NOTES ON 110.311

BENJAMIN DODSON

Contents

These notes come from [Tay19] and [SS03].

1. The complex plane $\mathbb{C}$

The complex plane arises naturally as a complete, algebraically closed field. One can easily obtain polynomials whose coefficients lie in the field of rational numbers $\mathbb{Q}$, but whose solutions do not lie in $\mathbb{Q}$. Take for example the equation

$$x^2 = 2. \tag{1.1}$$

It is a well known fact that $\sqrt{2}$ is not a rational number, although the coefficients of (1.1) are integers. However, it is easy to show using Newton's method that $\sqrt{2}$ is the limit of a sequence of rational numbers. Therefore, the real numbers $\mathbb{R}$ are defined to be the completion of the field of rational numbers, and $\mathbb{R}$ is a complete field.

A space is called complete if every Cauchy sequence converges. That is, if $\{z_n\}_{n=1}^{\infty}$ is a sequence such that for any $\epsilon > 0$ there exists $N(\epsilon) < \infty$ where $m, n \geq N(\epsilon)$ implies $|z_n - z_m| < \epsilon$.

This is not the end of the story, however, since it is still possible to obtain a polynomial whose coefficients lie in the field of real numbers, but whose solutions do not lie in $\mathbb{R}$. Consider the equation

$$x^2 = -1. \tag{1.2}$$

Since $x^2 \geq 0$ for every real number, (1.2) does not have any solutions in $\mathbb{R}$.

Therefore, define i to be the number satisfying $i^2 = -1$, and let $\mathbb{C}$ be the numbers of the form

$$z = x + iy, \qquad x, y \in \mathbb{R}, \qquad x = Re(z), \qquad y = Im(z). \tag{1.3}$$

The complex plane $\mathbb{C}$ is closed under multiplication and addition. Indeed, if $z = x + iy$ and $w = u + iv$, since $i^2 = -1$,

$$z + w = (u + x) + i(v + y), \qquad zw = (x + iy)(u + iv) = (xu - yv) + i(yu + xv). \tag{1.4}$$

These operations obey the commutative, associative, and distributive properties.

Using the Pythagorean theorem in the plane, $\mathbb{C}$ has the natural norm $|z| = \sqrt{x^2 + y^2}$. Let $\bar{z}$ denote the complex conjugate

$$\bar{z} = x - iy. \tag{1.5}$$

Then it is straightforward to verify from (1.4) that

$$|z|^2 = x^2 + y^2 = z\bar{z}. \tag{1.6}$$

Thus, $\mathbb{C}$ is a field, since for any $z \in \mathbb{C}$, $z \neq 0$, (1.6) implies that

$$\frac{1}{z} = \frac{\bar{z}}{|z|^2}. \tag{1.7}$$

The norm $|z|$ also obeys the triangle inequality.

Proposition 1 (Triangle inequality)**.**

$$|z + w| \leq |z| + |w|. \tag{1.8}$$

Proof. Calculating directly from (1.5),

$$\overline{z + w} = \bar{z} + \bar{w}, \qquad \overline{zw} = \bar{z}\bar{w}, \tag{1.9}$$

which implies that

$$|zw|^2 = zw\bar{z}\bar{w} = |z|^2|w|^2. \tag{1.10}$$

Therefore,

$$|z + w|^2 = (z + w)(\bar{z} + \bar{w}) = |z|^2 + |w|^2 + z\bar{w} + \bar{z}w \leq (|z| + |w|)^2, \tag{1.11}$$

which implies (1.8). □

Remark 1. *Note that $z\bar{w} + \bar{z}w$ is guaranteed to be a real number since*

$$2Re(z) = z + \bar{z}, \qquad \textit{and} \qquad 2Im(z) = z - \bar{z}. \tag{1.12}$$

Every Cauchy sequence in $\mathbb{R}$ converging implies that every Cauchy sequence in $\mathbb{C}$ converges under the norm $|z|$. Indeed, let $z_n = x_n + iy_n$ and $z = x + iy$. Then $z_n \to z$ if and only if $x_n \to x$ in $\mathbb{R}$ and $y_n \to y$ in $\mathbb{R}$. Similarly, $\{z_n\}_{n=1}^{\infty}$ is Cauchy in $\mathbb{C}$ if and only if $\{x_n\}_{n=1}^{\infty}$ and $\{y_n\}_{n=1}^{\infty}$ are Cauchy sequences in $\mathbb{R}$. Thus, $\mathbb{C}$ is a complete field.

Remark 2. *By standard properties of limits,* (1.3)*, and* (1.4)*, if $z_n \to z$ in $\mathbb{C}$ and $w_n \to w$ in $\mathbb{C}$, then $z_n + w_n \to z + w$ and $z_n w_n \to zw$ in $\mathbb{C}$.*

The fact that $\mathbb{C}$ is algebraically closed follows from the fundamental theorem of algebra.

Theorem 1 (Fundamental theorem of algebra)**.** *If $p(z)$ is a non-constant polynomial with complex coefficients, then $p(z)$ must have a complex root.*

Proof. Suppose that for some $n \geq 1$,

$$p(z) = a_n z^n + ... + a_1 z + a_0, \qquad a_n \neq 0, \qquad a_j \in \mathbb{C} \qquad \forall 0 \leq j \leq n. \tag{1.13}$$

Therefore, as $|z| \to \infty$,

$$p(z) = a_n z^n (1 + O(z^{-1})), \tag{1.14}$$

which implies that

$$\lim_{|z| \to \infty} |p(z)| = \infty, \tag{1.15}$$

so there exists $0 < R < \infty$ such that

$$\inf_{|z| > R} |p(z)| > |p(0)|, \tag{1.16}$$

and therefore,

$$\inf_{|z| \leq R} |p(z)| = \inf_{z \in \mathbb{C}} |p(z)|. \tag{1.17}$$

Since p is continuous, there exists $z_0 \in D_R$ which satisfies

$$|p(z_0)| = \inf_{z \in \mathbb{C}} |p(z)|, \tag{1.18}$$

where D_R refers to the disk of radius R, $D_R = \{z \in \mathbb{C} : |z| \leq R\}$.

Lemma 1. *If $p(z)$ is a non-constant polynomial and* (1.18) *holds, then $p(z_0) = 0$.*

Proof. Suppose by contradiction that $p(z_0) = a \neq 0$. Since a polynomial in z can easily be rewritten as a polynomial of the same degree in $(z - z_0)$ for any $z_0 \in \mathbb{C}$,

$$p(z_0 + \zeta) = a + q(\zeta), \qquad \zeta = z - z_0, \tag{1.19}$$

where q is a non-constant polynomial of order n. Therefore, for some $k \geq 1$, $b \neq 0$,

$$q(\zeta) = b\zeta^k + ... + b_n \zeta^n. \tag{1.20}$$

The term $b\zeta^k$ dominates the behavior of $q(\zeta)$ for $|\zeta|$ small,

$$q(\zeta) = b\zeta^k + O(\zeta^{k+1}), \qquad \text{as} \qquad \zeta \to 0. \tag{1.21}$$

Therefore, take $S^1 = \{\omega : |\omega| = 1\}$. For any fixed $\omega \in S^1$,

$$p(z_0 + \epsilon\omega) = a + b\omega^k \epsilon^k + O(\epsilon^{k+1}), \qquad \text{as} \qquad \epsilon \searrow 0. \tag{1.22}$$

Since $a \neq 0$ and $b \neq 0$, choose $\omega \in S^1$ such that

$$\frac{b}{|b|}\omega^k = -\frac{a}{|a|}. \tag{1.23}$$

Then,

$$p(z_0 + \epsilon\omega) = a(1 - |\frac{b}{a}|\epsilon^k) + O(\epsilon^{k+1}), \tag{1.24}$$

which contradicts the minimality of $p(z_0)$ when $\epsilon > 0$ is sufficiently small. □

Therefore, $p(z)$ has the root $p(z_0) = 0$. □

Rewriting $p(z)$ as a polynomial of order n in $(z - z_0)$, since $p(z_0) = 0$,

$$p(z) = a_n(z - z_0)^n + ... + \tilde{a}_1(z - z_0). \tag{1.25}$$

Dividing $p(z)$ by $(z - z_0)$ gives a polynomial of order $n - 1$. Using Theorem 1 and arguing by induction implies that $p(z)$ has n roots in $\mathbb{C}$.

2. The unit circle

To solve (1.23), define the curve

$$\gamma(t) = e^{it}, \qquad t \in \mathbb{R}. \tag{2.1}$$

Set

$$e^{it} = c(t) + is(t). \tag{2.2}$$

By the chain rule,

$$\frac{d}{dt}e^{it} = ie^{it}. \tag{2.3}$$

Observe that

$$i(x + iy) = -y + ix, \tag{2.4}$$

which is orthogonal to $x + iy$. Therefore, (2.1) travels on the unit circle, and by (1.10), $|(2.3)| = 1$. So $\gamma(t)$ travels around the circle at speed one in a counterclockwise direction.

Another way to show this is to calculate $|e^{it}|^2 = c(t)^2 + s(t)^2 = (e^{it})(\overline{e^{it}})$. Using the exponential function power series, for $z = it$, $t \in \mathbb{R}$,

$$\overline{e^z} = \sum_{k=0}^{\infty} \frac{\bar{z}^k}{k!} = e^{-it}. \tag{2.5}$$

Remark 3. *By the ratio test,* (2.5) *converges for any* $z \in \mathbb{C}$*.*

Therefore, $|e^{it}|^2 = 1$, and $t \mapsto \gamma(t)$ has the image in the unit circle centered at the origin. Also,

$$\gamma'(t) = ie^{it} \Rightarrow |\gamma'(t)| = 1. \tag{2.6}$$

Therefore, $\gamma(t)$ moves at unit speed on the unit circle, $\gamma(0) = 1$, $\gamma'(t) = i\gamma(t)$, so $\gamma(t)$ travels in a counterclockwise direction. From trigonometry,

$$\gamma(t) = \cos(t) + i\sin(t), \tag{2.7}$$

so

$$e^{it} = \cos t + i\sin t, \qquad \frac{d}{dt}e^{it} = -\sin t + i\cos t. \tag{2.8}$$

This gives the formula for the derivatives for the $\cos(t)$ and $\sin(t)$ functions. Next, using (2.5),

(2.9)
$$e^{is}e^{it} = e^{i(s+t)}, \qquad \Rightarrow \qquad \cos(s+t) = (\cos s)(\cos t) - (\sin s)(\sin t), \qquad \sin(s+t) = (\sin s)(\cos t) + (\cos s)(\sin t).$$

It is therefore possible to write any $z = x + iy \in \mathbb{C}$ in polar coordinates. Let $r = |z|$ and solve

$$\cos\theta = \frac{x}{|z|}, \qquad \sin\theta = \frac{y}{|z|}. \tag{2.10}$$

Equation (2.10) has a unique solution $\theta_0 \in [0, 2\pi)$, however, $\theta = \theta_0 + 2n\pi$ will also satisfy (2.10) for any $n \in \mathbb{Z}$. Therefore, define

$$Arg(z) = \theta_0, \qquad arg(z) = \theta_0 + 2\pi n. \tag{2.11}$$

By direct computation,

$$arg(z_1 z_2) = arg(z_1) + arg(z_2), \tag{2.12}$$

however it is not necessarily true that

$$Arg(z_1 z_2) = Arg(z_1) + Arg(z_2). \tag{2.13}$$

Remark 4. *Take* $z_1 = z_2 = -1$.

Since $arg(|z|^2) = 1$,

$$arg(\bar{z}) = -arg(z), \qquad \text{and} \qquad arg(\frac{1}{z}) = arg(\frac{\bar{z}}{|z|^2}) = -arg(z). \tag{2.14}$$

To solve (1.23), observe that the equation $z^k = 1$ has k unique solutions on the unit circle. Indeed,

$$|z^k| = |z|^k = 1, \tag{2.15}$$

so for any solution $|z| = 1$. Furthermore, $e^{ik\theta} = e^{2\pi i n}$, so $e^{i\frac{2\pi n}{k}}$ solves $z^k = 1$ for any n, which gives k unique solutions. Since the coefficient for z^{k-1} is equal to minus the sum of the roots,

Theorem 2. *For any k, the sum of k equidistant points on the unit circle is zero.*

The formula (2.7) may be used to compute the value of π. Let π be defined to be the smallest positive number such that $\gamma(2\pi) = 1$. Then $\gamma(\pi) = -1$ and $\gamma(\frac{\pi}{2}) = i$. Furthermore, from trigonometry,

$$\gamma(\frac{\pi}{3}) = \frac{1}{2} + \frac{\sqrt{3}}{2}i, \qquad \gamma(\frac{\pi}{6}) = \frac{\sqrt{3}}{2} + \frac{1}{2}i. \tag{2.16}$$

We can use this fact to determine the value of π. We know from (2.6) that the length of $\gamma(t)$ on $0 \leq t \leq \varphi$ is given by φ, so for $0 < \varphi < \frac{\pi}{2}$, parameterize this segment of the circle by

$$\sigma(s) = (\sqrt{1 - s^2}, s), \qquad 0 \leq s \leq \tau = \sin\varphi. \tag{2.17}$$

The length of this curve is given by

$$l = \int_0^\tau |\sigma'(s)| ds = \int_0^\tau \frac{ds}{\sqrt{1 - s^2}} = \varphi. \tag{2.18}$$

Therefore, from (2.16),

$$\frac{\pi}{6} = \int_0^{1/2} \frac{dx}{\sqrt{1 - x^2}}. \tag{2.19}$$

Making a power series expansion,

$$\frac{\pi}{6} = \sum_{n=0}^{\infty} \frac{a_n}{2n+1} (\frac{1}{2})^{2n+1}, \tag{2.20}$$

where a_n are defined recursively by

$$a_0 = 1, \qquad a_{n+1} = \frac{2n+1}{2n+2} a_n. \tag{2.21}$$

3. Matrix representation of complex numbers

The group of complex numbers can be represented by a two dimensional algebra of commuting matrices. Observe that for $c \in \mathbb{R}$, the operation $c : v \mapsto cv$ is represented by the dilation matrix

$$\begin{pmatrix} c & 0 \\ 0 & c \end{pmatrix}. \tag{3.1}$$

Next, the action of i on $x + iy$ is given by $i(x + iy) = -y + ix$, which is a ninety degree clockwise rotation, given by the matrix

$$J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}. \tag{3.2}$$

Lemma 2. *A matrix commutes with* (3.2) *if and only if it is of the form*

$$\begin{pmatrix} a & -b \\ b & a \end{pmatrix}. \tag{3.3}$$

Proof. By direct calculation,

$$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = \begin{pmatrix} d & -c \\ -b & a \end{pmatrix}. \tag{3.4}$$

Therefore, a matrix commutes with (3.2) if and only if $a = d$ and $b = -c$. □

By the distributive property, (3.1), and (3.2), $a + ib$ can be represented by the matrix

$$\begin{pmatrix} a & -b \\ b & a \end{pmatrix}. \tag{3.5}$$

Indeed,

$$\begin{pmatrix} a & -b \\ b & a \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} ax - by \\ bx + ay \end{pmatrix}, \tag{3.6}$$

which corresponds to

$$(a + ib)(x + iy) = (ax - by) + i(ay + bx). \tag{3.7}$$

Furthermore, since the column vectors in (3.6) are the vectors

$$\begin{pmatrix} a \\ b \end{pmatrix} \qquad \text{and} \qquad \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix}, \tag{3.8}$$

$$\begin{pmatrix} a & -b \\ b & a \end{pmatrix} \begin{pmatrix} x & -y \\ y & x \end{pmatrix} = \begin{pmatrix} ax - by & -bx - ay \\ bx + ay & ax - by \end{pmatrix}, \tag{3.9}$$

so the algebra of matrices of the form (3.6) corresponds to the algebra of complex numbers of the form $a + ib$.

