ORDINARY DIFFERENTIAL EQUATIONS

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Contents

These class notes are primarily taken from [BD65] and [Tay22].

1. The method of integrating factors

Let us consider an ordinary differential equation of the form

$$(4+t^2)\frac{dy}{dt} + 2ty = 4t. \tag{1.1}$$

Notice that by the product rule, (1.1) is equal to

$$\frac{d}{dt}((4+t^2)y(t)) = 4t. \tag{1.2}$$

Then by the fundamental theorem of calculus,

$$(4+t^2)y(t) = 2t^2 + c. \tag{1.3}$$

Usually, such equations do not fit into this framework exactly, but it may be possible to use an integrating factor. Consider the differential equation

$$\frac{dy}{dt} + \frac{1}{2}y = \frac{1}{2}e^{t/3}. \tag{1.4}$$

Let us multiply the left and right hand sides by the function $\mu(t) > 0$.

$$\mu(t)\frac{dy}{dt} + \frac{1}{2}\mu(t)y(t) = \frac{1}{2}\mu(t)e^{t/3}. \tag{1.5}$$

If $\mu(t)$ solves the equation

$$\frac{d}{dt}\mu(t) = \frac{1}{2}\mu(t), \tag{1.6}$$

then

$$\frac{d}{dt}(\mu(t)y(t)) = \frac{1}{2}\mu(t)e^{t/3}, \tag{1.7}$$

and we can therefore proceed as before. Computing,

$$\frac{1}{\mu(t)}\frac{d}{dt}\mu(t) = \frac{d}{dt}\ln\mu(t) = a, \tag{1.8}$$

and therefore,

$$\mu(t) = e^{at}. \tag{1.9}$$

Now let us try a more difficult problem.

$$t\frac{dy}{dt} + 2y(t) = 4t^2, \qquad y(1) = 2. \tag{1.10}$$

By direct computation, we need $\mu(t)$ such that

$$\frac{d}{dt}\mu(t) = \frac{2}{t}\mu(t). \tag{1.11}$$

Then we compute

$$\frac{d}{dt}\ln|\mu(t)| = \frac{2}{t}. \tag{1.12}$$

Integrating the left and right hand sides,

$$\ln|\mu(t)| = 2\ln t + c. \tag{1.13}$$

Therefore, we may set

$$\mu(t) = t^2. \tag{1.14}$$

For a general equation

$$\frac{dy}{dt} + p(t)y(t) = q(t), \tag{1.15}$$

we take the integrating factor

$$\mu(t) = \exp(\int p(t)dt). \tag{1.16}$$

2. Separable differential equations

Consider the general first-order differential equation

$$\frac{dy}{dx} = f(x, y). \tag{2.1}$$

Suppose such an equation is of the form

$$M(x) + N(y)\frac{dy}{dx} = 0. \tag{2.2}$$

Such an equation is separable, because it can be written in the differential form

$$M(x)dx + N(y)dy = 0. \tag{2.3}$$

For example, solve the equation

$$\frac{dy}{dx} = \frac{x^2}{1 - y^2}. \tag{2.4}$$

Then we can solve

$$\frac{1}{3}x^3 + c = y - \frac{y^3}{3}. \tag{2.5}$$

Next, solve

$$\frac{dy}{dx} = \frac{3x^2 + 4x + 2}{2(y-1)}, \qquad y(0) = -1. \tag{2.6}$$

Solving this equation,

$$y^2 - 2y = x^3 + 2x^2 + 2x + c. \tag{2.7}$$

In this case we take $c = 3$.

Now consider the separable differential equation

$$\frac{dy}{dx} = \frac{4x - x^3}{4 + y^3}, \qquad y(0) = 1. \tag{2.8}$$

In this case,

$$4y + \frac{y^4}{4} = 2x^2 - \frac{x^4}{4} + c, \tag{2.9}$$

and by direct computation, $c = \frac{17}{4}$.

