|d\theta. \tag{17.1}$$

Proof. The function

$$\frac{f(z)}{(z-z_1)\cdots(z-z_N)}, \tag{17.2}$$

is bounded near each z_j and is defined on $\Omega \setminus \{z_1, ..., z_N\}$. Then, each z_j is a removable singularity so

$$f(z) = (z-z_1)\cdots(z-z_N)g(z), \tag{17.3}$$

where $g(z)$ is holomorphic and nowhere vanishing on $\bar{D}_R$. Then in a slightly larger disk, Theorem 40 implies

$$g(z) = e^{h(z)}, \tag{17.4}$$

where $h(z)$ is holomorphic in D_R. Therefore,

$$|g(z)| = |e^{h(z)}| = e^{Re(h(z))}, \tag{17.5}$$

so $\log|g(z)| = Re(h(z))$, and

$$\log|g(0)| = Re(h(0)) = \frac{1}{2\pi}\int_0^{2\pi} Re(h(Re^{i\theta}))d\theta = \frac{1}{2\pi}\int_0^{2\pi} \log|g(Re^{i\theta})|d\theta. \tag{17.6}$$

Now consider the function $f(z) = z - w$, where $w \in D_R$ and $z \in C_R$. The function $f(z)$ has a simple zero at w, and

$$\begin{aligned} &\log(\frac{|w|}{R}) + \frac{1}{2\pi}\int_0^{2\pi} \log|Re^{i\theta} - w|d\theta \\ &= \log(|w|) - \log R + \frac{1}{2\pi}\int_0^{2\pi} \log R + \log|e^{i\theta} - \frac{w}{R}|d\theta \\ &= \log(|w|) + \frac{1}{2\pi}\int_0^{2\pi} \log|e^{i\theta} - \frac{w}{R}|d\theta. \end{aligned} \tag{17.7}$$

Now for $|a| < 1$,

$$\int_0^{2\pi} \log|e^{i\theta} - a|d\theta = \int_0^{2\pi} \log|1 - ae^{i\theta}|d\theta. \tag{17.8}$$

Since $1 - ae^{i\theta}$ lies to the right of the origin there exists some $G(z)$ such that

$$e^{G(z)} = F(z) = 1 - az, \tag{17.9}$$

for z in a disk of radius $R > 1$. Then $\log|F| = Re(G)$, so $F(0) = 1$ implies that $\log|F(0)| = 0$. Then by the mean value theorem

$$\int_0^{2\pi} Re(G(z))dz = 0. \tag{17.10}$$

□

Jensen's formula implies an identity linking the growth of a holomorphic function with the number of zeros inside a disk. Let f be a function that is holomorphic on the closure of a disk D_R and let $n_f(r)$ be the number of zeros of f in D_r. If $f(0) \neq 0$ and f does not vanish on a circle C_R, then

$$\int_0^R n(r)\frac{dr}{r} = \frac{1}{2\pi}\int_0^{2\pi} \log|f(Re^{i\theta})|d\theta - \log|f(0)|. \tag{17.11}$$

Indeed, (17.11) follows from (17.1) and

Lemma 15. *If $z_1, ..., z_N$ are the zeros of f inside the disk D_R, then*

$$\int_0^R n(r)\frac{dr}{r} = \sum_{k=1}^N \log(\frac{R}{|z_k|}). \tag{17.12}$$

Proof. Compute

$$\sum_{k=1}^N \log(\frac{R}{|z_k|}) = \sum_{k=1}^N \int_{|z_k|}^R \frac{dr}{r}. \tag{17.13}$$

Rearranging the order of sums,

$$\sum_{k=1}^N \int_{|z_k|}^R \frac{dr}{r} = \int_0^R n(r)\frac{dr}{r}. \tag{17.14}$$

□

It is possible to apply this formula to determine a relationship between the growth of a function and the number of zeros of the function in a compact disk.

Definition 38 (Order of a function)**.** *Suppose that $f(z)$ is an entire function, and that there exists a positive number ρ and constants $A, B > 0$ such that*

$$|f(z)| \leq Ae^{B|z|^\rho}, \qquad \forall z \in \mathbb{C}. \tag{17.15}$$

Then f has an order of growth $\leq \rho$. Let

$$\rho_f = \inf \rho \qquad \rho \text{ satisfies } (17.5), \tag{17.16}$$

is the order of growth of f.