A function f is called complex differentiable at z with $f'(z) = a + ib$, if and only if

$$\lim_{h\to 0} \frac{1}{h}[f(z+h) - f(z)] = f'(z) = a + ib. \tag{3.10}$$

Rewriting (3.10), if $h = h_1 + ih_2$,

$$f(z+h) = f(z) + f'(z)h + o(h) = f(z) + (a(z) + ib(z))(h_1 + ih_2) + o(h). \tag{3.11}$$

Rewriting (3.11) in matrix notation,

$$f(z+h) = f(z) + \begin{pmatrix} a & -b \\ b & a \end{pmatrix} \begin{pmatrix} h_1 \\ h_2 \end{pmatrix} + o(h), \tag{3.12}$$

which will be useful shortly.

Let $\Omega \subset \mathbb{C}$ be an open set. A set is called open if for all $z_0 \in \Omega$, there exists $\epsilon_0 > 0$ such that

$$D_{\epsilon_0}(z_0) = \{z : |z - z_0| < \epsilon_0\} \subset \Omega. \tag{3.13}$$

Definition 1 (Holomorphic)**.** *A function $f : \Omega \to \mathbb{C}$ is called holomorphic if and only if it is complex differentiable and f' is continuous on Ω. Another term for holomorphic is complex analytic.*

It is straightforward to verify that the function $f(z) = z$ is holomorphic, since

$$\frac{1}{h}[(z+h) - z] = 1. \tag{3.14}$$

On the other hand, $f(z) = \bar{z}$ is not holomorphic, since

$$\frac{1}{h}[f(z+h) - f(z)] = \frac{\bar{h}}{h}. \tag{3.15}$$

Of course, $f(z) = \bar{z}$ is a differentiable function from $\mathbb{R}^2 \to \mathbb{R}^2$,

$$f(x,y) = \begin{pmatrix} x \\ -y \end{pmatrix}. \tag{3.16}$$

Let $f : \mathbb{R}^2 \to \mathbb{R}^2$ be a differentiable function, $f(x,y) = \begin{pmatrix} u(x,y) \\ v(x,y) \end{pmatrix}$. It is known from multivariable calculus that if f is differentiable,

$$Df(x,y) = \begin{pmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{pmatrix}, \tag{3.17}$$

and furthermore,

$$f(x_0 + h_1, y_0 + h_2) = f(x_0, y_0) + Df(x_0, y_0) \begin{pmatrix} h_1 \\ h_2 \end{pmatrix} + o(h). \tag{3.18}$$

Comparing (3.15) to (3.18) yields a number of important facts about holomorphic functions.

Proposition 2. *If $f : \Omega \to \mathbb{C}$ is holomorphic, then $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ exist and are continuous on Ω, and*

$$\frac{\partial f}{\partial x} = \frac{1}{i}\frac{\partial f}{\partial y} = f'(z). \tag{3.19}$$

Proposition 3. *If $f : \Omega \to \mathbb{C}$ is C^1 and $\frac{\partial f}{\partial x} = \frac{1}{i}\frac{\partial f}{\partial y}$, then f is holomorphic.*

Proof. Propositions 2 and 3 follow from (3.8). ◻

Proposition 4. *If $f \in C^1(\Omega)$, then f is holomorphic if and only if for all $z \in \Omega$, $Df(z)$ and J commute.*

Proof. This follows from (3.4) and (3.8). □

4. Some holomorpic functions

We are now ready to show the existence of some more holomorphic functions.

Proposition 5. *If f and g are holomorphic on Ω, then so are $(fg)(z)$, $f(z) + g(z)$, and $cf(z)$, where $c \in \mathbb{C}$ is a constant. Furthermore,*

$$\frac{d}{dz}(fg)(z) = f'(z)g(z)+f(z)g'(z), \qquad \frac{d}{dz}(f+g)(z) = f'(z)+g'(z), \qquad \frac{d}{dz}(cf(z)) = cf'(z). \tag{4.1}$$

Proof. The proof uses the limit definition of the derivative and the usual computations from calculus. □

A corollary of this fact is that every polynomial is holomorphic.

Corollary 1. *Every polynomial is holomorphic.*

It is also possible to prove the usual chain rule computations to prove a chain rule.

Proposition 6 (Chain rule)**.** *Let Ω, $\mathcal{O}$ be open sets in $\mathbb{C}$. If $f : \Omega \to \mathbb{C}$ and $g : \mathcal{O} \to \Omega$ are holomorphic, then $f \circ g : \mathcal{O} \to \mathbb{C}$ is holomorphic, and*

$$\frac{d}{dz} f(g(z)) = f'(g(z))g'(z). \tag{4.2}$$

Combining the chain rule with the computation

$$\frac{1}{z+h} - \frac{1}{z} = -\frac{h}{z(z+h)} = -\frac{h}{z^2} + o(h), \tag{4.3}$$

Proposition 7. *If $f : \Omega \to \mathbb{C}$ is holomorphic, then $\frac{1}{f(z)}$ is holomorphic on $\Omega \setminus S$, where $S = \{z \in \Omega : f(z) = 0\}$, and on $\Omega \setminus S$,*

$$\frac{d}{dz}\frac{1}{f(z)} = -\frac{f'(z)}{f(z)^2}. \tag{4.4}$$

Moving on from polynomials, next consider the power series. First, let

$$\sum_{k=0}^{\infty} z_k \tag{4.5}$$

denote a series. Then define the sequence $s_n = \sum_{k=0}^{n} z_k$. The sequence s_n converges as a sequence if and only if $\sum_{k=0}^{\infty} z_k$ converges.

Lemma 3. *Assume that*

$$\sum_{k=0}^{\infty} |z_k| < \infty. \tag{4.6}$$

Then s_n is a Cauchy sequence.

Proof. Since $\sum_{k=0}^{n} |z_k| \leq A$, for any n,

$$s_n = \sum_{k=0}^{n} |z_k| \tag{4.7}$$

is a bounded, monotone sequence. Since a bounded, monotone sequence converges in $\mathbb{R}$, for any $\epsilon > 0$, there exists $M(\epsilon) < \infty$ such that for any n,

$$\sum_{k=M(\epsilon)}^{n} |z_k| < \epsilon. \tag{4.8}$$

Then by the triangle inequality, (4.8) implies that for any $m, n \geq M(\epsilon)$, $|s_n - s_m| < \epsilon$, and therefore s_n is a Cauchy sequence. □

Definition 2. *A series that satisfies* (4.6) *is absolutely convergent.*

A power series has the form

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n. \tag{4.9}$$

Any such series has a radius of convergence, some $0 \leq R \leq \infty$ such that (3.1) converges absolutely on the disk $D_R(z_0) = \{z : |z - z_0| < R\}$ and diverges for z such that $|z - z_0| > R$. Let

$$\frac{1}{R} = \limsup_{n \to \infty} |a_n|^{1/n}, \tag{4.10}$$

where $R = \infty$ if (4.10) $= 0$, and $R = 0$ if (4.10) $= \infty$.

5. Holomorphic functions defined by power series

Proposition 8. *The series* (4.9) *converges whenever $|z - z_0| < R$ and diverges whenever $|z - z_0| > R$, where R is given by* (4.10)*. If $R > 0$, then the series converges uniformly on any $D_{R'}(z_0)$ for $R' < R$. Thus, when $R > 0$, the series* (4.9) *defines a continuous function on $D_R(z_0)$,*

$$f : D_R(z_0) \to \mathbb{C}. \tag{5.1}$$

Proof. When $R = 0$, Proposition 8 is true. For any $R' < R$, there exists $\epsilon > 0$ and N sufficiently large such that

$$\sup_{n \geq N} |a_n|^{1/n} \leq \frac{1}{R' + \epsilon}. \tag{5.2}$$

Doing some algebra, for any $n \geq N$,

$$|a_n| \leq \frac{1}{(R' + \epsilon)^n}. \tag{5.3}$$

Therefore, for $z \in D_{R'}(z_0)$,

$$|a_n (z - z_0)^n| \leq (\frac{R'}{R' + \epsilon_0})^n. \tag{5.4}$$

Therefore, (4.9) converges uniformly on $D_{R'}(z_0)$.

Meanwhile, for $R < \infty$, for any $z \in \mathbb{C}$ satisfying $|z - z_0| > R$, there exists a subsequence $m(n) \nearrow \infty$ such that

$$|a_{m(n)} (z - z_0)^{m(n)}| \geq 1, \tag{5.5}$$

so (4.9) fails to converge. □

Proposition 9. *If $R > 0$, the function defined by* (3.1) *is holomorphic on $D_R(z_0)$, with derivative given by*

$$f'(z) = \sum_{n=1}^{\infty} n a_n (z - z_0)^{n-1}. \tag{5.6}$$

Proof. Following (4.10),

$$\limsup_{n \to \infty} |n a_n|^{\frac{1}{n-1}} = \lim_{n \to \infty} n^{\frac{1}{n-1}} \cdot \limsup_{n \to \infty} |a_n|^{\frac{1}{n} \cdot \frac{n}{n-1}} = \frac{1}{R}. \tag{5.7}$$

Therefore, the right hand side of (4.9) converges on $D_R(z_0)$ and diverges for $|z - z_0| > R$. A related theorem is the following.

Theorem 3. *If the power series*

$$f(z) = \sum_{k=0}^{\infty} a_k (z - z_0)^k, \tag{5.8}$$

converges for some $z_1 \neq z_0$, then either (5.8) *converges for all $z \in \mathbb{C}$, or* (5.8) *converges on a disk of radius $0 < R < \infty$.*

Proof. Since $a_k(z_1 - z_0)^k \to 0$, there exists a constant C such that $|a_k(z_1 - z_0)| \leq C$. Therefore, the series will converge for $|z - z_0| < |z_1 - z_0|$. □

Therefore, it only remains to prove that the right hand side of (3.8) is equal to $f'(z)$.

Proposition 10. *If* (5.8) *has a radius of convergence $R > 0$ and $z_1 \in D_R(z_0)$, then $f(z)$ has a convergent power series about z_1,*

$$f(z) = \sum_{k=0}^{\infty} b_k (z - z_1)^k, \qquad \textit{for} \qquad |z - z_1| < R - |z_1 - z_0|. \tag{5.9}$$

Proof. Suppose without loss of generality that $z_0 = 0$. Setting $f_{z_1}(\zeta) = f(z_1 + \zeta)$ when $|\zeta| < R - |z_1|$, using the binomial formula,

$$f_{z_1}(\zeta) = \sum_{n=0}^{\infty} a_n (z_1 + \zeta)^n = \sum_{n=0}^{\infty} \sum_{k=0}^{n} a_n \binom{n}{k} \zeta^k z_1^{n-k}, \tag{5.10}$$

which converges absolutely by the binomial formula. Therefore,

$$f_{z_1}(\zeta) = \sum_{k=0}^{\infty} (\sum_{n=k}^{\infty} a_n \binom{n}{k} z_1^{n-k}) \zeta^k. \tag{5.11}$$

Therefore, (5.9) holds with

$$b_k = \sum_{n=k}^{\infty} a_n \binom{n}{k} z_1^{n-k}. \tag{5.12}$$

□

Now then, to prove (5.6), (5.9) implies

$$f(z_1 + h) = b_0 + b_1 h + \sum_{k=2}^{\infty} b_k h^k. \tag{5.13}$$

Therefore,

$$\frac{f(z_1 + h) - f(z_1)}{h} = b_1 + o(h). \tag{5.14}$$

By the limit definition of the derivative,

$$f'(z_1) = b_1 = \sum_{n=1}^{\infty} n a_n z_1^{n-1}. \tag{5.15}$$

□

6. Integrating along curves

Turning from integrating on a circle to an integral on a general curve in $\mathbb{C}$, recall the fundamental theorem of calculus in one variable.

Theorem 4. *If $f \in C^1([a, b])$, then*

$$\int_a^b f'(t)dt = f(b) - f(a). \tag{6.1}$$

Furthermore, if $g \in C([a, b])$, then

$$\frac{d}{dt} \int_a^t g(s)ds = g(t). \tag{6.2}$$

In the study of the holomorphic functions on the open set $\Omega \subset \mathbb{C}$, consider the integral along the path

$$\gamma : [a, b] \to \Omega. \tag{6.3}$$

Then if $f : \Omega \to \mathbb{C}$ is continuous,

$$\int_\gamma f(z)dz = \int_a^b f(\gamma(t))\gamma'(t)dt. \tag{6.4}$$

Proposition 11. *If f is holomorphic on Ω and $\gamma : [a, b] \to \mathbb{C}$ is a C^1 path, then*

$$\int_\gamma f'(z)dz = f(\gamma(b)) - f(\gamma(a)). \tag{6.5}$$

Proof. The proof uses the following chain rule.

Proposition 12. *If $f : \Omega \to \mathbb{C}$ is holomorphic and $\gamma : [a, b] \to \mathbb{C}$ is C^1, then for $a < t < b$,*

$$\frac{d}{dt} f(\gamma(t)) = f'(\gamma(t))\gamma'(t). \tag{6.6}$$

Proof. This follows from the the chain rule in (4.2). □

Then by the fundamental theorem of calculus, the proof of (4.5) is complete. □

It is possible to use these computations in connection with an antiderivative.