Remark: If we have an equation of the form

$$\frac{dy}{dx} = f(y)g(x), \tag{2.10}$$

if $f(y_0) = 0$, then the solution to (2.10) is of the form $y(x) = y_0$. In this case we would not want to divide by $f(y)$.

3. Linear and nonlinear differential equations

Theorem 1. *If the functions p and g are continuous on an open interval $I : \alpha < t < \beta$ containing the point $t = t_0$, then there exists a unique function $y = \phi(t)$ that satisfies the differential equation*

$$\frac{dy}{dt} + p(t)y(t) = g(t), \tag{3.1}$$

for each $t \in I$, and that also satisfies the initial condition $y(t_0) = y_0$.

Proof. Since $p(t)$ is continuous, $p(t)$ is integrable on a subinterval of I. Therefore,

$$\mu(t) = \exp(\int p(t)dt), \tag{3.2}$$

is well-defined.

Now then, suppose (3.1) has two solutions, $y_1(t)$ and $y_2(t)$. Then, let $y_1(t) - y_2(t) = y(t)$. Then,

$$\frac{dy}{dt} + p(t)y(t) = 0, \qquad y(t_0) = 0. \tag{3.3}$$

We can show that the only solution to (3.3) is $y(t) = 0$. □

Theorem 2. *Suppose f and $\frac{\partial f}{\partial y}$ are continuous in some rectangle $\alpha < t < \beta$, $\gamma < y < \delta$, containing (t_0, y_0). Then there exists some interval $t_0 - h < t < t_0 + h$ contained in $\alpha < t < \beta$, there is a unique solution $y(t) = \phi(t)$ of the initial value problem*

$$\frac{dy}{dt} = f(t, y), \qquad y(t_0) = y_0. \tag{3.4}$$

We can apply this theorem to the initial value problem

$$ty'(t) + 2y(t) = 4t^2, \qquad y(1) = 2. \tag{3.5}$$

Doing some algebra, $p(t) = \frac{2}{t}$, which is continuous on $t \neq 0$.

Now consider the initial value problem

$$\frac{dy}{dx} = \frac{3x^2 + 4x + 2}{2(y-1)}, \qquad y(0) = -1. \tag{3.6}$$

In this case, f and $\frac{\partial f}{\partial y}$ are continuous on any rectangle that does not contain $y = 1$. If $x = 0$ and $y = 1$, we obtain

$$y^2 - 2y = x^3 + 2x^2 + 2x + c, \qquad c = -1. \tag{3.7}$$

For the equation,

$$\frac{dy}{dt} = y^{1/3}, \qquad y(0) = 0, \tag{3.8}$$

we do not have a unique solution.

The initial value problem,

$$\frac{dy}{dt} = y^2, \qquad y(0) = 1, \tag{3.9}$$

has a solution on the interval $(0, 1)$.

4. Exact differential equations and integrating factors

Now consider the differential equation

$$2x + y^2 + 2xy\frac{dy}{dx} = 0. \tag{4.1}$$

Observe that if we did not have $2x$,

$$y^2 + 2xy\frac{dy}{dx} = 0, \tag{4.2}$$

is a separable equation.

Here, notice that $2x + y^2 = \frac{\partial\psi}{\partial x}$ and $2xy = \frac{\partial\psi}{\partial y}$, where $\psi(x, y) = x^2 + xy^2$. Then,

$$\frac{\partial\psi}{\partial x} + \frac{\partial\psi}{\partial y}\frac{dy}{dx} = 0. \tag{4.3}$$

Then, (4.3) has the form,

$$\frac{d\psi}{dx}(x, y) = \frac{d}{dx}(x^2 + xy^2) = 0. \tag{4.4}$$

Therefore,

$$\psi(x, y) = x^2 + xy^2 = c. \tag{4.5}$$

How do we know in general if this is possible? Observe that if

$$\frac{\partial\psi}{\partial x}(x, y) = M(x, y), \qquad \frac{\partial\psi}{\partial y}(x, y) = N(x, y), \tag{4.6}$$

then

$$\frac{\partial M}{\partial y}(x, y) = \frac{\partial N}{\partial x}(x, y). \tag{4.7}$$