For example e^{z^2} has an order of growth of 2.

Theorem 60. *If f is an entire function that has order of growth $\leq \rho$, then*

(1) *For some $C > 0$, r sufficiently large, $n(r) \leq Cr^\rho$,*

(2) *If $z_1, ..., z_N$ are the zeros of f with $z_k \neq 0$ then for all $s > \rho$,*

$$\sum_{k=1}^{\infty} \frac{1}{|z_k|^s} < \infty. \tag{17.17}$$

Proof. The function f has at most a zero of finite order at 0, so without loss of generality suppose $f(0) \neq 0$. Then by (17.2),

$$\int_0^R n(x) \frac{dx}{x} = \frac{1}{2\pi} \int_0^{2\pi} \log |f(Re^{i\theta})| d\theta - \log |f(0)|. \tag{17.18}$$

Choosing $R = 2r$ and using the fact that $n(r) \geq 0$,

$$\int_r^{2r} n(x) \frac{dx}{x} \leq \frac{1}{2\pi} \int_0^{2\pi} \log |f(Re^{i\theta})| d\theta - \log |f(0)|. \tag{17.19}$$

Since $n(r)$ is increasing,

$$\int_r^{2r} n(x) \frac{dx}{x} \geq n(r) \int_r^{2r} \frac{dx}{x} = n(r) \log(2). \tag{17.20}$$

Plugging (17.20) into the left hand side of (17.19),

$$n(r) \log(2) \leq \int_0^{2\pi} \log |f(Re^{i\theta})| d\theta \leq \int_0^{2\pi} \log |Ae^{BR^\rho}| d\theta \leq C' r^\rho. \tag{17.21}$$

Therefore, $n(r) \leq C r^\rho$.

Now to prove (17.17). Since only finitely many zeros lie in the ball $|z| \leq 1$, $\sum_{|z_k| \leq 1, z_k \neq 0} |z_k|^{-s} < \infty$ for any $s \in \mathbb{R}$. Now then, for $|z_k| > 1$, using (17.21),

$$\sum_{|z_k| \geq 1} |z_k|^{-s} = \sum_{j=0}^{\infty} (\sum_{2^j \leq |z| \leq 2^{j+1}} |z_k|^{-s}) \leq \sum_{j=0}^{\infty} 2^{-js} n(2^{j+1}) \leq C \sum_{j=0}^{\infty} 2^{-js} 2^{(j+1)\rho} \leq C' \sum_{j=0}^{\infty} 2^{j(\rho - s)} < \infty. \tag{17.22}$$

□

Remark 10. *This estimate cannot be improved. For example, take*

$$f(z) = \frac{e^{i\pi z} - e^{-i\pi z}}{2i} = \sin(\pi z). \tag{17.23}$$

Then $|f(z)| \leq e^{\pi |z|}$, so f has an order of growth ≤ 1, and

$$\sum_{n \in \mathbf{Z}} \frac{1}{n^s} < \infty, \tag{17.24}$$

precisely when $s > 1$.

For another example, consider the function defined by

$$f(z) = \cos(z^{1/2}) = \sum_{n=0}^{\infty} (-1)^n \frac{z^n}{(2n)!}. \tag{17.25}$$

In this case then f is entire, $|f(z)| \leq e^{|z|^{1/2}}$, and the growth of f is order $\frac{1}{2}$. In this case f vanishes when $z_n = ((n + \frac{1}{2})\pi)^2$, and

$$\sum_n \frac{1}{|z_n|^s} < \infty \tag{17.26}$$

exactly when $s > \frac{1}{2}$.

18. Infinite products

The next question is whether it is possible to define a holomorphic function with zeros precisely at $z_1, z_2, ...$ with

$$\lim_{k \to \infty} |z_k| = \infty, \tag{18.1}$$

and perhaps with $|z_k|$ satisfying a summation condition like (17.26).

For finitely many zeros, it is known that for finitely many zeros z_1, ..., z_N,

$$f(z) = (z - z_1) \cdots (z - z_k). \tag{18.2}$$

The infinite product $\prod_{n=1}^{\infty}(1 + a_n)$ converges if

$$\lim_{N \to \infty} \prod_{n=1}^{N} (1 + a_n) \tag{18.3}$$

exists.