Definition 3 (Anti-derivative)**.** *A holomorphic function $g : \Omega \to \mathbb{C}$ is said to have an anti-derivative f on Ω provided $f : \Omega \to \mathbb{C}$ is holomorphic and $f' = g$.*

Each holomorphic function $g : \Omega \to \mathbb{C}$ has an antiderivative for a class of sets $\Omega \subset \mathbb{C}$ which satisfy the following property: If $a + ib \in \Omega$ and $x + iy \in \Omega$, then the vertical line from $a + ib$ to $a + iy$ and the horizontal line from $a + iy$ to $x + iy$ lie in Ω.

Proposition 13. *If $\Omega \subset \mathbb{C}$ is an open set satisfying the above property, and $g : \Omega \to \mathbb{C}$ is holomorphic, then there exists a holomorphic $f : \Omega \to \mathbb{C}$ such that $f' = g$.*

Proof. By the fundamental theorem of calculus,

$$\frac{\partial f}{\partial x}(z) = g(z). \tag{6.7}$$

Also,

$$\frac{1}{i}\frac{\partial f}{\partial y} = g(a + iy) + \lim_{h \to 0} \frac{1}{ih} \int_a^x [g(t + iy + ih) - g(t + iy)]dt. \tag{6.8}$$

Since g is holomorphic,

$$(6.8) = g(x + iy), \tag{6.9}$$

so f is holomorphic. □

This computation may be used to give a second proof of

Proposition 14. *If $R > 0$, the function defined by*

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n \tag{6.10}$$

is holomorphic on $D_R(z_0)$ with derivative given by

$$f'(z) = \sum_{n=1}^{\infty} n a_n (z - z_0)^{n-1}. \tag{6.11}$$

Proof. For any k, consider

$$f_k(z) = \sum_{n=0}^{k} a_n (z - z_0)^n, \qquad g_k(z) = \sum_{n=1}^{k} n a_n (z - z_0)^{n-1}. \tag{6.12}$$

Then $f_k \to f$ and $g_k \to g$ locally uniformly on $D_R(z_0)$. Also, for each k, $f_k'(z) = g_k(z)$. Therefore, for any $z \in D_R(z_0)$,

$$f_k(z) = a_0 + \int_{\sigma_z} g_k(\zeta) d\zeta, \tag{6.13}$$

where σ_z is a path from z_0 to z.

Making use of local uniform convergence,

$$f(z) = a_0 + \int_{\sigma_z} g(\zeta) d\zeta. \tag{6.14}$$

Taking σ_z to be a path that approaches z horizontally, $z = x + iy$, $z_0 = x_0 + iy_0$,

$$f(z) = a_0 + \int_{y_0}^{y} g(x_0 + it) i dt + \int_{x_0}^{x} g(t + iy) dt, \tag{6.15}$$

$$\frac{\partial f}{\partial x}(z) = g(z). \tag{6.16}$$

Meanwhile, taking σ_z to be a path that approaches z vertically,

$$f(z) = a_0 + \int_{x_0}^{x} g(t + iy_0)dt + \int_{y_0}^{y} g(x + it)idt, \tag{6.17}$$

so therefore,

$$\frac{1}{i}\frac{\partial f}{\partial y}(z) = g(z). \tag{6.18}$$

Since each g_k is holomorphic, and therefore by Proposition 14 the integrals of each g_k are path independent, and $g_k \to g$ locally uniformly, the proof is complete. □

7. Square roots and logs

Recall the inverse function theorem for functions from $\mathbb{R}^n$ to $\mathbb{R}^n$.

Theorem 5. *Let $\Omega \subset \mathbb{R}^n$ be open and let $f : \Omega \to \mathbb{R}^n$ be a C^1 map. Take $p \in \Omega$ and assume $Df(p)$ is an invertible linear transformation on $\mathbb{R}^n$. Then there exists a neighborhood $\mathcal{O}$ of p and a neighborhood U of $q = f(p)$ such that $f : \mathcal{O} \to U$ is one-to-one and onto, the inverse $g = f^{-1} : U \to \mathcal{O}$ is C^1, and for $x \in \mathcal{O}$, $y = f(x)$,*

$$Dg(y) = Df(x)^{-1}. \tag{7.1}$$

This result has the following consequence for holomorphic functions.

Theorem 6. *Let $\Omega \subset \mathbb{C}$ be open and let $f : \Omega \to \mathbb{C}$ be holomorphic. Take $p \in \Omega$ and assume $f'(p) \neq 0$. Then there exists a neighborhood $\mathcal{O}$ of p and a neighborhood U of $q = f(p)$ such that $f : \mathcal{O} \to U$ is one-to-one and onto, the inverse $g = f^{-1} : U \to \mathcal{O}$ is holomorphic, and, for $z \in \mathcal{O}$, $w = f(z)$,*

$$g'(w) = \frac{1}{f'(z)}. \tag{7.2}$$

Proof. Taking the matrix representation of the derivative, $Df(x)$ is of the form

$$\begin{pmatrix} a & -b \\ b & a \end{pmatrix}. \tag{7.3}$$

The inverse of this matrix is given by

$$\frac{1}{a^2 + b^2}\begin{pmatrix} a & b \\ -b & a \end{pmatrix}, \tag{7.4}$$

which satisfies (7.2). □

This theorem can be applied to give an inverse function in the case when $f : \Omega \to \mathcal{O}$ is a bijection. Consider for example the function $f(z) = z^2$. In polar coordinates, if $z = re^{i\theta}$, $z^2 = r^2 e^{2i\theta}$. Therefore, $f(z)$ maps the right half plane

$$H = \{z \in \mathbb{C} : Re(z) > 0\}, \tag{7.5}$$

bijectively onto $\mathbb{C} \setminus \mathbb{R}^-$. Since $f'(z) = 2z$ vanishes only at the origin, we have a holomorphic inverse

$$Sqrt : \mathbb{C} \setminus \mathbb{R}^- \to H, \tag{7.6}$$

which is given by

$$Sqrt(re^{i\theta}) = r^{1/2}e^{i\theta/2}, \qquad r > 0, \qquad -\pi < \theta < \pi. \tag{7.7}$$

We can also write

$$\sqrt{z} = z^{1/2} = Sqrt(z). \tag{7.8}$$

Next, consider the inverse of the exponential function $\exp(z) = e^z$. Consider the strip

$$\Sigma = \{x + iy : \qquad x \in \mathbb{R}, \qquad -\pi < y < \pi\}. \tag{7.9}$$

Since $e^{x+iy} = e^x e^{iy}$, we have a bijective map

$$\exp : \Sigma \to \mathbb{C} \setminus \mathbb{R}^-. \tag{7.10}$$

Since $\frac{d}{dz}e^z = e^z$ is nowhere vanishing, (5.10) has a holomorphic inverse denoted as log.

$$\log : \mathbb{C} \setminus \mathbb{R}^- \to \Sigma. \tag{7.11}$$

Taking $\log 1 = 0$ and since

$$\frac{d}{dz}e^z = e^z \Rightarrow \frac{d}{dz}\log z = \frac{1}{z}. \tag{7.12}$$

Thus,

$$\log z = \int_1^z \frac{1}{\zeta} d\zeta, \tag{7.13}$$

where the integral is along any path from 1 to z in $\mathbb{C} \setminus \mathbb{R}^-$. However, observe that since $\int_{\mathcal{C}} \frac{1}{\zeta} d\zeta = 2\pi i$ when $\mathcal{C}$ is a circle around the origin, we cannot use (7.13) to define the log globally.

Then, given $a \in \mathbb{C}$, define

$$z^a = Pow_a(z), \qquad Pow_a : \mathbb{C} \setminus \mathbb{R}^- \to \mathbb{C}, \tag{7.14}$$

by

$$z^a = e^{a \log z}. \tag{7.15}$$

Since $e^{u+v} = e^u e^v$,

$$z^{a+b} = z^a z^b. \tag{7.16}$$

In particular, (7.16) implies that for any $n \in \mathbb{Z}$, $n \neq 0$,

$$(z^{1/n})^n = z. \tag{7.17}$$

Then by (7.12),

$$\frac{d}{dz} z^a = a z^{a-1}. \tag{7.18}$$

8. The inverse sine function

Defining the inverse sine function uses a global inverse function theorem.

Theorem 7. *Suppose $\Omega \subset \mathbb{C}$ is convex. Assume f is holomorphic in Ω, and there exists $a \in \mathbb{C}$ such that*

$$Re(af'(z)) > 0, \qquad on \qquad \Omega. \tag{8.1}$$

Then f maps Ω one to one onto its image $f(\Omega)$.

Proof. Take two distinct points $z_0,\ z_1 \in \Omega$. By convexity, $\sigma(t) = (1-t)z_0 + tz_1$ lies in Ω for all $t \in [0,1]$. Then

$$a\frac{f(z_1) - f(z_0)}{z_1 - z_0} = \int_0^1 af'((1-t)z_0 + tz_1)dt. \tag{8.2}$$

Then by (8.1), (8.2) $\neq 0$. □

Remark 5. *Compare this result to the global inverse function theorem when $f : \mathbb{R} \to \mathbb{R}$ when f is monotone increasing or decreasing.*

For example, consider the strip

$$\tilde{\Sigma} = \{x + iy : -\frac{\pi}{2} < x < \frac{\pi}{2}, \qquad y \in \mathbb{R}\}. \tag{8.3}$$

Take $f(z) = \sin z,\ f'(z) = \cos z$. Then for $z \in \tilde{\Sigma}$,

$$Re \cos z = \cos x \cosh y \qquad \text{for} \qquad z \in \tilde{\Sigma}. \tag{8.4}$$

Indeed,

$$\begin{aligned} Re \cos(x+iy) &= Re\frac{e^{i(x+iy)} + e^{-i(x+iy)}}{2} = Re\frac{e^{ix}e^{-y} + e^{-ix}e^{y}}{2} \\ &= \frac{e^{ix}e^{-y} + e^{-ix}e^{-y} + e^{-ix}e^{y} + e^{ix}e^{y}}{4} = \cos(x)\cosh(y). \end{aligned} \tag{8.5}$$

Therefore, f maps $\tilde{\Sigma}$ one to one onto its image.

Theorem 8. *The function* sin *maps $\tilde{\Sigma}$ one-to-one onto the set*

$$\mathbb{C} \setminus \{(-\infty, -1] \cup [1, \infty)\}. \tag{8.6}$$

Proof. To see this, observe that $\sin(z) = g(e^{iz})$, where $g(\zeta) = \frac{1}{2i}(\zeta - \frac{1}{\zeta})$. Observe that the image of $\tilde{\Sigma}$ under the map $z \mapsto e^{iz}$ is the right half plane H. Next, the image of H under g is

$$\mathbb{C} \setminus \{(-\infty, -1] \cup [1, \infty)\}. \tag{8.7}$$

Proposition 15. *Let*

$$h(\zeta) = g(i\zeta) = \frac{1}{2}(\zeta + \frac{1}{\zeta}). \tag{8.8}$$

Since $g(\zeta) = h(-i\zeta)$,

$$h(-i\zeta) = \frac{1}{2}(\frac{\zeta}{i} + \frac{1}{-i\zeta}) = \frac{1}{2i}(\zeta - \frac{1}{\zeta}). \tag{8.9}$$

The function h given by (8.8) *maps both the upper half plane* $U = \{\zeta : Im\zeta > 0\}$ *and the lower half plane* $U^* = \{\zeta : Im(\zeta) > 0\}$ *one-to-one and onto*

$$\mathbb{C} \setminus \{(-\infty, -1] \cup [1, \infty)\}. \tag{8.10}$$

Proof. Observe that $h : \mathbb{C} \setminus 0 \to \mathbb{C}$, and

$$h(\frac{1}{\zeta}) = h(\zeta). \tag{8.11}$$

Solving for $h(\zeta) = w$, if $\zeta \neq 0$,

$$\zeta^2 - 2w\zeta + 1 = 0, \tag{8.12}$$

which has the solutions

$$\zeta = w \pm \sqrt{w^2 - 1}. \tag{8.13}$$

Then for each $w \in \mathbb{C}$, there are two solutions, except for $w = \pm 1$.

Then h maps $\mathbb{R} \setminus 0$ onto $(-\infty, 1] \cup [1, \infty)$ two to one, except at $x = \pm 1$. This takes care of the two images on the real line with $|x| \geq 1$. Therefore, given $\zeta \in \mathbb{C} \setminus 0$, $h(\zeta) = w$ belongs to $(-\infty, -1] \cup [1, \infty)$ if and only if $\zeta \in \mathbb{R}$.

Therefore, if $w \in \mathbb{C} \setminus \{(-\infty, -1] \cup [1, \infty)\}$, then $h(\zeta) = w$ has two solutions, both in $\mathbb{C} \setminus \mathbb{R}$. Furthermore, the two solutions are reciprocals of each other, so given $\zeta \in \mathbb{C} \setminus \mathbb{R}$, $\zeta \in U \Leftrightarrow \frac{1}{\zeta} \in U^*$. □

The inverse function is denoted

$$\sin^{-1} : \mathbb{C} \setminus \{(-\infty, -1] \cup [1, \infty)\} \to \tilde{\Sigma}. \tag{8.14}$$

For $z \in \tilde{\Sigma}$, $\sin^2(z) \in \mathbb{C} \setminus [1, \infty)$, and therefore,

$$\cos(z) = (1 - \sin^2 z)^{1/2}, \qquad z \in \tilde{\Sigma}. \tag{8.15}$$

Therefore, by the inverse function theorem, $g(z) = \sin^{-1} z$ satisfies,

$$g'(z) = (1 - z^2)^{-1/2}, \qquad z \in \mathbb{C} \setminus \{(-\infty, -1] \cup [1, \infty)\}. \tag{8.16}$$

Therefore,

$$\sin^{-1} z = \int_0^z (1 - \zeta^2)^{-1/2} d\zeta, \tag{8.17}$$

where the integral is along any path from 0 to z in $\mathbb{C} \setminus \{(-\infty, -1] \cup [1, \infty)\}$. □

9. Harmonic functions on a planar domain

Suppose $f \in C^{\infty}(\Omega)$ is a holomorphic function. Applying $(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y})$ to the Cauchy–Riemann equations, implies

$$(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y})(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y})f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = 0, \tag{9.1}$$

on the open set $\Omega \subset \mathbb{C}$.