Theorem 3. *Suppose the functions M, N, M_y, and N_x are continuous in the rectangular region $\alpha < x < \beta$, $\gamma < y < \delta$. Then,*

$$M(x, y) + N(x, y)\frac{dy}{dx} = 0, \tag{4.8}$$

is an exact differential equation in R if and only if

$$M_y(x, y) = N_x(x, y). \tag{4.9}$$

Proof. We can try integrating in x or in y. Take

$$\psi(x, y) = Q(x, y) + h(y), \qquad Q(x, y) = \int_{x_0}^{x} M(s, y)ds. \tag{4.10}$$

Differentiating (4.10) with respect to y,

$$\psi_y(x, y) = \frac{\partial Q}{\partial y}(x, y) + h'(y) = \int_{x_0}^{x} N_x(s, y)ds + h'(y) = N(x, y) - N(x_0, y) + h'(y). \tag{4.11}$$

Now then, since we want $\psi_y(x, y) = N(x, y)$, we need to solve $h'(y) = N(x_0, y)$. So take $h(y) = \int_{y_0}^{y} N(x_0, s)ds$. □

First consider the equation

$$(y\cos x + 2xe^{y}) + (\sin x + x^{2}e^{y} - 1)\frac{dy}{dx} = 0. \tag{4.12}$$

In this case, $\psi(x, y) = y\sin x + x^{2}e^{y} - y$.

It is sometimes possible to convert a differential equation that is not exact to an exact differential equation by multiplying by a suitable integrating factor. Indeed, suppose we have the equation

$$M(x, y) + N(x, y)\frac{dy}{dx} = 0. \tag{4.13}$$

Multiplying by $\mu(x, y)$,

$$\mu(x, y)M(x, y) + \mu(x, y)N(x, y)\frac{dy}{dx} = 0. \tag{4.14}$$

Then (4.14) is exact if and only if

$$(\mu(x, y)M(x, y))_{y} = (\mu(x, y)N(x, y))_{x}. \tag{4.15}$$

Computing,

$$M\mu_{y} - N\mu_{x} + (M_{y} - N_{x})\mu = 0. \tag{4.16}$$

For example, consider the equation

$$(3xy + y^{2}) + (x^{2} + xy)\frac{dy}{dx} = 0. \tag{4.17}$$

Then we wish to solve

$$(3xy + y^{2})\mu_{y} - (x^{2} + xy)\mu_{x} + (3x + 2y - 2x - y)\mu = 0. \tag{4.18}$$

Simplifying by setting $\mu_{y} = 0$,

$$\frac{\mu_{x}}{\mu} = \frac{x + y}{x(x + y)} = \frac{1}{x}, \qquad \mu = x. \tag{4.19}$$

5. Second order equations - reducible cases

Second order differential equations have the form

$$\frac{dy^{2}}{dt^{2}} = f(t, y, \frac{dy}{dt}), \qquad y(t_{0}) = y_{0}, \qquad y'(t_{0}) = v_{0}. \tag{5.1}$$

There are some cases which reduce to first order equations for

$$v(t) = \frac{dy}{dt}. \tag{5.2}$$

For example, consider

$$y'' = f(t, y'). \tag{5.3}$$

In this case, let $v = y'$,

$$\frac{dv}{dt} = f(t, v), \qquad v(t_{0}) = v_{0}. \tag{5.4}$$

Solving for $v(t)$,

$$y(t) = y_{0} + \int_{t_{0}}^{t} v(s)ds. \tag{5.5}$$

For example, consider the equation

$$\frac{d^2y}{dt^2} = t\frac{dy}{dt}. \tag{5.6}$$

$$\frac{dv}{dt} = tv, \tag{5.7}$$

so $v(t) = e^{t^2/2}$ and $y(t) = y_0 + \int_0^t e^{s^2/2} ds$.