Proposition 27. *If $\sum |a_n| < \infty$ then the product $\prod_{n=1}^{\infty}(1 + a_n)$ converges. Moreover, the product converges to zero if and only if one of its factors is zero.*

Proof. If $|z| < \frac{1}{2}$ then $1 + z = e^{\log(1+z)}$. Therefore,

$$\prod_{n=1}^{N} (1 + a_n) = e^{\sum_{n=1}^{N} \log(1+a_n)}. \tag{18.4}$$

Since $|\log(1 + z)| \leq 2|z|$ if $|z| < \frac{1}{2}$, $\sum_{n=1}^{N} |a_n|$ converges implies $\sum_{n=1}^{N} \log(1 + a_n)$ converges. Also, if $1 + a_n \neq 0$ for all n then the product converges. $\square$

Now consider and infinite product of holomorphic functions.

Proposition 28. *Suppose $\{F_n\}$ is a sequence of holomorphic functions on an open set Ω. If there exists $c_n > 0$ such that $\sum c_n < \infty$ and $|F_n(z) - 1| \leq c_n$ for all $z \in \Omega$, then*

1. *$\prod_{n=1}^{\infty} F_n(z)$ converges uniformly in Ω to a holomorphic function $F(z)$,*
2. *If $F_n(z)$ does not vanish for any n then*

$$\frac{F'(z)}{F(z)} = \sum_{n=1}^{\infty} \frac{F_n'(z)}{F_n(z)}. \tag{18.5}$$

Proof. To prove the first statement, write $F_n(z) = 1 + a_n(z)$. Then $|a_n(z)| \leq c_n$ for any $z \in \Omega$, so the product converges uniformly on Ω. Thus, $\prod_{n=1}^{\infty} F_n(z)$ is holomorphic on Ω.

Now let

$$G_N(z) = \prod_{n=1}^{N} F_n(z). \tag{18.6}$$

$G_N \to F$ uniformly on any compact set $K \subset \Omega$, which implies that $G'_N \to F'$ uniformly on any compact $K \subset \Omega$. Since G_N is uniformly bounded below on K,

$$\frac{G'_N}{G_N} \to \frac{F'}{F} \tag{18.7}$$

uniformly on any compact $K \subset \Omega$. Then since

$$\frac{G'_N}{G_N} = \sum_{n=1}^{N} \frac{F'_n}{F_n}, \tag{18.8}$$

the proof is complete. □

Several examples of infinite products arise in the study of trigonometric functions.

Lemma 16.

$$\pi \cot(\pi z) = \sum_{n=-\infty}^{\infty} \frac{1}{z+n} = \lim_{N\to\infty} \sum_{|n|\leq N} \frac{1}{z+n} = \frac{1}{z} + \sum_{n=1}^{\infty} \frac{2z}{z^2 - n^2}. \tag{18.9}$$

Proof. The function $F(z) = \pi \cot(\pi z)$ satisfies the following properties:

(1) $F(z+1) = F(z)$ whenever z is not an integer,
(2) $F(z) = \frac{1}{z} + F_0(z)$, where $F_0(z)$ is analytic near 0,
(3) $F(z)$ has simple poles at $\mathbb{Z}$ and nowhere else.

The sum

$$\lim_{N\to\infty} \sum_{|n|\leq N} \frac{1}{z+n}, \tag{18.10}$$

satisfies the same properties, and thus, so does

$$\Delta(z) = F(z) - \sum_{n=-\infty}^{\infty} \frac{1}{z+n}. \tag{18.11}$$

Remark 11. *This sum converges since for large n,*

$$\frac{1}{z+n} + \frac{1}{z+(-n)} = \frac{1}{n+z} - \frac{1}{n-z} = \frac{-2z}{n^2 - z^2}. \tag{18.12}$$