Such a function is called harmonic. More generally, if $\mathcal{O}$ is an open set in $\mathbb{R}^n$, a function $f \in C^2(\mathcal{O})$ is said to be harmonic on $\mathcal{O}$ if $\Delta f = 0$ on $\mathcal{O}$, where

$$\Delta f = \frac{\partial^2 f}{\partial x_1^2} + ... + \frac{\partial^2 f}{\partial x_n^2} = 0. \tag{9.2}$$

Taking the real and imaginary parts of (9.1), $f = u + iv$,

$$\Delta u = 0, \qquad \Delta v = 0. \tag{9.3}$$

Therefore, if $f \in C^\infty(\Omega)$ is a holomorphic function, the real and imaginary parts of f are harmonic functions on Ω.

Many domains $\Omega \subset \mathbb{C}$ have the property that if $u \in C^2(\Omega)$ is a real-valued, harmonic function, then there exists a real-valued harmonic function $v \in C^2(\Omega)$ such that $f = u + iv$ is holomorphic on Ω.

Definition 4. *v is said to be the harmonic conjugate of u.*

Given $\alpha = a + ib$ and $z = x + iy$, let $\gamma_{\alpha z}$ denote a path from $a + ib$ to $a + iy$, and then the horizontal line from $a + iy$ to $x + iy$. Next, let $\sigma_{\alpha z}$ denote the horizontal line segment from $a + ib$ to $x + ib$, and then the vertical line segment from $x + ib$ to $x + iy$. Let $R_{\alpha z}$ denote the rectangle bounded for the four line segments.

Proposition 16. *Let $\Omega \subset \mathbb{C}$ be open, $\alpha = a+ib \in \Omega$, and assume that the following property holds: If $z \in \Omega$, then $R_{\alpha z} \subset \Omega$. Let $u \in C^2(\Omega)$ be harmonic. Then u has a harmonic conjugate $v \in C^2(\Omega)$.*

Proof. For $z \in \Omega$, set

$$v(z) = \int_{\gamma_{\alpha z}} (-\frac{\partial u}{\partial y}dx + \frac{\partial u}{\partial x}dy) = \int_b^y \frac{\partial u}{\partial x}(a, s)ds - \int_a^x \frac{\partial u}{\partial y}(t, y)dt. \tag{9.4}$$

Also set

$$\tilde{v}(z) = \int_{\sigma_{\alpha z}} (-\frac{\partial u}{\partial y}dx + \frac{\partial u}{\partial x}dy) = -\int_a^x \frac{\partial u}{\partial y}(t, b)dt + \int_b^y \frac{\partial u}{\partial x}(x, s)ds. \tag{9.5}$$

By the fundamental theorem of calculus,

$$\frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}(z), \qquad \frac{\partial \tilde{v}}{\partial y}(z) = \frac{\partial u}{\partial x}(z). \tag{9.6}$$

Furthermore, since $R_{\alpha z} \subset \Omega$, by Green's theorem, since u is a harmonic function,

$$\tilde{v}(z) - v(z) = \int_{\partial R_{\alpha z}} (-\frac{\partial u}{\partial y}dx + \frac{\partial u}{\partial x}dy) = \int\int_{R_{\alpha z}} (\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2})dxdy = 0. \tag{9.7}$$

Therefore, u and v satisfy the Cauchy–Riemann equations. □

It is possible to prove this this proposition without Green's theorem.

Proposition 17. *Let $\Omega \subset \mathbb{C}$ be open, $\alpha = a + ib \in \Omega$, and assume the following property holds: If also $z \in \Omega$ then $\gamma_{\alpha z} \subset \mathbb{C}$.*

Let $u \in C^2(\Omega)$ be harmonic. Then u has a harmonic conjugate $v \in C^2(\Omega)$.

Proof. Define v as in (9.4). Then $\frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}$. Also, by (9.4),

$$\frac{\partial v}{\partial y}(z) = \frac{\partial u}{\partial x}(a, y) - \int_a^x \frac{\partial^2 u}{\partial y^2}(t, y)dt = \frac{\partial u}{\partial x}(a, y) + \int_a^x \frac{\partial^2 u}{\partial x^2}(t, y)dt = \frac{\partial u}{\partial x}(z). \tag{9.8}$$

Therefore, u and v satisfy the Cauchy–Riemann equations. □

Proposition 18 (Mean value theorem for harmonic functions)**.** *If $u \in C^2(\Omega)$ is harmonic, $z_0 \in \Omega$, and $\overline{D_r(z_0)} \subset \Omega$, then*

$$u(z_0) = \frac{1}{2\pi}\int_0^{2\pi} u(z_0 + re^{i\theta})d\theta. \tag{9.9}$$

Proof. Since u is a continuous function,

$$\lim_{r \searrow 0} \frac{1}{2\pi} \int_0^{2\pi} u(z_0 + re^{i\theta}) d\theta = u(z_0). \tag{9.10}$$

Taking the derivative with respect to r,

$$\frac{d}{dr} \frac{1}{2\pi} \int_0^{2\pi} u(z_0 + re^{i\theta}) d\theta = \frac{1}{2\pi} \int_0^{2\pi} u_r(z_0 + re^{i\theta}) d\theta. \tag{9.11}$$

By Green's theorem,

$$\frac{1}{2\pi} \int_0^{2\pi} u_r(z_0 + re^{i\theta}) d\theta = \frac{1}{2\pi} \int_{D_r(z_0)} \Delta u dx dy = 0. \tag{9.12}$$

This proves (9.9). □

Writing (9.9) in polar coordinates,

$$u(z_0) = \frac{1}{\pi r^2} \int \int_{D_r(z_0)} u(z) dx dy. \tag{9.13}$$

With this, we can establish a maximum principle for harmonic functions.

Proposition 19. *Let $\Omega \subset \mathbb{C}$ be a connected open set. If $u : \Omega \to \mathbb{R}$ is harmonic on Ω, then given $z_0 \in \Omega$,*

$$u(z_0) = \sup_{z \in \Omega} u(z) \qquad \Rightarrow \qquad u \qquad \text{is constant on} \qquad \Omega. \tag{9.14}$$

If, in addition, Ω is bounded and $u \in C(\bar{\Omega})$, then

$$\sup_{z \in \bar{\Omega}} u(z) = \sup_{z \in \partial\Omega} u(z). \tag{9.15}$$

Proof. Equation (9.15) follows from (9.14) if Ω is bounded, since u must achieve a maximum somewhere on $\bar{\Omega}$. Thus, assume there exists $z_0 \in \Omega$ such that the hypotheses of (9.14) hold. Set

$$\mathcal{O} = \{\zeta \in \Omega : u(\zeta) = u(z_0)\}. \tag{9.16}$$

Since $z_0 \in \mathcal{O}$, $\mathcal{O}$ is not empty. Moreover, by continuity, $\mathcal{O}$ is a closed subset of Ω. Moreover, by (9.13), if there exists a a disk of radius ρ, $\overline{D_\rho(\zeta_0)} \subset \Omega$, since u is the supremum, $u(z) = u(\zeta_0)$ for all $z \in D_\rho(\zeta_0)$. □

10. More harmonic functions

Corollary 2. *If $f(z)$ is a holomorphic function, and $f \in C^\infty(\Omega)$, given $z_0 \in \Omega$,*

$$|f(z_0)| = \sup_{z \in \Omega} |f(z)| \qquad \Rightarrow f \qquad \text{is constant on} \qquad \Omega. \tag{10.1}$$

If, in addition, Ω is bounded, and $f \in C(\bar{\Omega})$, then

$$\sup_{z \in \bar{\Omega}} |f(z)| = \sup_{z \in \partial\Omega} |f(z)|. \tag{10.2}$$

Proof. If $f = u + iv$, and u and v are harmonic functions, by the product rule,

$$\Delta(u^2 + v^2) = |\nabla u|^2 + |\nabla v|^2. \tag{10.3}$$

Plugging this fact into the proof of Proposition 18,

$$u(z_0)^2 + v(z_0)^2 \leq \frac{1}{\pi r^2} \int_{D_r(z_0)} (u(z)^2 + v(z)^2) dx dy. \tag{10.4}$$

Moreover, equality holds if and only if $|\nabla u| = 0$ and $|\nabla v| = 0$ on $D_r(z_0)$. □

Next, Liouville's theorem for harmonic functions on $\mathbb{C}$.

Proposition 20. *If $u \in C^2(\Omega)$ is bounded and harmonic on all of $\mathbb{C}$, then u is constant.*

Proof. Choose any two points $p, q \in \mathbb{C}$. For all $r > 0$,

$$u(p) - u(q) = \frac{1}{\pi r^2} [\int \int_{D_r(p)} u(z) dx dy - \int_{D_r(q)} u(z) dx dy]. \tag{10.5}$$

Hence,

$$|u(p) - u(q)| \leq \frac{1}{\pi r^2} \int \int_{\Delta(p,q,r)} |u(z)| dx dy, \tag{10.6}$$

where $\Delta(p, q, r)$ is the set of points contained in $D_r(p)$ or $D_r(q)$, but not both. Therefore, $\Delta(p, q, r) \sim r$ as $r \to \infty$. Taking $r \to \infty$ in (10.6), since $|u|$ is bounded, $u(p) - u(q) = 0$. □

Corollary 3. *If $f : \mathbb{C} \to \mathbb{C}$ is holomorphic and bounded, and $f \in C^\infty(\Omega)$, then f is constant.*

Proof. Since f is holomorphic, $f = u + iv$, where u and v are harmonic functions. Since $|f|$ is uniformly bounded, $|u|$ and $|v|$ are uniformly bounded, and therefore, by Proposition 20, u and v are constant. □

If $f \in C^\infty(\Omega)$ is a holomorphic function, then since $f = u + iv$, where u and v are harmonic functions, so

$$f(z_0) = \frac{1}{2\pi} \int_0^{2\pi} f(z_0 + re^{i\theta}) d\theta. \tag{10.7}$$

However, it is possible to prove (10.7) without making the a priori assumption that $f \in C^\infty(\Omega)$.

Theorem 9 (Cauchy integral formula)**.** *If f is holomorphic on the open set $\Omega \subset \mathbb{C}$, and $\overline{D_r(z_0)} \subset \Omega$, then*

$$f(z_0) = \frac{1}{2\pi} \int_0^{2\pi} f(z_0 + re^{i\theta}) d\theta = \frac{1}{2\pi i} \int_{\partial D_r(z_0)} \frac{f(\zeta)}{\zeta - z_0} d\zeta. \tag{10.8}$$

Proof. As in the proof of Proposition 18,

$$\lim_{r \searrow 0} \frac{1}{2\pi} \int_0^{2\pi} f(z_0 + re^{i\theta}) d\theta = f(z_0). \tag{10.9}$$

Taking a derivative with respect to r,

$$\frac{d}{dr} \frac{1}{2\pi} \int_0^{2\pi} f(z_0 + re^{i\theta}) d\theta = \frac{1}{2\pi} \int_0^{2\pi} f'(z_0 + re^{i\theta}) e^{i\theta} d\theta = \frac{1}{2\pi i r} \int_{\partial D_r(z_0)} f'(z_0 + \zeta) d\zeta = 0. \tag{10.10}$$

□

Corollary 4. *If f is holomorphic on Ω, then $f \in C^{\infty}(\Omega)$.*

Proof. We can compute the derivative of (10.8) directly.

$$(\frac{d}{dz})^n f(z) = \frac{(-1)^n n!}{2\pi i} \int_{\partial D_r(z)} \frac{f(\zeta)}{(\zeta - z)^{n+1}} d\zeta. \tag{10.11}$$

This integral converges for any n. □

11. Consequences of the Cauchy integral formula

The Cauchy integral formula may be extended to an integral on the boundary of Ω, where Ω is a bounded region.

Theorem 10. *If $f \in C^1(\bar{\Omega})$ is holomorphic on Ω, then*

$$\int_{\partial \Omega} f(z) dz = 0. \tag{11.1}$$

Proof. It is possible to take "bites" out of Ω with sets of the form $R_{\alpha z}$. □

Then the integral in Theorem 9 on $D_r(z_0)$ can be moved out to $\partial\Omega$, since $\frac{1}{\zeta - z_0}$ is holomorphic on $\mathbb{C} \setminus \{z_0\}$.

Theorem 11. *If $f \in C^1(\bar{\Omega})$ is holomorphic, then for $z \in D_r(z_0) \subset \Omega$, $f(z)$ has the convergent power series expansion*

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n, \tag{11.2}$$

with

$$a_n = \frac{1}{2\pi i} \int_{\partial\Omega} \frac{f(\zeta)}{(\zeta - z_0)^{n+1}} d\zeta = \frac{f^{(n)}(z_0)}{n!}. \tag{11.3}$$

Proof. Suppose $z \in D_r(z_0)$. By Theorem 9,

$$f(z) = \frac{1}{2\pi i} \int_{\partial\Omega} \frac{f(\zeta)}{(\zeta - z_0) - (z - z_0)} d\zeta. \tag{11.4}$$

Making the infinite series expansion, since $|z - z_0| < |\zeta - z_0|$,

$$\frac{1}{(\zeta - z_0) - (z - z_0)} = \frac{1}{\zeta - z_0} \sum_{n=0}^{\infty} (\frac{z - z_0}{\zeta - z_0})^n. \tag{11.5}$$

Plugging this series into (10.8) with $\partial D_r(z_0)$ replaced by $\partial\Omega$ gives (11.2) and (11.3). □

Proposition 21 (Schwarz lemma)**.** *Suppose f is holomorphic on the unit disk $D_1(0)$. Assume $|f(z)| \leq 1$ for $|z| < 1$, and $f(0) = 0$. Then,*

$$|f(z)| \leq |z|. \tag{11.6}$$

Furthermore, equality holds in (11.6), for some $z \in D_1(0) \setminus 0$, if and only if $f(z) = cz$ for some constant of absolute value one.

Proof. The hypotheses imply that $g(z) = \frac{f(z)}{z}$ is a holomorphic function on $D_1(0)$. Therefore, $|g(z)| \leq \frac{1}{a}$ for $z \in D_a(0)$, $0 < a < 1$. Using the maximum principle, $|g(z)| \leq \frac{1}{a}$ for all $z \in D_a(0)$. Taking $a \nearrow 1$,

$$|g(z)| \leq 1, \qquad \forall z \in D_1(0). \tag{11.7}$$

Therefore, (11.6) holds. Next, suppose that $|f(z_0)| = |z_0|$ at some $z_0 \in D_1(0) \setminus 0$. Then g attains a maximum at z_0, which implies $g(z)$ is constant on $D_1(0)$, so $f(z) = cz$. □

It is also possible to prove the fundamental theorem of algebra using the maximum principle.