Now, consider the equation,

$$y'' = f(y, y'). \tag{5.8}$$

Taking $v(t) = \frac{dy}{dt}$,

$$\frac{dv}{dt} = f(y, v), \tag{5.9}$$

which contains too many variables. Rewriting the equation as one for v as a function of y,

$$\frac{dv}{dt} = \frac{dv}{dy}\frac{dy}{dt} = v\frac{dv}{dx}. \tag{5.10}$$

Substituting (5.10) into (5.8),

$$\frac{dv}{dy} = \frac{f(y, v)}{v}, \qquad v(y_0) = v_0. \tag{5.11}$$

For example, consider the equation

$$y'' = f(y). \tag{5.12}$$

In this case,

$$\frac{dv}{dy} = \frac{f(y)}{v}. \tag{5.13}$$

This equation is separable,

$$vdv = f(y)dy. \tag{5.14}$$

Therefore,

$$\frac{1}{2}v^2 = g(y) + C, \qquad \int f(y)dx = g(y) + C. \tag{5.15}$$

Therefore,

$$\frac{dy}{dt} = v(t) = \pm\sqrt{2g(x) + 2C}. \tag{5.16}$$

This equation is also separable:

$$\pm \int \frac{dy}{\sqrt{2g(y) + 2C}} = t + C_2. \tag{5.17}$$

Take

$$\frac{d^2y}{dt^2} = y^2. \tag{5.18}$$

Then,

$$\frac{dv}{dy} = \frac{y^2}{v}. \tag{5.19}$$

Therefore,

$$\frac{1}{2}v^2 = \frac{1}{3}y^3 + C. \tag{5.20}$$

Therefore,

$$\frac{dy}{dt} = v = \pm\sqrt{\frac{2}{3}y^3 + 2C}. \tag{5.21}$$

$$\pm\int \frac{dy}{\sqrt{\frac{2}{3}y^3 + 2C}} = t + C_2. \tag{5.22}$$

6. Homogeneous differential equations with constant coefficients

Consider the constant coefficient, second order linear differential equation

$$ay'' + by' + cy = 0. \tag{6.1}$$

Taking $y(t) = e^{rt}$, $y'(t) = re^{rt}$, and $y''(t) = r^2e^{rt}$. Substituting this into (6.1),

$$(ar^2 + br + c)e^{rt} = 0. \tag{6.2}$$

This condition is only satisfied when $ar^2 + br + c = 0$. This equation is called the characteristic equation.

For example, take

$$y'' - y = 0, \qquad y(0) = 2, \qquad y'(0) = -1. \tag{6.3}$$

The characteristic equation is $r^2 - 1 = 0$, which has solutions $r = \pm 1$. A general solution of (6.3) is given by

$$y(t) = c_1e^t + c_2e^{-t}. \tag{6.4}$$

Now then, solving $c_1 + c_2 = 2$, $c_1 - c_2 = -1$, so $c_1 = \frac{1}{2}$, $c_2 = \frac{3}{2}$.

Solve

$$y'' + 5y' + 6y = 0, \qquad y(0) = 2, \qquad y'(0) = 3. \tag{6.5}$$

Solve

$$4y'' - 8y' + 3y = 0, \qquad y(0) = 2, \qquad y'(0) = \frac{1}{2}. \tag{6.6}$$

7. Repeated roots: reduction of order

Suppose now that the characteristic equation has a repeated root. This occurs when the discriminant is zero,

$$b^2 - 4ac = 0. \tag{7.1}$$

In this case,

$$r_1 = r_2 = -\frac{b}{2a}. \tag{7.2}$$

Let us first suppose that $r_1 = 0$. In that case, we have the equation,

$$y''(t) = 0. \tag{7.3}$$

We know how to solve this equation,

$$y(t) = c_1 t + c_2. \tag{7.4}$$

Notice that in this case, $e^{r_1 t}$ is a constant function.