Since $F(z)$ and $\frac{1}{z}$ both have simple poles of residue 1 at $z = 0$, the singularity at $z = 0$ for $\Delta(z)$ is removable, and so by periodicity $\Delta(z)$ is an entire function. Moreover, since $\Delta(z)$ is holomorphic, Δ is bounded on $|Im(z)| \leq 1$. For $|Im(z)| > 1$, if $z = x + iy$,

$$\cot(\pi z) = i\frac{e^{i\pi z} + e^{-i\pi z}}{e^{i\pi z} - e^{-i\pi z}} = i\frac{e^{-2\pi y} + e^{-2\pi i x}}{e^{-2\pi y} - e^{-2\pi i x}}. \tag{18.13}$$

This ratio is bounded for $|Im(z)| > 1$. Also,

$$\frac{1}{z} + \sum_{n=1}^{\infty} \frac{2z}{z^2 - n^2} = \frac{1}{x+iy} + \sum_{n=1}^{\infty} \frac{2(x+iy)}{x^2 - y^2 - n^2 + 2ixy}. \tag{18.14}$$

This sum is bounded if $|Im(z)| > 1$. To see why, suppose without loss of generality that $-\frac{1}{2} \leq x \leq \frac{1}{2}$. Then, when $|y| \geq 1$,

$$\sum_{n \leq |y|} |\frac{2(x+iy)}{x^2 - y^2 - n^2 + 2ixy}| \leq \sum_{n \leq |y|} \frac{2(1+|y|)}{\frac{1}{2}y^2} \leq 10, \tag{18.15}$$

and

$$\sum_{n \geq |y|} |\frac{2(x+iy)}{x^2 - y^2 - n^2 + 2ixy}| \leq \sum_{n \geq |y|} \frac{2(1+|y|)}{\frac{1}{2}n^2} \leq 10. \tag{18.16}$$

Therefore, by Liouville's theorem, $\Delta(z)$ is constant, so since $\Delta(z)$ is also odd, $\Delta(z) \equiv 0$. □

Lemma 17. *The product formula for the sine function holds,*

$$\frac{\sin \pi z}{\pi} = z \prod_{n=1}^{\infty} (1 - \frac{z^2}{n^2}). \tag{18.17}$$

Proof. Let $G(z) = \frac{\sin(\pi z)}{\pi}$ and $P(z) = z \prod_{n=1}^{\infty}(1 - \frac{z^2}{n^2})$. Away from the integers,

$$\frac{P'(z)}{P(z)} = \frac{1}{z} + \sum_{n=1}^{\infty} \frac{2z}{z^2 - n^2}, \tag{18.18}$$

and

$$\frac{G'(z)}{G(z)} = \pi \cot(\pi z). \tag{18.19}$$

Then by Lemma 16,

$$(\frac{P(z)}{G(z)})' = \frac{P(z)}{G(z)}[\frac{P'(z)}{P(z)} - \frac{G'(z)}{G(z)}] = 0, \tag{18.20}$$

so then $\frac{P(z)}{G(z)}$ is constant. Taking $z \to 0$ shows that $\frac{P(z)}{G(z)} = 1$. □

19. Hadamard factorization theorem

Now turn to the Weierstrass construction of an entire function with prescribed zeros.

Proposition 29. *Given any sequence $\{a_n\}$ of complex numbers with $|a_n| \to \infty$ as $n \to \infty$, there exists an entire function f that vanishes at all $z = a_n$ and nowhere else. Any other such entire function is of the form $f(z)e^{g(z)}$, where $g(z)$is entire.*

Since the sequence could allow for repetitions, the theorem actually guarantees the existence of entire functions with prescribed zeros with the desired multiplicities.

Proof. First observe that if f_1 and f_2 are entire functions with the same zeros with the same multiplicities, $\frac{f_1}{f_2}$ has removable singularities at all the points a_n. Therefore, $\frac{f_1}{f_2}$ is a nowhere vanishing entire function. Then by Theorem 40, $f_1(z) = f_2(z)e^{g(z)}$. Therefore, it only remains to prove the existence of an entire function that vanishes at all the points a_n and nowhere else.

For each integer $k \geq 0$, define the canonical factors by

$$E_0(z) = 1 - z, \qquad E_k(z) = (1-z)e^{z + z^2/2 + ... + z^k/k}, \qquad \text{for} \qquad k \geq 1. \tag{19.1}$$

The integer k is called the degree of the canonical factor. Note that $\log(1-z) + [z + \frac{z^2}{2} + ... + \frac{z^k}{k}] = O(z^{k+1})$. Therefore, $E_k(z) = e^{O(z^{k+1})}$.