Theorem 12 (Fundamental theorem of algebra). *If $p(z) = a_n z^n + ... + a_1 z + a_0$, $a_n \neq 0$ for some $n \geq 1$ is a polynomial of degree n, then $p(z)$ must vanish somewhere on $\mathbb{C}$.*

Proof. If $p(z)$ does not vanish on $\mathbb{C}$, then $f(z) = \frac{1}{p(z)}$ is an entire function on $\mathbb{C}$. Furthermore,

$$\frac{1}{p(z)} = \frac{1}{z^n} \frac{1}{a_n + a_{n-1} z^{-1} + ... + a_0 z^{-n}}. \tag{11.8}$$

Then

$$\lim_{z \to \infty} |\frac{1}{p(z)}|, \tag{11.9}$$

exists and is uniformly bounded. Then, by Liouville's theorem, $\frac{1}{p(z)}$ is constant. □

12. Morera's theorem and Goursat's theorem

Let Ω be a connected open set in $\mathbb{C}$. If $f : \Omega \to \mathbb{C}$ is holomorphic, then the Cauchy integral formula and Cauchy integral theorem hold for f. Here, we establish a converse of the Cauchy integral theorem, Morera's theorem.

Theorem 13 (Morera's theorem). *Assume $g : \Omega \to \mathbb{C}$ is continuous and*

$$\int_\gamma g(z) dz = 0, \tag{12.1}$$

whenever $\gamma = \partial R$, where R is a rectangle with sides parallel to the real and imaginary axes. Then g is holomorphic.

Proof. Holomorphicity is a local property, so assume without loss of generality that Ω is a rectangle. Fix $\alpha = a + ib$ in Ω. Given $z \in \Omega$, let $\gamma_{\alpha z}$ and $\sigma_{\alpha z}$ be piecewise linear paths from α to z. Then

$$f(z) = \int_{\gamma_{\alpha z}} g(\zeta) d\zeta = i \int_b^y g(a + is) ds + \int_a^x g(t + iy) dt, \tag{12.2}$$

and

$$f(z) = \int_{\sigma_{\alpha z}} g(\zeta) d\zeta = \int_a^x g(s + ib) ds + i \int_b^y g(x + it) dt. \tag{12.3}$$

By (12.1), (12.2) and (12.3) are equal. Therefore,

$$\frac{\partial f}{\partial x}(z) = \frac{1}{i} \frac{\partial f}{\partial y}(z) = g(z). \tag{12.4}$$

Thus, $f : \Omega \to \mathbb{C}$ is C^1 and satisfies the Cauchy–Riemann equations, so f is holomorphic. Therefore, g is holomorphic. □

Next, prove Goursat's theorem, which shows that if f is merely complex differentiable, f is holomorphic.

Theorem 14 (Goursat's theorem)**.** *If $f : \Omega \to \mathbb{C}$ is complex differentiable at each point of Ω, then f is holomorphic, so $f \in C^1(\Omega)$, and in fact $f \in C^\infty(\Omega)$.*

Proof. It is enough to show that the hypotheses yield

$$\int_{\partial R} f(z)dz = 0, \tag{12.5}$$

for every rectangle $R \subset \Omega$.

Given a rectangle $R \subset \Omega$, set $a = \int_{\partial R} f(z)dz$. Divide R into four rectangles of equal size. The integral over R is equal to the sum of the integrals over all four rectangles. Therefore, there must exist one rectangle R_1 such that

$$|\int_{\partial R_1} f(z)dz| \geq \frac{|a|}{4}. \tag{12.6}$$

Then, divide R_1 into four equal rectangles. One of them, R_2, must have the property that

$$|\int_{\partial R_2} f(z)dz| \geq 4^{-2}|a|. \tag{12.7}$$

Thus, there exists a sequence of nested rectangles R_k with perimeter ∂R_k of length $2^{-k} l(\partial R) = 2^{-k} b$ such that

$$|\int_{\partial R_k} f(z)dz| \geq 4^{-k}|a|. \tag{12.8}$$

The rectangles shrink to a point $p \in \Omega$. Since f is complex differentiable,

$$f(z) = f(p) + f'(p)(z - p) + o(|z - p|). \tag{12.9}$$

Now then,

$$\int_{\partial R_k} f(p)dz = \int_{\partial R_k} f'(p)(z - p)dz = 0. \tag{12.10}$$

Therefore,

$$|\int_{\partial R_k} f(z)dz| \leq C\delta_k 2^{-k} 2^{-k}. \tag{12.11}$$

Plugging (12.11) into (12.8), $a = 0$. □

13. More theorems

Set $L = \Omega \cap \mathbb{R}$ and set $\Omega^\pm = \{z \in \Omega : \pm Im(z) > 0\}$.

Proposition 22 (Schwarz reflection principle)**.** *Assume $f : \Omega^+ \cup L \to \mathbb{C}$ is continuous, holomorphic in Ω^+, and real valued on L. Then define $g : \Omega \to \mathbb{C}$ by*

$$g(z) = f(z), \qquad z \in \Omega^+ \cup L, \qquad g(z) = \overline{f(\bar{z})}, \qquad z \in \Omega^-. \tag{13.1}$$

Then g is holomorphic on Ω.

Proof. It can be verified that g is C^1 on Ω^- and satisfies the Cauchy–Riemann equations, so g is holomorphic on $\Omega \setminus L$. Also, g is continuous on Ω.

To show that g is holomorphic on all of Ω, g satisfies (12.1) when $\gamma = \partial R$, and $R \subset \Omega^+$. The same is also true if $R \subset \Omega^-$. Finally, if R intersects L, it is possible to split the integral on ∂R into two integrals on rectangles. □

The Cauchy integral formula yields a locally uniform convergence result.

Proposition 23. *Let $\Omega \subset \mathbb{C}$ be an open set and let $f_\nu : \Omega \to \mathbb{C}$ be holomorphic. Assume $f_\nu \to f$ locally uniformly (i.e. uniformly on each compact subset of Ω). Then $f : \Omega \to \mathbb{C}$ is holomorphic, and $f'_\nu \to f'$ locally uniformly on Ω.*

Proof. Let $K \subset \Omega$ be a compact set. Then choose a smoothly bounded $\mathcal{O}$ such that $K \subset \mathcal{O} \subset \bar{\mathcal{O}} \subset \Omega$. Then, by the Cauchy integral formula,

$$f_\nu(z) = \frac{1}{2\pi i} \int_{\partial \mathcal{O}} \frac{f_\nu(\zeta)}{\zeta - z} d\zeta, \tag{13.2}$$

$$f'_\nu(z) = \frac{1}{2\pi i} \int_{\partial \mathcal{O}} \frac{f_\nu(\zeta)}{(\zeta - z)^2} d\zeta. \tag{13.3}$$

Since $f_\nu \to f$ locally uniformly on $\partial \mathcal{O}$, the integrands in (13.2) and (13.3) converge uniformly on $\bar{\mathcal{O}}$. Therefore, for any $z \in \mathcal{O}$,

$$f(z) = \frac{1}{2\pi i} \int_{\partial \mathcal{O}} \frac{f(\zeta)}{\zeta - z} d\zeta, \qquad \forall z \in \mathcal{O}, \tag{13.4}$$

so f is holomorphic on $\mathcal{O}$, and

$$f'(z) = \frac{1}{2\pi i} \int_{\partial \mathcal{O}} \frac{f(\zeta)}{(\zeta - z)^2} d\zeta, \tag{13.5}$$

so $f'_\nu \to f'$ locally uniformly on $\mathcal{O}$. □

It is also possible to produce an integral for the inverse of a holomorphic map.

Proposition 24. *Suppose f is holomorphic and one-to-one on a neighborhood of $\bar{\Omega}$, the closure of a piecewise, smoothly bounded domain $\Omega \subset \mathbb{C}$. Set $g = f^{-1} : f(\Omega) \to \Omega$. Then*

$$g(w) = \frac{1}{2\pi i} \int_{\partial \Omega} \frac{z f'(z)}{f(z) - w} dz, \qquad \forall w \in f(\Omega). \tag{13.6}$$

Proof. Set $\zeta = g(w)$, so that $h(z) = f(z) - w$ has one zero in $\bar{\Omega}$, at $z = \zeta$, and $h'(\zeta) \neq 0$. Indeed, if h has a zero of order k at z_0, then

$$h(z) = (z - z_0)^k \varphi(z)^k \tag{13.7}$$

for some holomorphic function $\varphi(z)$ that is nonvanishing in a neighborhood of $\bar{\Omega}$. Therefore, if f is one to one, then $f(z) - w = (z - z_0)\varphi(z)$ for some holomorphic function $\varphi(z)$ on a neighborhood of $\bar{\Omega}$. Therefore,

$$\frac{1}{2\pi i} \int_{\partial \Omega} z \frac{h'(z)}{h(z)} dz = \frac{1}{2\pi i} \int_{\partial \Omega} z \left(\frac{1}{z - z_0} + \frac{\varphi'(z)}{\varphi(z)}\right) dz = \zeta. \tag{13.8}$$

□

14. Laurent series

The Laurent series is a generalization of the power series expansion, which works for functions holomorphic in an annulus. Let

$$\mathcal{A} = \{z \in \mathbb{C} : r_0 < |z - z_0| < r_1\} \tag{14.1}$$

be such an annulus, where $0 < r_0 < r_1 < \infty$. Let γ_j be the counterclockwise circles $\{|z - z_0| = r_j\}$, so that $\partial\mathcal{A} = \gamma_1 - \gamma_0$. Then for any $f \in C^1(\mathcal{A})$ holomorphic in $\mathcal{A}$, the Cauchy integral formula implies that for $z \in \mathcal{A}$,

$$f(z) = \frac{1}{2\pi i}\int_{\gamma_1} \frac{f(\zeta)}{\zeta - z} d\zeta - \frac{1}{2\pi i}\int_{\gamma_0} \frac{f(\zeta)}{\zeta - z} d\zeta. \tag{14.2}$$

Now then, for $\zeta \in \gamma_1$, since $|z - z_0| < |\zeta - z_0|$,

$$\frac{1}{\zeta - z} = \frac{1}{(\zeta - z_0) - (z - z_0)} = \frac{1}{\zeta - z_0}\sum_{j=0}^{\infty}(\frac{z - z_0}{\zeta - z_0})^j. \tag{14.3}$$

Meanwhile, for $\zeta \in \gamma_0$, since $|\zeta - z_0| < |z - z_0|$,

$$\frac{1}{\zeta - z} = -\frac{1}{(z - z_0) - (\zeta - z_0)} = \frac{-1}{z - z_0}\sum_{j=0}^{\infty}(\frac{\zeta - z_0}{z - z_0})^j. \tag{14.4}$$

Therefore,

$$f(z) = \sum_{n=-\infty}^{\infty} a_n (z - z_0)^n, \qquad z \in \mathcal{A}, \tag{14.5}$$

where for $n \geq 0$,

$$a_n = \frac{1}{2\pi i}\int_{\gamma_1} \frac{f(\zeta)}{(\zeta - z_0)^{n+1}} d\zeta, \tag{14.6}$$

and for $n < 0$,

$$a_n = \frac{1}{2\pi i}\int_{\gamma_0} f(\zeta)(\zeta - z_0)^{-n-1} d\zeta. \tag{14.7}$$

Therefore,

Proposition 25. *Given $0 \leq r_0 < r_1 \leq \infty$, let $\mathcal{A}$ be the annulus* (14.1)*. If $f : \mathcal{A} \to \mathbb{C}$ is holomorphic, then it is given by the absolutely convergent series* (14.5)*, with*

$$a_n = \frac{1}{2\pi i}\int_{\gamma} \frac{f(\zeta)}{(\zeta - z_0)^{n+1}} d\zeta, \qquad n \in \mathbb{Z}, \tag{14.8}$$

where γ is any counterclockwise oriented circle centered at z_0 of radius $r_0 < r < r_1$.

Proof. The preceding argument can be applied to any annulus

$$\mathcal{A}^b = \{z \in \mathbb{C} : r_0' < |z - z_0| < r_1'\}, \tag{14.9}$$

where $r_0 < r_0' < r_1' < r_1$. Since f is holomorphic on $\mathcal{A}^b$, the integrals on γ_0 and γ_1 can be moved to γ. □

When f has an isolated singularity at z_0, we can take $r_0 = 0$. For example, take

$$f(z) = \frac{3z^2 + 2z + 6}{(z-2)^3}. \tag{14.10}$$

Making a partial fraction decomposition,

$$f(z) = \frac{22}{(z-2)^3} + \frac{14}{(z-2)^2} + \frac{3}{z-2}. \tag{14.11}$$

In general, if

$$f(z) = \frac{p(z)}{(z-z_0)^n} = \sum_{j=1}^{n-1} \frac{a_j}{(z-z_0)^j}, \qquad a_j = \frac{1}{(n-j)!} \lim_{z \to z_0} (\frac{d}{dz})^{n-j} p(z). \tag{14.12}$$

15. More Laurent series

Now consider the function

$$f(z) = \frac{3z^2 + 4z + 5}{(z-2)(z+1)(z+3)} = \frac{1}{z-2} + \frac{1}{z+1} + \frac{1}{z+3}. \tag{15.1}$$

Since $f(z)$ is analytic in the disk, $\{z : |z| < 1\}$, the Laurent series in this region is a power series.

For an annulus centered at $z_0 = 2$, $f(z)$ is analytic on the annulus $\{z : 0 < |z-2| < 3\}$. The $\frac{1}{z-2}$ term is okay. Now then, $\frac{1}{z+1} + \frac{1}{z+3}$ is analytic on $\{z : |z-2| < 3\}$. Then,

$$\frac{1}{z+1} = \frac{1}{3 + (z-2)} = \frac{1}{3} \sum_{n=0}^{\infty} \frac{(-1)^n (z-2)^n}{3^n}. \tag{15.2}$$

Similarly,

$$\frac{1}{z+3} = \frac{1}{5 + (z-2)} = \frac{1}{5} \sum_{n=0}^{\infty} \frac{(-1)^n (z-2)^n}{5^n}. \tag{15.3}$$

Therefore, for $n \geq 0$,

$$a_n = \frac{(-1)^n}{5^{n+1}} + \frac{(-1)^n}{3^{n+1}}. \tag{15.4}$$

Many of the same computations that we have done for convergent power series may also be utilized for convergent Laurent series.