For a general equation with $r_1 = r_2$, we have an equation of the form

$$y'' - 2r_1 y' + r_1^2 y = 0. \tag{7.5}$$

In this case, $y_1(t) = e^{r_1 t}$ is a solution to our equation. Now let us try $y_2(t) = v(t)y_1(t) = e^{r_1 t}v(t)$. In this case, by the product rule,

$$\begin{aligned} &v''(t)y_1(t) + 2v'(t)y_1'(t) + v(t)y_1''(t) - 2r_1 v'(t)y_1(t) - 2r_1 v(t)y_1'(t) + r_1^2 v(t)y_1(t) \\ &\qquad = v''(t)y_1(t) + 2v'(t)y_1'(t) - 2r_1 v'(t)y_1(t) = v''(t)y_1(t) = 0. \end{aligned} \tag{7.6}$$

Therefore, in this case, $y_2(t) = c_2 t e^{r_1 t}$.

We can use the reduction of order for a general equation of the form

$$y'' + p(t)y' + q(t)y = 0. \tag{7.7}$$

Suppose we know that there exists a solution $y_1(t)$ to (7.7), not everywhere zero. Set $v(t)y_1(t) = y(t)$. Plugging this into (7.7),

$$\begin{aligned} &v''(t)y_1(t) + 2v'(t)y_1'(t) + v(t)y_1''(t) + p(t)v'(t)y_1(t) + p(t)v(t)y_1'(t) + q(t)v(t)y_1(t) \\ &\qquad = v''(t)y_1(t) + (2y_1'(t) + p(t)y_1(t))v'(t) = 0. \end{aligned} \tag{7.8}$$

This equation is actually first order, if we substitute $w(t) = v'(t)$.

Consider, for example, the equation

$$2t^2 y'' + 3ty' - y = 0, \qquad t > 0. \tag{7.9}$$

Now then, we know that $y_1(t) = t^{-1}$ is a solution of (7.9). Now, set $y(t) = v(t)t^{-1}$. Then,

$$v''(t)t^{-1} - 2t^{-2}v'(t) + p(t)t^{-1}v'(t) = v''(t)t^{-1} - 2t^{-2}v'(t) + \frac{3}{2}t^{-2}v'(t) = 0. \tag{7.10}$$

Therefore,

$$2tv'' - v' = 0. \tag{7.11}$$

Setting $w = v'$, we wish to solve,

$$w' - \frac{1}{2t}w = 0. \tag{7.12}$$

Therefore,

$$w(t) = ct^{1/2}, \qquad \text{and} \qquad v(t) = \frac{2}{3}ct^{3/2} + k. \tag{7.13}$$

8. Complex roots of the characteristic equation

Now consider the second order differential equation,

$$ay'' + by' + cy = 0. \tag{8.1}$$

Suppose this equation has the complex roots,

$$r_1 = \lambda + i\mu, \qquad r_2 = \lambda - i\mu. \tag{8.2}$$

In this case, we can try

$$y_1(t) = c_1 e^{r_1 t}, \qquad y_2(t) = c_2 e^{r_2 t}. \tag{8.3}$$

Now let us make sense of e^{it}. Observe that

$$\frac{d}{dt}(e^{it}) = ie^{it}. \tag{8.4}$$

Therefore, $e^{it} = c(t) + is(t)$ solves an ordinary differential equation

$$\frac{d}{dt}y = iy, \qquad y(0) = 1. \tag{8.5}$$

Since i rotates by ninety degrees, $y(t)$ travels at speed one counterclockwise along the unit circle. Thus,

$$e^{it} = \cos(t) + i\sin(t). \tag{8.6}$$

Therefore, the general solution has the form

$$y(t) = c_1 e^{-\lambda t}(\cos(\mu t) + i\sin(\mu t)) + c_2 e^{-\lambda t}(\cos(\mu t) - i\sin(\mu t)). \tag{8.7}$$