Lemma 18. *If* $|z| \leq \frac{1}{2}$*, then* $|1 - E_k(z)| \leq c|z|^{k+1}$ *for some* $c > 0$.

Proof. This follows from the fact that $E_k(z) = e^{O(z^{k+1})}$. Since $\log(1-z) = -\sum_{j=1}^{\infty} \frac{z^j}{j}$ converges uniformly when $|z| \leq \frac{1}{2}$, $\log(1-z) + \sum_{j=1}^{k} \frac{z^j}{j} = -\sum_{j=k+1}^{\infty} \frac{z^j}{j} \leq 2|z|^{k+1}$. In particular, this implies

$$|1 - E_k(z)| \leq c|z|^{k+1}. \tag{19.2}$$

The constant c is independent of k. We can take $c' = e$ and then $c = 2e$. □

Suppose that we are given a zero of order m at the origin and that $a_1, a_2, ...$ are all non–zero. Then define the Weierstrass product by

$$f(z) = z^m \prod_{n=1}^{\infty} E_n(\frac{z}{a_n}). \tag{19.3}$$

This function has the required properties: it is entire with a zero of order m at the origin and zeros at each point in the sequence $\{a_n\}$, and f vanishes nowhere else. To show this, it suffices to show that the infinite product (19.3) converges uniformly on any disk $|z| < R$. Consider two cases separately: $|a_n| \leq 2R$ and $|a_n| > 2R$. Since there are only finitely many $|a_n| \leq 2R$,

$$z^m \prod_{|a_n| \leq 2R} E_n(\frac{z}{a_n}), \tag{19.4}$$

is an entire function. On the other hand, when $|z| < R$ and $|a_n| > 2R$, $|\frac{z}{a_n}| < \frac{1}{2}$,

$$|1 - E_n(\frac{z}{a_n})| \leq c|\frac{z}{a_n}|^{n+1} \leq \frac{c}{2^{n+1}}. \tag{19.5}$$

Therefore, the product

$$\prod_{|a_n| \geq 2R} E_n(\frac{z}{a_n}), \tag{19.6}$$

defines a holomorphic function on $|z| < R$, and that vanishes only at $z = a_n$ for some n. □

Hadamard refined this result by showing that in the case of functions of finite order, the degree of the canonical factors can be taken to be constant, and then g is a polynomial. This is because Theorem 60 implies that if f has order of growth $\leq \rho$, $n(r) \leq Cr^{\rho}$ for all large r and $\sum |a_n|^{-s} < \infty$.

Theorem 61. *Suppose* f *is entire and has growth of order* ρ_0*. Let* k *be the integer such that* $k \leq \rho_0 < k+1$*. If* $a_1, a_2, ...$ *denote the non–zero zeros of* f*, then*

$$f(z) = e^{P(z)} z^m \prod_{n=1}^{\infty} E_k(\frac{z}{a_n}), \tag{19.7}$$

where P *is a polynomial of degree* $\leq k$ *and* m *is the order of the zero of* f *at* $z = 0$.

The proof of Theorem 61 uses a few lemmas.

Lemma 19. *The canonical products satisfy*

$$|E_k(z)| \geq e^{-c|z|^{k+1}}, \qquad \text{if} \qquad |z| \leq \frac{1}{2}, \tag{19.8}$$

and

$$|E_k(z)| \geq |1-z| e^{-c'|z|^k}, \qquad \text{if} \qquad |z| \geq \frac{1}{2}. \tag{19.9}$$

Proof. If $|z| \leq \frac{1}{2}$ then make use of the power series expansion of $\log(1-z)$,

$$E_k(z) = e^{\log(1-z)+\sum_{j=1}^{k} z^j/j} = e^{-\sum_{j=k+1}^{\infty} z^j/j}. \tag{19.10}$$

Since $|e^w| \geq e^{-|w|}$ and $\sum_{j=k+1}^{\infty} z^j/j \leq c|z|^{k+1}$ when $|z| \leq \frac{1}{2}$, which proves (19.8).