Proposition 26. *Assume $f(z)$ is given by the series* (10.5) *converging for $z \in \mathcal{A}$, for $r_0 < |z - z_0| < r_1$. Then f is holomorphic on $\mathcal{A}$, and*

$$f'(z) = \sum_{n=-\infty}^{\infty} n a_n (z-z_0)^{n-1}, \qquad z \in \mathcal{A}. \tag{15.5}$$

Proof. Choose R_1 such that

$$\frac{1}{R_1} = \limsup_{n \to \infty} |a_n|^{1/n}, \tag{15.6}$$

and

$$R_0 = \limsup_{n \to \infty} |a_{-n}|^{1/n}. \tag{15.7}$$

As before,

$$\sum_{n=0}^{\infty} a_n (z - z_0)^n, \tag{15.8}$$

converges absolutely for $|z - z_0| < R_1$ and diverges for $|z - z_0| > R_1$, and

$$\sum_{n=-\infty}^{-1} a_n (z - z_0)^n, \tag{15.9}$$

converges absolutely for $|z-z_0| > R_0$ and diverges for $|z-z_0| < R_0$. Once again, since $\lim_{n\to\infty} n^{1/n} = 1$, the same computations may be made on $\mathcal{A}$.

Taking

$$f_\nu(z) = \sum_{n=-\nu}^{\nu} a_n (z - z_0)^n, \tag{15.10}$$

$f_\nu \to f$ locally uniformly on $\mathcal{A}$, so the limit f is holomorphic on $\mathcal{A}$, and f_ν' converges locally uniformly to f'. □

For example, do the Laurent expansion for

$$f(z) = e^{1/z} = \sum_{n=0}^{\infty} \frac{1}{n!} \frac{1}{z^n}. \tag{15.11}$$

Clearly $R_1 = \infty$ since $a_n = 0$ for $n > 0$. Now then,

$$\lim_{n\to\infty} |a_{-n}|^{-1/n} = \lim_{n\to\infty} (n!)^{1/n} = \infty, \tag{15.12}$$

so $R_0 = 0$.

16. Singularities

The function $f(z) = \frac{1}{z}$ is holomorphic on $\mathbb{C} \setminus \{0\}$ and has a singularity at $z = 0$.

Definition 5 (Isolated singularity)**.** *A point $p \in \mathbb{C}$ is an isolated singularity if there is a neighborhood U of p such that f is holomorphic on $U \setminus \{p\}$.*

So then, 0 is an isolated singularity for $f(z) = \frac{1}{z}$. An isolated singularity is said to be removable if there exists a function $\tilde{f}$ holomorphic on U, where $\tilde{f} = f$ on $U \setminus \{p\}$. If p is a removable singularity, then f is bounded near p. The converse is also true.

Theorem 15. *If $p \in \Omega$ and f is holomorphic on $\Omega \setminus \{p\}$ and bounded, then p is a removable singularity.*

Proof. Consider the function $g : \Omega \to \mathbb{C}$ defined by

$$g(z) = (z - p)^2 f(z), \qquad z \in \Omega \setminus \{p\}, \qquad g(p) = 0. \tag{16.1}$$

Since f is bounded, g is continuous on Ω. Also, g is complex differentiable at each point of Ω, since

$$g'(z) = 2(z - p) f(z) + (z - p)^2 f'(z), \qquad z \in \Omega \setminus \{p\}, \qquad g'(p) = 0. \tag{16.2}$$

Therefore, by Goursat's theorem, g is holomorphic on Ω, so on a neighborhood U of p, g has the convergent power series

$$g(z) = \sum_{n=0}^{\infty} a_n (z-p)^n, \qquad z \in U. \tag{16.3}$$

Since $g(p) = g'(p) = 0$, $a_0 = a_1 = 0$, so

$$g(z) = (z-p)^2 h(z), \qquad h(z) = \sum_{n=0}^{\infty} a_{n+2}(z-p)^n, \qquad z \in U. \tag{16.4}$$

Comparing (16.4) to (16.1), $h(z) = f(z)$ on $U \setminus \{p\}$, so set

$$\tilde{f}(z) = f(z), \qquad z \in \Omega \setminus \{p\}, \qquad \tilde{f}(p) = h(p). \tag{16.5}$$

□

An isolated singularity p is said to be a pole if $|f(z)| \to \infty$ as $z \to p$. Therefore, there exists a neighborhood U centered at p such that $|f(z)| \geq 1$ on $U \setminus \{p\}$. Thus, $g(z) = \frac{1}{f(z)}$ is holomorphic on $U \setminus \{p\}$, and $g(z) \to 0$ as $z \to p$, so g has a removable singularity on U. Therefore, g has a convergent power series expansion on U,

$$g(z) = \sum_{n=k}^{\infty} a_n (z-p)^n, \tag{16.6}$$

where a_k is the first nonzero coefficient in the power series. Therefore,

$$g(z) = (z-p)^k h(z), \qquad h(p) = a_k \neq 0, \tag{16.7}$$

with h holomorphic on U.

Proposition 27. *If f is holomorphic on $\Omega \setminus \{p\}$ with a pole at p, then there exists $k \in \mathbb{Z}^+$ such that*

$$f(z) = (z-p)^{-k} F(z), \tag{16.8}$$

on $\Omega \setminus \{p\}$, with F holomorphic on Ω, and $F(p) \neq 0$.

If $k = 1$, f has a simple pole at p.

A function holomorphic on Ω except for a set of poles is said to be meromorphic on Ω. One example of such a function is

$$\tan z = \frac{\sin z}{\cos z}, \tag{16.9}$$

which is meromorphic on $\mathbb{C}$, with poles at $\{(k + \frac{1}{2})\pi : k \in \mathbb{Z}\}$.

17. More singularities and zeros

Proposition 28. *If $f : \mathbb{C} \to \mathbb{C}$ is holomorphic and $|f(z)| \to \infty$ as $|z| \to \infty$, then $f(z)$ is a polynomial.*

Proof. Define the function $g : \mathbb{C} \setminus \{0\} \to \mathbb{C}$ defined by

$$g(z) = f(\frac{1}{z}). \tag{17.1}$$

Since $|g(z)| \to \infty$ as $z \to \infty$, g has a pole at 0. Then, by Proposition 27,

$$g(z) = z^{-k} G(z), \tag{17.2}$$

on $\mathbb{C} \setminus \{0\}$ for some $k \in \mathbb{Z}^+$, with G holomorphic on $\mathbb{C}$ and $G(0) \neq 0$. Then,

$$G(z) = \sum_{j=0}^{k-1} g_j z^j + z^k h(z), \tag{17.3}$$

and therefore,

$$g(z) = \sum_{j=0}^{k-1} g_j z^{j-k} + h(z). \tag{17.4}$$

Therefore,

$$f(z) = \sum_{j=0}^{k-1} g_j z^{k-j} + h(\frac{1}{z}), \tag{17.5}$$

so

$$f(z) - \sum_{j=0}^{k-1} g_j z^{k-j}, \tag{17.6}$$

is holomorphic on $\mathbb{C}$, and approaches $h(0)$ as $|z| \to \infty$. Therefore, by Liouville's theorem, the difference is constant, so $f(z)$ is a polynomial. □

An isolated singularity of a function that is not a pole or a removable singularity is called an essential singularity. An example of an essential singularity is the function $f(z) = e^{\frac{1}{z}}$.

Proposition 29 (Casorati-Weierstrass theorem)**.** *Suppose $f : \Omega \setminus \{p\} \to \mathbb{C}$ has an essential singularity at p. Then for any neighborhood U of p, the image of $U \setminus \{p\}$ is dense in $\mathbb{C}$.*

Proof. Suppose there exists a neighborhood U of p such that the image of $U \setminus \{p\}$ omits a neighborhood of $w_0 \in \mathbb{C}$. Replacing $f(z)$ by $f(z) - w_0$, suppose without loss of generality $w_0 = 0$. Then

$$g(z) = \frac{1}{f(z)} \tag{17.7}$$

is holomorphic and bounded on $U \setminus \{p\}$, so $g(z)$ has a removable singularity at p, so $\tilde{g}(z)$ has a holomorphic extension on U. If $\tilde{g}(p) \neq 0$, then p is a removable singularity for f. If $\tilde{g}(p) = 0$, then p is a pole of f. □

Definition 6 (Zeros)**.** *An analytic function is said to have a zero of order m at z_0 if*

$$f(z_0) = ... = f^{(m-1)}(z_0) = 0, \tag{17.8}$$

and

$$f^{(m)}(z_0) \neq 0. \tag{17.9}$$

Suppose f has a zero of order m. Then

$$f(z) = \sum_{j=m}^{\infty} f^{(j)}(z_0)\frac{1}{j!}(z-z_0)^j = (z-z_0)^m g(z), \tag{17.10}$$

where $g(z) \neq 0$ in a neighborhood of z_0. Therefore, if a sequence of zeros of f converges, and f is analytic, then f must be identically zero.

18. Residue calculus

Suppose f is holomorphic on an open set Ω, except for isolated singularities at points $p_j \in \Omega$. Each p_j is contained in a disk $D_j \subset\subset \Omega$ on a neighborhood of which f has a Laurent series

$$f(z) = \sum_{n=-\infty}^{\infty} a_n(p_j)(z-p_j)^n. \tag{18.1}$$

Definition 7 (Residue)**.** *The coefficient $a_{-1}(p_j)$ of $(z-p_j)^{-1}$ is called the residue of f at p_j and is denoted $Res_{p_j}(f)$. Then*

$$Res_{p_j}(f) = \frac{1}{2\pi i}\int_{\partial D_j} f(z)dz. \tag{18.2}$$

If, in addition Ω is bounded, with piecewise smooth boundary, and $f \in C(\bar{\Omega}, \{p_j\})$, assuming $\{p_j\}$ is a finite set, then by the Cauchy integral formula,

$$\int_{\partial\Omega} f(z)dz = \sum_j \int_{\partial D_j} f(z)dz = 2\pi i \sum_j Res_{p_j}(f). \tag{18.3}$$

The residue formula has a number of applications. For example, we can compute

$$\int_{-\infty}^{\infty} \frac{dx}{1+x^2}. \tag{18.4}$$

The function $f(z) = (1+z^2)^{-1}$ is a meromorphic function, with simple poles at $z = \pm i$. The residue at i may be computed

$$\lim_{z\to i}(z-i)\frac{1}{z^2+1} = \lim_{z\to i}\frac{1}{z+i} = \frac{1}{2i}. \tag{18.5}$$

Therefore, if γ is a positively oriented contour that contains i in its interior, but not $-i$, then

$$\int_\gamma \frac{1}{z^2+1}dz = \pi. \tag{18.6}$$

In particular, let γ_R denote the contour from $-R$ to R, and then the semicircle $Re^{i\theta}$, where $0 \leq \theta \leq \pi$. Then

$$\int_{\gamma_R} \frac{1}{z^2+1}dz = \pi. \tag{18.7}$$

Then,

$$\int_0^\pi \frac{R}{1+R^2e^{2i\theta}} ie^{i\theta}d\theta = O(\frac{1}{R}), \tag{18.8}$$

so

$$\int_{-R}^{R} \frac{1}{1+x^2} dx = \pi + O(\frac{1}{R}). \tag{18.9}$$

Taking $R \to \infty$,

$$\int_{-\infty}^{\infty} \frac{1}{1+x^2} dx = \pi. \tag{18.10}$$

We also know the antiderivative of $\frac{1}{1+x^2}$, which is $\arctan(x)$.

Next, consider the integral

$$\int_{-\infty}^{\infty} \frac{1}{1+x^4} dx. \tag{18.11}$$

The function $\frac{1}{1+z^4}$ has four simple poles, at $z = e^{i\frac{\pi}{4}}$, $e^{i\frac{3\pi}{4}}$, $e^{i\frac{5\pi}{4}}$, and $e^{i\frac{7\pi}{4}}$. Since

$$\lim_{z \to e^{i\frac{\pi}{4}}} z^2 + i = 2i, \qquad \text{and} \qquad \lim_{z \to e^{i\frac{3\pi}{4}}} z^2 - i = -2i, \tag{18.12}$$

$$\lim_{z \to e^{i\frac{\pi}{4}}} \frac{1}{(z^2+i)(z+e^{i\frac{\pi}{4}})} = \frac{1}{4i}\frac{1}{e^{i\frac{\pi}{4}}} = \frac{1}{4}e^{-\frac{3\pi i}{4}}, \tag{18.13}$$

$$\lim_{z \to e^{i\frac{3\pi}{4}}} \frac{1}{(z^2-i)(z+e^{i\frac{3\pi}{4}})} = \frac{-1}{4i}\frac{1}{e^{i\frac{\pi}{4}}} = \frac{1}{4}e^{-\frac{\pi i}{4}}. \tag{18.14}$$

Therefore, for any γ_R,

$$\int_{\gamma_R} \frac{1}{z^4+1} dz = \frac{\pi}{\sqrt{2}}. \tag{18.15}$$

Since

$$\int \frac{1}{1+R^4 e^{4i\theta}} Rie^{i\theta} d\theta = O(\frac{1}{R^3}), \tag{18.16}$$

$$\int_{-\infty}^{\infty} \frac{1}{1+x^4} dx = \frac{\pi}{\sqrt{2}}. \tag{18.17}$$

19. More residue calculus

The evaluation of Fourier transforms provides a rich source of examples to which to apply residue calculus. For example, consider the problem of computing

$$\int_{-\infty}^{\infty} \frac{e^{ix\xi}}{1+x^2} dx. \tag{19.1}$$

This integral has simple poles at $z = \pm i$. Moreover, the residue at $z = i$ is $\frac{e^{-\xi}}{2i}$ and the residue at $z = -i$ is $-\frac{e^{\xi}}{2i}$. Then for $\xi \geq 0$,

$$\int_{\gamma_R} \frac{e^{iz\xi}}{1+z^2} dz = \pi e^{-\xi}. \tag{19.2}$$

Taking $R \to \infty$,

$$\int_{-\infty}^{\infty} \frac{e^{ix\xi}}{1+x^2} dx = \pi e^{-\xi}. \tag{19.3}$$

Making the same computation for $\xi \le 0$,

$$\int_{-\infty}^{\infty} \frac{e^{ix\xi}}{1+x^2} dx = \pi e^{-|\xi|}. \tag{19.4}$$