Doing some algebra,

$$y(t) = c_1 e^{-\lambda t}\cos(\mu t) + c_2 e^{-\lambda t}\sin(\mu t). \tag{8.8}$$

We can also use the power series expansion to obtain (8.6). In this case,

$$e^{it} = \sum_{k=0}^{\infty} \frac{i^k t^k}{k!} = \sum_{n=0}^{\infty} \frac{(-1)^n t^{2n}}{(2n)!} + i\sum_{n=0}^{\infty} \frac{t^{2n+1}}{(2n+1)!} = \cos(t) + i\sin(t). \tag{8.9}$$

9. The Wronskian

Let us define the concept of a differential operator. Suppose $p(t)$ and $q(t)$ are continuous functions. Then let

$$L[\phi] = \phi''(t) + p(t)\phi'(t) + q(t)\phi(t). \tag{9.1}$$

With this equation, we associate a set of initial conditions,

$$y(t_0) = y_0, \qquad y'(t_0) = y_0'. \tag{9.2}$$

We have the existence and uniqueness theorem.

Theorem 4. *Consider the initial value problem*

$$y'' + p(t)y' + q(t)y = g(t), \qquad y(t_0) = y_0, \qquad y'(t_0) = y_0', \tag{9.3}$$

where p, q, and g are continuous on an open interval I that contains the point t_0. This problem has exactly one solution $y(t) = \phi(t)$, and the solution exists throughout the interval I.

Now then, $L[cy] = cL[y]$ and $L[c_1y_1 + c_2y_2] = c_1L[y_1] + c_2L[y_2]$. Therefore, if $L[y_1] = 0$ and $L[y_2] = 0$, then $L[c_1y_1 + c_2y_2] = 0$.

Now we need to solve the system of equations,

$$\begin{aligned} c_1y_1(t_0) + c_2y_2(t_0) &= y_0, \\ c_1y_1'(t_0) + c_2y_2'(t_0) &= y_0'. \end{aligned} \tag{9.4}$$

This system is solvable if and only if

$$det \begin{pmatrix} y_1(t_0) & y_2(t_0) \\ y_1'(t_0) & y_2'(t_0) \end{pmatrix} \neq 0. \tag{9.5}$$

Then we solve

$$\begin{pmatrix} y_1(t_0) & y_2(t_0) \\ y_1'(t_0) & y_2'(t_0) \end{pmatrix}^{-1} \begin{pmatrix} y_0 \\ y_0' \end{pmatrix} = det \begin{pmatrix} y_1(t_0) & y_2(t_0) \\ y_1'(t_0) & y_2'(t_0) \end{pmatrix}^{-1} \begin{pmatrix} y_2'(t_0) & -y_2(t_0) \\ -y_1'(t_0) & y_1(t_0) \end{pmatrix} \begin{pmatrix} y_0 \\ y_0' \end{pmatrix}. \tag{9.6}$$

Theorem 5. *Suppose that y_1 and y_2 are two solutions to $L[y] = 0$, and that the initial conditions $y(t_0) = y_0$ and $y'(t_0) = y_0'$ are assigned. Then it is possible to choose constants c_1, c_2 so that $y(t) = c_1y_1(t) + c_2y_2(t)$ satisfies the differential equation and the initial conditions if and only if the Wronskian $W[y_1, y_2]$ is not zero at t_0.*

Theorem 6 (Abel's theorem)**.** *If y_1 and y_2 are solutions of the second order linear differential equation,*

$$L[y] = y'' + p(t)y' + q(t)y = 0, \tag{9.7}$$

where p and q are continuous on an open interval I, then the Wronskian $W[y_1, y_2](t)$ is given by

$$W[y_1, y_2](t) = c\exp(-\int p(t)dt). \tag{9.8}$$

Furthermore, $W[y_1, y_2](t)$ is either zero for all $t \in I$ or else is never zero on I.