For the second inequality, observe that if $|z| \geq \frac{1}{2}$,

$$|E_k(z)| = |1-z||e^{z+z^2/2+...+z^k/k}|, \tag{19.11}$$

and there exists $c' \geq 0$ such that

$$|e^{z+z^2/2+...+z^k/k}| \geq e^{-c'|z|^k}. \tag{19.12}$$

□

To prove Hadamard's theorem, it is necessary to find a lower bound for the product of the canonical factors when z stays away from the zeros $\{a_n\}$.

Lemma 20. *For any s with $\rho_0 < s < k+1$, we have*

$$|\prod_{n=1}^{\infty} E_k(\frac{z}{a_n})| \geq e^{-c|z|^s}, \tag{19.13}$$

except possibly when z belongs to the union of disks centered at a_n of radius $|a_n|^{-k-1}$ for $n = 1, 2, 3, ...$

Proof. First write

$$\prod_{n=1}^{\infty} E_k(\frac{z}{a_n}) = \prod_{|a_n| \leq 2|z|} E_k(\frac{z}{a_n}) \prod_{|a_n| > 2|z|} E_k(\frac{z}{a_n}). \tag{19.14}$$

By (19.8),

$$|\prod_{|a_n|>2|z|} E_k(\frac{z}{a_n})| = \prod_{|a_n|>2|z|} |E_k(\frac{z}{a_n})| \geq \prod_{|a_n|>2|z|} e^{-c|z/a_n|^{k+1}} \geq e^{-c|z|^{k+1} \sum_{|a_n|>2|z|} |a_n|^{-k-1}}. \tag{19.15}$$

Since $|a_n|^{-k-1} = |a_n|^{-s}|a_n|^{s-k-1} \leq C|a_n|^{-s}|z|^{s-k-1}$. Therefore, since $\sum |a_n|^{-s}$ converges,

$$|\prod_{|a_n|>2|z|} E_k(\frac{z}{a_n})| \geq e^{-c|z|^s}, \tag{19.16}$$

for some $c > 0$.

To estimate the first product, write

$$|\prod_{|a_n| \leq 2|z|} E_k(\frac{z}{a_n})| \geq \prod_{|a_n| \leq 2|z|} |1 - \frac{z}{a_n}| \prod_{|a_n| \leq 2|z|} e^{-c'|\frac{z}{a_n}|^k}. \tag{19.17}$$

Now note that

$$\prod_{|a_n| \leq 2|z|} e^{-c'|z/a_n|^k} = e^{-c'|z|^k \sum_{|a_n| \leq 2|z|} |a_n|^{-k}}. \tag{19.18}$$

Since $s > \rho_0 \geq k$, $|a_n|^{-k} = |a_n|^{-s}|a_n|^{s-k} \leq C|a_n|^{-s}|z|^{s-k}$, thereby proving

$$\prod_{|a_n| \leq 2|z|} e^{-c'|z/a_n|^k} \geq e^{-c|z|^s}. \tag{19.19}$$

Now then, when z does not belong to a disk of a radius $|a_n|^{-k-1}$ centered at a_n, $|a_n - z| \geq |a_n|^{-k-1}$. Therefore,

$$\prod_{|a_n|\leq 2|z|} |1 - \frac{z}{a_n}| = \prod_{|a_n|\leq 2|z|} |\frac{a_n - z}{a_n}| \geq \prod_{|a_n|\leq 2|z|} |a_n|^{-k-1}|a_n|^{-1} = \prod_{|a_n|\leq 2|z|} |a_n|^{-k-2}. \tag{19.20}$$

The estimate for the first product follows from the fact that

$$(k+2) \sum_{|a_n|\leq 2|z|} \log |z_n| \leq (k+2)n(2|z|)\log(2|z|) \leq c|z|^{\rho}\log(2|z|) \leq c'|z|^s, \tag{19.21}$$

if we take $\rho_0 < \rho < s$. Therefore, $\prod_{|a_n|\leq 2|z|} |1 - \frac{z}{a_n}| \geq e^{-c'|z|^s}$. □