It is also possible to use residue calculus to compute trigonometric integrals. Take for example,

$$\int_0^{2\pi} \frac{\sin^2\theta}{5+4\cos\theta} d\theta = \int_0^{2\pi} \frac{\frac{(e^{i\theta}-e^{-i\theta})^2}{-4}}{5+2(e^{i\theta}+e^{-i\theta}} d\theta. \tag{19.5}$$

Since $e^{i\theta}$ travels around a circle of radius 1 when θ travels from 0 to 2π, and if $z = e^{i\theta}$, $d\theta = \frac{dz}{iz}$,

$$= -\frac{1}{4}\int_{|z|=1} \frac{(z-\frac{1}{z})^2}{5+2(z+\frac{1}{z})} \frac{dz}{iz} = -\frac{1}{4i}\int_{|z|=1} \frac{(z^2-1)^2}{5z^3+2z^4+2z^2} dz. \tag{19.6}$$

Factoring the denominator,

$$2z^4 + 5z^3 + 2z^2 = 2z^2(z+\frac{1}{2})(z+2). \tag{19.7}$$

Therefore, (19.6) has a double pole at $z = 0$ and simple poles at $z = -\frac{1}{2}$ and $z = -2$. Therefore,

$$(19.6) = -\frac{\pi}{2}[Res|_{z=0} + Res|_{z=-\frac{1}{2}}] = -\frac{\pi}{2}[\frac{d}{dz}\frac{(z^2-1)^2}{2z^2+5z+2}|_{z=0} + \frac{(z^2-1)^2}{2z^2(z+2)}|_{z=-\frac{1}{2}}] \tag{19.8}$$

$$= -\frac{\pi}{2}(\frac{2(-1)(0)}{2} - \frac{5(-1)^2}{(2)^2} + \frac{(\frac{1}{4}-1)^2}{2(\frac{1}{4})(\frac{3}{2})}) = \frac{\pi}{4}. \tag{19.9}$$

20. Residue calculus using algebra of paths

Next, consider the Fourier transform

$$A = \int_{-\infty}^{\infty} \frac{e^{ix\xi}}{2\cosh\frac{x}{2}} dx. \tag{20.1}$$

Take the integral over the the contour $\gamma(x) = x + 2\pi i$. Then compute

$$2\cosh\frac{x-2\pi i}{2} = -(e^{x/2} + e^{-x/2}) = -2\cosh(\frac{x}{2}). \tag{20.2}$$

The poles of $\frac{1}{2\cosh\frac{z}{2}}$ are exactly the points where $\cosh(\frac{z}{2}) = 0$, or $e^{z/2} = -e^{-z/2}$, so then $e^z = -1$, $z = i\pi + i2n\pi$. Therefore,

$$(1+e^{-2\pi\xi})\int_{-\infty}^{\infty} \frac{e^{ix\xi}}{2\cosh\frac{x}{2}} dx = 2\pi i \lim_{z\to\pi i} \frac{(z-\pi i)e^{iz\xi}}{2\cosh(\frac{z}{2})} = 2\pi i \frac{e^{-\xi}}{\sinh(\frac{\pi i}{2})} = \pi e^{-\xi}. \tag{20.3}$$

Doing some algebra,

$$\int_{-\infty}^{\infty} \frac{e^{ix\xi}}{2\cosh\frac{x}{2}} dx = \frac{\pi}{\cosh\pi\xi}. \tag{20.4}$$

It is possible to use the algebra of paths to compute the integral

$$\int_{-\infty}^{\infty} \frac{e^{ax}}{1+e^x} dx, \tag{20.5}$$

when $0 < a < 1$. Let γ_R denote the contour from $-R$ to R, then up to $R + 2i\pi$, then left to $-R + 2i\pi$, then down to $-R$. Since $\frac{e^{az}}{1+e^z}$ has a simple pole at $z = i\pi + 2ni\pi$, for all n,

$$\int_{\gamma_R} \frac{e^{az}}{1+e^z} dz = \frac{e^{ia\pi}}{e^{i\pi}} = -e^{ia\pi}. \tag{20.6}$$

Furthermore, we can show that when $0 < a < 1$,

$$\lim_{R \to \infty} \int_R^{R+2\pi i} \frac{e^{az}}{1+e^z} dz, \qquad \int_{-R}^{-R+2\pi i} \frac{e^{az}}{1+e^z} dz = 0. \tag{20.7}$$

Therefore, let

$$A = \int_{-\infty}^{\infty} \frac{e^{ax}}{1+e^x} dx, \tag{20.8}$$

$$A - e^{2\pi i a} A = -e^{ia\pi}. \tag{20.9}$$

Doing some algebra,

$$A = \frac{\pi}{\sin \pi a}. \tag{20.10}$$

21. More residue calculus using algebra of paths

Now apply residue calculus to an integrand with a double pole. For example, consider the integral

$$u(\xi) = \int_{-\infty}^{\infty} \frac{e^{ix\xi}}{(1+x^2)^2} dx, \qquad \xi \in \mathbb{R}. \tag{21.1}$$

Then,

$$Res_i \frac{e^{iz\xi}}{(1+z^2)^2} = g'(i), \tag{21.2}$$

where

$$g(z) = \frac{e^{i\xi z}}{(z+i)^2}. \tag{21.3}$$

Then,

$$g'(i) = -\frac{i}{4}(1+\xi)e^{-\xi}. \tag{21.4}$$

Therefore, when $\xi > 0$,

$$u(\xi) = \lim_{R \to \infty} \int_{\gamma_R} \frac{e^{i\xi z}}{(1+z^2)^2} dz = \frac{\pi}{2}(1+\xi)e^{-\xi}. \tag{21.5}$$

Therefore, since $u(\xi)$ is an even function of ξ,

$$\int_{-\infty}^{\infty} \frac{e^{ix\xi}}{(1+x^2)^2} dx = \frac{\pi}{2}(1+|\xi|)e^{-|\xi|}, \qquad \forall \xi \in \mathbb{R}. \tag{21.6}$$

Another example of this kind is the integral

$$B = \int_0^{\infty} \frac{x^\alpha}{1+x^2} dx, \tag{21.7}$$

for some $0 < \alpha < 1$. Then define $z^\alpha = r^\alpha e^{i\alpha\theta}$ for $0 < \theta < 2\pi$. Then z^α is holomorphic on $\mathbb{C} \setminus \mathbb{R}^+$. Moreover, $(x+iy)^\alpha$ has distinct boundary values when $x > 0$ as $y \searrow 0$ and $y \nearrow 0$. Then let γ_R be the curve going from 0 to $Re^{i\epsilon}$ along the curve $re^{i\epsilon}$, followed by the counterclockwise circle from $Re^{i\epsilon}$ to $Re^{i(2\pi-\epsilon)}$, and then from $Re^{i(2\pi-\epsilon)}$ to zero along the path $re^{i(2\pi-\epsilon)}$, r going from R to 0.

Then for any $R > 1$,

(21.8)
$$\int_{\gamma_R} \frac{z^\alpha}{1+z^2}dz = 2\pi i Res_{z=i}(\frac{z^\alpha}{1+z^2}) + 2\pi i Res_{z=-i}(\frac{z^\alpha}{1+z^2}) = \frac{2\pi i}{2i}e^{i\alpha\frac{\pi}{2}} - \frac{2\pi i}{2i}e^{\frac{3\pi i\alpha}{2}} = \pi(e^{i\pi\alpha/2} - e^{3\pi i\alpha/2}).$$

Next,

$$\int_\epsilon^{2\pi-\epsilon} \frac{R^\alpha e^{i\theta\alpha}}{1+R^2e^{2i\theta}} Rie^{i\theta}d\theta = O(R^{\alpha-1}). \tag{21.9}$$

Also, for any $R > 0$ fixed,

$$\lim_{\epsilon\searrow 0}\int_0^R \frac{r^\alpha e^{i\epsilon\alpha}}{1+r^2e^{2i\epsilon}}e^{i\epsilon}dr = \int_0^R \frac{x^\alpha}{1+x^2}dx. \tag{21.10}$$

Meanwhile,

$$-\lim_{\epsilon\searrow 0}\int_0^R \frac{r^\alpha e^{i(2\pi-\epsilon)\alpha}}{1+r^2e^{2i(2\pi-\epsilon)}}dr = -e^{2\pi i\alpha}\int_0^R \frac{x^\alpha}{1+x^2}dx. \tag{21.11}$$

Therefore, taking $R \to \infty$,

$$(1-e^{2\pi i\alpha})B = \pi(e^{\pi i\alpha/2} - e^{3\pi i\alpha/2}). \tag{21.12}$$

Doing some algebra,

$$B = \pi\frac{\sin(\pi\alpha/2)}{\sin\pi\alpha}. \tag{21.13}$$

22. The argument principle

Suppose $\Omega \subset \mathbb{C}$ is a bounded domain with piecewise smooth boundary and $f \in C^2(\bar\Omega)$ is holomorphic on Ω, and nowhere zero on $\partial\Omega$. The number of zeros, counted with multiplicity, may be expressed in terms of the behavior of f on $\partial\Omega$. We say that $p_j \in \Omega$ is a zero of multiplicity k provided,

$$f^{(l)}(p_j) = 0, \qquad 0 \le l \le k-1, \qquad f^{(k)}(p_j) \neq 0. \tag{22.1}$$

Proposition 30. *Under the hypotheses stated above, the number $\nu(f,\Omega)$ of zeros of f in Ω, counted with multiplicity, is given by*

$$\nu(f,\Omega) = \frac{1}{2\pi i}\int_{\partial\Omega}\frac{f'(z)}{f(z)}dz. \tag{22.2}$$

Proof. Let D_j be small, disjoint disks around $p_j \in \Omega$. Then, by the Cauchy integral formula,

$$\frac{1}{2\pi i}\int_{\partial\Omega}\frac{f'(z)}{f(z)}dz = \frac{1}{2\pi i}\int_{\partial D_j}\frac{f'(z)}{f(z)}dz. \tag{22.3}$$

In a neighborhood $\bar D_j$ of p_j,

$$f(z) = (z-p_j)^{m_j}g(z), \tag{22.4}$$

with $g(z)$ non-vanishing on $\bar D_j$. Therefore, on $\bar D_j$,

$$\frac{f'(z)}{f(z)} = \frac{m_j}{(z-p_j)} + \frac{g'(z)}{g(z)}. \tag{22.5}$$

Since $\frac{g'(z)}{g(z)}$ is holomorphic on $\bar{D}_j$,

$$\frac{1}{2\pi i}\int_{\partial D_j}\frac{f'(z)}{f(z)}dz = \frac{m_j}{2\pi i}\int_{\partial D_j}\frac{dz}{z-p_j} = m_j. \tag{22.6}$$

□

Proposition 30 has an interpretation in terms of winding numbers. Let C_j denote the connected components of $\partial\Omega$, with proper orientation, and suppose C_j is parameterized by $\varphi_j : S^1 \to C_j$. Then,

$$f \circ \varphi_j : S^1 \to \mathbb{C} \setminus \{0\} \tag{22.7}$$

parameterizes the image curve $\gamma_j = f(C_j)$.

Proposition 31. *With C_j and γ_j as above,*

$$\frac{1}{2\pi i}\int_{C_j}\frac{f'(z)}{f(z)}dz = \frac{1}{2\pi i}\int_{\gamma_j}\frac{dz}{z}. \tag{22.8}$$

Proof. In general,

$$\int_{C_j} u(z)dz = \int_0^{2\pi} u(\varphi_j(t))\varphi_j'(t)dt, \tag{22.9}$$

and

$$\int_{\gamma_j} v(z)dz = \int_0^{2\pi} v(f(\varphi_j(t)))\frac{d}{dt}f\circ\varphi_j(t)dt = \int_0^{2\pi} v(f(\varphi_j(t)))f'(\varphi_j(t))\varphi_j'(t)dt. \tag{22.10}$$

In particular, taking $v(z) = \frac{1}{z}$,

$$\begin{aligned}\int_{C_j}\frac{f'(z)}{f(z)}dz = \int_0^{2\pi}\frac{f'(\varphi_j(t))}{f(\varphi_j(t))}\varphi_j'(t)dt = \int_0^{2\pi}\frac{1}{f(\varphi_j(t))}f'(\varphi_j(t))\varphi_j'(t)dt \\ = \int_0^{2\pi} v(f(\varphi_j(t)))f'(\varphi_j(t))\varphi_j'(t)dt = \int_{\gamma_j}\frac{1}{z}dz.\end{aligned} \tag{22.11}$$

□

Suppose γ is an arbitrary continuous, piecewise C^1 curve in $\mathbb{C}\setminus\{0\}$, say,

$$\gamma : [0, 2\pi] \to \mathbb{C}\setminus\{0\}, \qquad \gamma(t) = r(t)e^{i\theta(t)}, \tag{22.12}$$

where $r(t)$ and $\theta(t)$ are continuous, piecewise C^1, real valued functions of t, and $r(t) > 0$. Then,

$$\gamma'(t) = [r'(t) + ir(t)\theta'(t)]e^{i\theta(t)}. \tag{22.13}$$

Computing,

$$\frac{1}{2\pi i}\int_\gamma\frac{dz}{z} = \frac{1}{2\pi i}\int_0^{2\pi}\frac{\gamma'(t)}{\gamma(t)}dt = \frac{1}{2\pi i}\int_0^{2\pi}[\frac{r'(t)}{r(t)} + i\theta'(t)]dt. \tag{22.14}$$

Since $r(0) = e^{i\theta(0)} = r(2\pi)e^{i\theta(2\pi)}$,

$$\int_0^{2\pi}\frac{r'(t)}{r(t)}dt = \log r(2\pi) - \log r(0) = 0, \tag{22.15}$$

and

$$\frac{1}{2\pi}\int_0^{2\pi}\theta'(t)=\frac{1}{2\pi}[\theta(2\pi)-\theta(0)]=n(\gamma,0)\in\mathbb{Z}. \tag{22.16}$$

This integer is called the winding number.

Since continuous, integer valued functions must be constant, the winding number is stable.

Proposition 32. *If γ_0 and γ_1 are smoothly homotopic in $\mathbb{C}\setminus\{0\}$, then*

$$n(\gamma_0,0)=n(\gamma_1,0). \tag{22.17}$$

Proof. If γ_s is a smooth family of curves in $\mathbb{C}\setminus\{0\}$, for $0\le s\le 1$, then

$$n(\gamma_s,0)=\frac{1}{2\pi}\int_{\gamma_s}d\theta, \tag{22.18}$$

is a continuous function of $s\in[0,1]$, taking values in $\mathbb{Z}$. Hence it is constant. □

23. Rouche's theorem

Proposition 33 (Argument principle)**.** *If C_j denote the connected components of $\partial\Omega$,*

$$\nu(f,\Omega)=\sum_j n(\gamma_j,0),\qquad \gamma_j=f(C_j). \tag{23.1}$$

That is, the total number of zeros of f in Ω, counting multiplicity, is equal to the sum of the winding numbers of $f(C_j)$ about 0.