Proof. By direct computation,

$$(y_1y_2'' - y_1''y_2) + p(t)(y_1y_2' - y_1'y_2) = 0. \tag{9.9}$$

Now then, observe that $W' = y_1y_2'' - y_1''y_2$, proving that

$$W' + p(t)W = 0. \tag{9.10}$$

Thus,

$$W(t) = c\exp(-\int p(t)dt). \tag{9.11}$$

□

10. Nonhomogeneous equations: method of undetermined coefficients

Now turn attention to the nonhomogeneous second-order linear differential equations

$$L[y] = y'' + p(t)y' + q(t)y = g(t). \tag{10.1}$$

The equation,

$$L[y] = y'' + p(t)y' + q(t)y = 0, \tag{10.2}$$

is called the homogeneous equation.

Theorem 7. *If Y_1 and Y_2 are two solutions of the nonhomogeneous linear differential equation* (10.1)*, then their difference $Y_1(t) - Y_2(t)$ is a solution to the corresponding homogeneous differential equation* (10.2)*. Then,*

$$Y_1(t) - Y_2(t) = c_1 y_1(t) + c_2 y_2(t). \tag{10.3}$$

Proof. Indeed,

$$L[Y_1](t) - L[Y_2](t) = g(t) - g(t) = 0. \tag{10.4}$$

□

Theorem 8. *The general solution of the nonhomogeneous equation* (10.1) *can be written in the form,*

$$y(t) = c_1 y_1(t) + c_2 y_2(t) + Y(t), \tag{10.5}$$

where $Y(t)$ is any solution to the nonhomogeneous equation (10.1)*.*

Definition 1. *The solution $Y(t)$ is called the particular solution.*

$$y'' - 3y' - 4y = 3e^{2t}. \tag{10.6}$$

Take the particular solution $Y(t) = Ae^{2t}$. In this case, $Y(t) = -\frac{1}{2}e^{2t}$.

$$y'' - 3y' - 4y = 2\sin t. \tag{10.7}$$

In this case, use the particular solution $Y(t) = A\sin t + B\cos t$. Indeed, we can decompose

$$2\sin t = \frac{1}{i}e^{it} + \frac{1}{i}e^{-it}. \tag{10.8}$$

Find the particular solution,

$$y'' - 3y' - 4y = -8e^t\cos(2t). \tag{10.9}$$

In this case,

$$Y(t) = Ae^t\cos(2t) + Be^t\sin(2t). \tag{10.10}$$

Here is a table.

$$\begin{aligned} P_n(t) = a_0 t^n + ... + a_n, &\qquad t^s(A_0 t^n + ... + A_n), \\ P_n(t)e^{\alpha t}, &\qquad t^s(A_0 t^n + ... + A_n)e^{\alpha t}, \\ P_n e^{\alpha t}(A_1\sin(\beta t) + A_2\cos(\beta t)), &\qquad t^s(A_0 t^n + ... + A_n)e^{\alpha t}\cos(\beta t) + t^s(B_0 t^n + ... + B_n)e^{\alpha t}\sin(\beta t). \end{aligned} \tag{10.11}$$

11. Variation of parameters

Consider the nonhomogeneous second order linear differential equation,

$$y''(t) + p(t)y'(t) + q(t)y(t) = g(t). \tag{11.1}$$

Now then, reducing to a first order equation,

$$\frac{d}{dt}\begin{pmatrix} y(t) \\ y'(t) \end{pmatrix} + \begin{pmatrix} 0 & -1 \\ q(t) & p(t) \end{pmatrix}\begin{pmatrix} y(t) \\ y'(t) \end{pmatrix} = \begin{pmatrix} 0 \\ g(t) \end{pmatrix}. \tag{11.2}$$