Corollary 21. *There exists a sequence of radii r_1, r_2, ... with $r_m \to \infty$ such that*

$$|\prod_{n=1}^{\infty} E_k(\frac{z}{a_n})| \geq e^{-c|z|^s}, \qquad \textit{for} \qquad |z| = r_m. \tag{19.22}$$

Proof. Since $\sum |a_n|^{-k-1} < \infty$, there exists an integer N such that $\sum_{n=N}^{\infty} |a_n|^{-k-1} < \frac{1}{2}$. Therefore, for any two consecutive large integers L and $L+1$, there exists some $L < r < L+1$ such that the circle of radius r does not intersect the forbidden disks in the previous lemma. Indeed, this follows from the fact that the union of the intervals $I_n = [|a_n| - \frac{1}{|a_n|^{k+1}}, |a_n| + \frac{1}{|a_n|^{k+1}}]$, $n \geq N$ has measure ≤ 1. Therefore, apply Lemma 20 with $|z| = r$ to prove the corollary. □

Proof of Theorem 61. Let

$$E(z) = z^m \prod_{n=1}^{\infty} E_k(\frac{z}{a_n}). \tag{19.23}$$

To prove that E is entire, observe that by Lemma 18, (19.23) is entire. Moreover, E has the same zeros with the same multiplicities as f, so $\frac{f}{E}$ is holomorphic and nowhere vanishing. Therefore,

$$\frac{f(z)}{E(z)} = e^{g(z)}, \tag{19.24}$$

for some entire function g. Furthermore, since f has growth order ρ_0 and using the estimate from below from Corollary 21,

$$e^{Re(g(z))} = |\frac{f(z)}{E(z)}| \leq c'e^{c|z|^s}, \tag{19.25}$$

when $|z| = r_m$. This proves that

$$Re(g(z)) \leq C|z|^s, \qquad \text{for} \qquad |z| = r_m. \tag{19.26}$$

The proof of Hadamard's theorem follows from the following lemma.

Lemma 21. *Suppose g is entire and $u = Re(g)$ satisfies*

$$u(z) \leq Cr^s, \qquad \textit{whenever} \qquad |z| = r_m, \tag{19.27}$$

for a sequence of positive real numbers $r_m \to \infty$. Then g is a polynomial of degree $\leq s$.

Therefore, (19.26) implies that g is a polynomial of degree $\leq s$. □

Proof of Lemma 21. Since g is entire, expand g in a power series centered at the origin,

$$g(z) = \sum_{n=0}^{\infty} a_n z^n. \tag{19.28}$$

The Cauchy integral formula implies

$$\frac{1}{2\pi} \int_0^{2\pi} g(re^{i\theta}) e^{-in\theta} d\theta = \begin{cases} a_n r^n & \text{if } n \geq 0, \\ 0 & \text{if } n < 0. \end{cases} \tag{19.29}$$

Taking the complex conjugate,

$$\frac{1}{2\pi} \int_0^{2\pi} \overline{g(re^{i\theta})} e^{-in\theta} d\theta = \frac{1}{2\pi} \overline{\int_0^{2\pi} g(re^{i\theta}) e^{in\theta} d\theta} = 0, \tag{19.30}$$

whenever $n > 0$. When $n = 0$, taking the real part of both sides of (19.29) gives

$$2Re(a_0) = \frac{1}{\pi} \int_0^{2\pi} u(re^{i\theta}) d\theta. \tag{19.31}$$

Now then, since when $n \neq 0$, $\int_0^{2\pi} e^{-in\theta} d\theta = 0$,

$$a_n = \frac{1}{\pi r^n} \int_0^{2\pi} [u(re^{i\theta}) - Cr^s] e^{-in\theta} d\theta, \qquad \text{when} \qquad n > 0. \tag{19.32}$$

Therefore, since $u(z) \leq Cr^s$,

$$|a_n| \leq \frac{1}{\pi r^n} \int_0^{2\pi} [Cr^s - u(re^{i\theta})] d\theta \leq 2Cr^{s-n} - 2Re(a_0) r^{-n}. \tag{19.33}$$

Remark 12. *Equation* (19.27) *says nothing about a lower bound for* $u(z)$*, and* $u(z)$ *could be very negative.*

Letting $r \to \infty$ along the sequence $r_m \to \infty$ proves that $a_n = 0$ for $n > s$. □

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