Proof. For any C_j,

$$\frac{1}{2\pi i}\int_{C_j}\frac{f'(z)}{f(z)}dz=n(\gamma_j,0),\qquad \gamma_j=f(C_j). \tag{23.2}$$

□

The argument principle also holds for meromorphic functions. If f has a pole of order m_j at p_j, then (19.6) would give $-m_j$ for a disk of small radius around p_j.

Proposition 34. *Assume f is meromorphic on a bounded domain Ω, and C^1 in a neighborhood of $\partial\Omega$. Then the number of zeros of f minus the number of poles of f (counting multiplicity) in Ω is equal to the sum of the winding numbers of $f(C_j)$ about 0, where the C_j are connected components of $\partial\Omega$.*

Many times, the right hand side is more readily calculable than the left hand side. For example, a useful corollary to the above result is Rouche's theorem.

Proposition 35 (Rouche's theorem)**.** *Let $f,g\in C^1(\bar{\Omega})$ be holomorphic in Ω and nowhere zero on $\partial\Omega$. Also assume that*

$$|f(z)-g(z)|<|f(z)|,\qquad \forall z\in\partial\Omega. \tag{23.3}$$

Then,

$$\nu(f,\Omega)=\nu(g,\Omega). \tag{23.4}$$

Proof. Inequality (23.3) implies that f and g are smoothly homotopic as maps from $\partial\Omega$ to $\mathbb{C} \setminus \{0\}$. Indeed, take

$$f_\tau(z) = f(z) - \tau[f(z) - g(z)], \qquad 0 \leq \tau \leq 1. \tag{23.5}$$

Therefore, $f|_{C_j}$ and $g|_{C_j}$ have the same winding numbers about 0, for each boundary component C_j. $\square$

Rouche's theorem gives another proof of the fundamental theorem of algebra. Let

$$f(z) = z^n, \qquad \text{and} \qquad g(z) = z^n + a_{n-1}z^{n-1} + ... + a_0. \tag{23.6}$$

For z sufficiently large, $|f(z) - g(z)| < |f(z)|$, and therefore, f and g have the same number of zeros inside the disk $\{z : |z| \leq R\}$. It is clear that f has n zeros inside this disk, so g must have n zeros as well.

Proposition 36. *Suppose Ω is as in Proposition* 30 *and let $f \in C^1(\bar{\Omega})$ be holomorphic on Ω. Suppose $S \subset \mathbb{C}$ is connected, and $S \cap f(\partial\Omega) = \emptyset$. Then,*

$$\nu(f - q, \Omega) \qquad \text{is independent of} \qquad q \in S. \tag{23.7}$$

Proof. Define the function,

$$\varphi(q) = \nu(f - q, \Omega) = \frac{1}{2\pi i} \int_{\partial\Omega} \frac{f'(z)}{f(z) - q} dz. \tag{23.8}$$

This function is a continuous function of q, so since $\varphi : S \to \mathbb{Z}$ and S is connected, φ must be constant. $\square$

Proposition 37 (Open mapping theorem)**.** *If $\Omega \subset \mathbb{C}$ is open and connected, and $f : \Omega \to \mathbb{C}$ is holomorphic and nonconstant, then f maps open sets to open sets.*

Proof. Suppose $p \in \Omega$ and $q = f(p)$. Then we have a power series expansion

$$f(z) = f(p) + \sum_{n=k}^{\infty} a_n (z - p)^n, \tag{23.9}$$

where $a_k \neq 0$. Therefore, there exists a disk $D_\rho(p)$ such that $f|_{D_\rho(p)}$ is bounded away from q. Applying Proposition 36 to $S = D_\epsilon(q)$ for some $\epsilon > 0$, for all $q' \in D_\epsilon(q)$,

$$\nu(f - q', D_\rho(p)) = \nu(f - q,_\rho (p)) = k. \tag{23.10}$$

Therefore, such points q' are contained in the range of f, and are hit exactly k times, counting multiplicity. $\square$

Proposition 38 (Hurwitz theorem)**.** *Assume f_n are holomorphic on each connected region Ω and $f_n \to f$ locally uniformly on Ω. Assume each f_n is nowhere vanishing in Ω. Then f is either nowhere vanishing or identically zero in Ω.*

Proof. Since $f_n \to f$ locally uniformly, f is holomorphic on Ω and $f_n' \to f'$ locally uniformly on Ω. If f is not identically zero on Ω, then the only zeros of f in Ω are isolated. Let D be a disk in Ω for which f has zeros in D, but not in ∂D. Then $\frac{1}{f_n} \to \frac{1}{f}$ locally uniformly on ∂D. By (22.2),

$$\frac{1}{2\pi i} \int_{\partial D} \frac{f_n'(z)}{f_n(z)} dz = 0, \qquad \text{for all} \qquad n. \tag{23.11}$$

Passing to the limit,

$$\nu(f,D) = \frac{1}{2\pi i}\int_{\partial D}\frac{f'(z)}{f(z)} = 0, \tag{23.12}$$

so f does not have any zeros in D. $\square$

24. Infinite products

In addition to represented holomorphic functions as an infinite sum of functions, it is also useful to represent a function as an infinite product of functions.

First, consider the product of numbers

$$\prod_{k=1}^{\infty}(1+a_k). \tag{24.1}$$

Disregarding the case when $a_k = -1$ for some k, convergence of $\prod_{k=1}^{M}(1+a_k)$ as $M \to \infty$ amounts to the convergence of

$$\lim_{M\to\infty}\prod_{k=M}^{N}(1+a_k) = 1, \qquad \text{uniformly in} \qquad N > M. \tag{24.2}$$

In particular, we require $a_k \to 0$ as $k \to \infty$.

Writing out the product

$$\prod_{k=M}^{N}(1+a_k) = 1 + \sum_{j=M}^{N} a_j + \sum_{M\le j_1<j_2\le N} a_{j_1}a_{j_2} + ... + a_M\cdots a_N, \tag{24.3}$$

$$|\prod_{k=M}^{N}(1+a_k) - 1| \le \prod_{k=M}^{N}(1+|a_k|) - 1. \tag{24.4}$$

Now then,

$$\log\prod_{k=M}^{N}(1+|a_k|) = \sum_{k=M}^{N}\log(1+|a_k|). \tag{24.5}$$

Since $x \ge 0$ implies $\log(1+x) \le x$, and $0 \le x \le 1$ implies $\log(1+x) \ge \frac{x}{2}$,

$$\frac{1}{2}\sum_{k=M}^{N}|a_k| \le \log\prod_{k=M}^{N}(1+|a_k|) \le \sum_{k=M}^{N}|a_k|. \tag{24.6}$$

Therefore, $\lim_{M\to\infty}\prod_{k=M}^{N}(1+|a_k|) = 1$ uniformly for $N > M$, if and only if $\sum_{k=M}^{N}|a_k| \to 0$ uniformly in $M \to \infty$.

Another consequence is the following,

$$\text{If} \qquad 1+a_k \ne 0, \qquad \text{for all} \qquad k, \qquad \text{then} \qquad \sum_k |a_k| < \infty, \Rightarrow \prod_{k=1}^{\infty}(1+a_k) \ne 0. \tag{24.7}$$

Now replace the sequence (a_k) of complex numbers by a sequence (f_k) of holomorphic functions.

Proposition 39. *Let $f_k : \Omega \to \mathbb{C}$ be holomorphic. Assume that for each compact set $K \subset \Omega$ there exist $M_k(K)$ such that*

$$\sup_{z \in K} |f_k(z)| \leq M_k(K), \qquad \text{and} \qquad \sum_k M_k(K) < \infty. \tag{24.8}$$

Then we have a convergent infinite product

$$\prod_{k=1}^{\infty}(1 + f_k(z)) = F(z). \tag{24.9}$$

In fact,

$$\prod_{k=1}^{n}(1 + f_k(z)) \to F(z), \qquad \text{as} \qquad n \to \infty, \tag{24.10}$$

uniformly on compact subsets of Ω. Therefore, F is holomorphic on Ω. If $z_0 \in \Omega$ and $1 + f_k(z_0) \neq 0$ for all k, then $F(z_0) \neq 0$.

Now assume that f_k, $g_k : \Omega \to \mathbb{C}$ are holomorphic, and assume in addition that $\sup_K |g_k| \leq M_k(K)$. Then one has the convergent infinite product

$$\prod_{k=1}^{\infty}(1 + g_k(z)) = G(z), \tag{24.11}$$

with G holomorphic on Ω. Now then,

$$(1 + f_k(z))(1 + g_k(z)) = 1 + f_k(z) + g_k(z) + f_k(z)g_k(z), \tag{24.12}$$

so

$$|f_k(z) + g_k(z) + f_k(z)g_k(z)| \leq 2M_k(K) + M_k(K)^2. \tag{24.13}$$

Therefore,

$$\prod_{k=1}^{\infty}(1 + h_k(z)) = H(z), \tag{24.14}$$

is a convergent infinite product, with $H(z)$ holomorphic on Ω. Moreover, for any n,

$$\prod_{k=1}^{n}(1 + f_k(z))(1 + g_k(z)) = \prod_{k=1}^{n}(1 + f_k(z)) \cdot \prod_{k=1}^{n}(1 + g_k(z)), \tag{24.15}$$

so therefore,

$$F(z)G(z) = H(z). \tag{24.16}$$

Consider the infinite product

$$S(z) = z \prod_{k=1}^{\infty}(1 - \frac{z^2}{k^2}). \tag{24.17}$$

If K is contained in the set $\{z : |z| \leq R\}$, $M_k(K) \leq \frac{R^2}{k^2}$, so $S(z)$ is holomorphic on all of $\mathbb{C}$. Furthermore, $S(z) = 0$ if and only if $z \in \mathbb{Z}$. Also, all zeros of $S(z)$ are simple.

A familiar function which has the same zeros as $S(z)$ is $\sin(\pi z)$. Since

$$\lim_{z \to 0} \frac{1}{z} S(z) = 1, \tag{24.18}$$

compare $S(z)$ to

$$s(z) = \frac{1}{\pi}\sin(\pi z). \tag{24.19}$$

Lemma 4. *For $S(z)$ as in* (24.17)*,*

$$S(z-1) = -S(z). \tag{24.20}$$

Proof. Since $S(z) = \lim_{n\to\infty} S_n(z)$, where

$$\begin{aligned} S_n(z) &= z\prod_{k=1}^{n}(1-\frac{z^2}{k^2}) = z\prod_{k=1}^{n}(1-\frac{z}{k})(1+\frac{z}{k}) \\ &= z\prod_{k=1}^{n}(\frac{k-z}{k})\cdot(\frac{k+z}{k}) = \frac{(-1)^n}{(n!)^2}(z-n)(z-n+1)\cdots(z+n-1)(z+n). \end{aligned} \tag{24.21}$$

Plugging in $z-1$,

$$S_n(z-1) = \frac{(-1)^n}{(n!)^2}(z-1-n)(z-n)\cdots(z+n-2)(z+n-1) = \frac{z-n-1}{z+n}S_n(z). \tag{24.22}$$

Taking $n\to\infty$,

$$S(z-1) = -S(z). \tag{24.23}$$

□

Since $\sin(\pi(z-1)) = -\sin(\pi z)$, $s(z-1) = -s(z)$. Now take

$$f(z) = \frac{1}{S(z)} - \frac{1}{s(z)}. \tag{24.24}$$

This function is holomorphic on $\mathbb{C}\setminus\mathbb{Z}$ and satisfies $f(z-1) = -f(z)$. Furthermore,

$$S(z) = zH(z), \qquad s(z) = zh(z), \tag{24.25}$$

with H and h holomorphic on $\mathbb{C}$, with $H(0) = h(0) = 1$. Therefore, on some neighborhood $\mathcal{O}$ of 0,

$$\frac{1}{H(z)} = 1 + zA(z), \qquad \frac{1}{h(z)} = 1 + za(z), \tag{24.26}$$

Consequently, on $\mathcal{O}\setminus 0$,

$$\frac{1}{S(z)} - \frac{1}{s(z)} = \frac{1}{z}(1+zA(z)) - \frac{1}{z}(1+za(z)) = A(z) - a(z). \tag{24.27}$$

Thus, $f(z)$ has a removable singularity at $z = 0$, setting $f(0) = A(0) - a(0)$. Setting $f(-k) = (-1)^k[A(0) - a(0)]$ for each $k\in\mathbb{Z}$,

$$f:\mathbb{C}\to\mathbb{C}, \qquad \text{holomorphic.} \tag{24.28}$$

Lemma 5. *We have $f(z)\to 0$ as $|z|\to\infty$, uniformly on the set*

$$\{z\in\mathbb{C} : 0\le Re(z)\le 1\}. \tag{24.29}$$

Proof. Since

$$\sin(x + iy) = \frac{1}{2i}(e^{-y+ix} - e^{y-ix}), \tag{24.30}$$

$|\sin(x + iy)| \to \infty$ as $|y| \to \infty$. Meanwhile, for $S(z)$,

$$|1 - \frac{z^2}{k^2}| \geq 1 + \frac{y^2 - x^2}{k^2} \geq 1 + \frac{y^2 - 1}{k^2}, \tag{24.31}$$

so $|Re(z)| \leq 1$ and $|Im(z)| \geq 1$ implies $|S(z)| \geq |z|$. Therefore, $|S(z)| \to \infty$ as $|z| \to \infty$. □

Therefore, $f(z)$ is bounded. Since f is holomorphic on $\mathbb{C}$, f is constant. Finally, since $f(z) \to 0$ as $|z| \to \infty$, $0 \leq Re(z) \leq 1$, $f(z) = 0$. Therefore,

Proposition 40. *For* $z \in \mathbb{C}$,

$$\sin \pi z = \pi z \cdot \prod_{k=1}^{\infty}(1 - \frac{z^2}{k^2}). \tag{24.32}$$

References

[SS03] EB Saff and AD Snider. *Complex Analysis.* Prentice Hall, Pearson Education Inc, 2003.

[Tay19] Michael E. Taylor. *Introduction to complex analysis*, volume 202 of *Graduate Studies in Mathematics.* American Mathematical Society, Providence, RI, 2019.