To simplify notation, let

$$A(t) = \begin{pmatrix} 0 & -1 \\ q(t) & p(t) \end{pmatrix}. \tag{11.3}$$

Now let $\mu(t) = \exp(\int_0^t A(s)ds)$ to be the integrating factor. Then,

$$\frac{d}{dt}(\mu(t)\begin{pmatrix} y(t) \\ y'(t) \end{pmatrix}) = \mu(t)\begin{pmatrix} 0 \\ g(t) \end{pmatrix}. \tag{11.4}$$

Therefore,

$$\begin{pmatrix} y(t) \\ y'(t) \end{pmatrix} = \mu(t)^{-1}\begin{pmatrix} y_0 \\ y_0' \end{pmatrix} + \mu(t)^{-1}(\int_0^t \mu(s)\begin{pmatrix} 0 \\ g(s) \end{pmatrix} ds). \tag{11.5}$$

Now we need to compute $\mu(t)$ and $\mu(t)^{-1}$. First observe that

$$\mu(t)^{-1}\begin{pmatrix} y_0 \\ y_0' \end{pmatrix} \tag{11.6}$$

gives the solution to

$$y''(t) + p(t)y'(t) + q(t)y(t) = 0, \qquad y(0) = y_0, \qquad y(0) = y_0'. \tag{11.7}$$

Thus, if $y_1(t)$ and $y_2(t)$ are solutions to (11.7) with nonzero Wronskian,

$$\mu(t)^{-1} = \mu(t)^{-1}\begin{pmatrix} y_1(0) & y_2(0) \\ y_1'(0) & y_2'(0) \end{pmatrix}\begin{pmatrix} y_1(0) & y_2(0) \\ y_1'(0) & y_2'(0) \end{pmatrix}^{-1} = \begin{pmatrix} y_1(t) & y_2(t) \\ y_1'(t) & y_2'(t) \end{pmatrix}\begin{pmatrix} y_1(0) & y_2(0) \\ y_1'(0) & y_2'(0) \end{pmatrix}^{-1}. \tag{11.8}$$

Then, doing some algebra,

$$\mu(s) = \begin{pmatrix} y_1(0) & y_2(0) \\ y_1'(0) & y_2'(0) \end{pmatrix}\begin{pmatrix} y_1(s) & y_2(s) \\ y_1'(s) & y_2'(s) \end{pmatrix}^{-1}. \tag{11.9}$$

Therefore,

$$\int_0^t \mu(t)^{-1}\mu(s)\begin{pmatrix} 0 \\ g(s) \end{pmatrix} ds = \int_0^t \begin{pmatrix} y_1(t) & y_2(t) \\ y_1'(t) & y_2'(t) \end{pmatrix}\begin{pmatrix} y_1(s) & y_2(s) \\ y_1'(s) & y_2'(s) \end{pmatrix}^{-1}\begin{pmatrix} 0 \\ g(s) \end{pmatrix} ds. \tag{11.10}$$

(11.11)

$$= \int_0^t \begin{pmatrix} y_1(t) & y_2(t) \\ y_1'(t) & y_2'(t) \end{pmatrix} W(s)^{-1}\begin{pmatrix} y_2'(s) & -y_2(s) \\ -y_1'(s) & y_1(s) \end{pmatrix}\begin{pmatrix} 0 \\ g(s) \end{pmatrix} ds = \int_0^t \begin{pmatrix} y_1(t) & y_2(t) \\ y_1'(t) & y_2'(t) \end{pmatrix} W(s)^{-1}\begin{pmatrix} -y_2(s)g(s) \\ y_1(s)g(s) \end{pmatrix} ds.$$

Therefore, we have a particular solution,

$$Y(t) = -y_1(t)\int_0^t \frac{y_2(s)g(s)}{W(s)}ds + y_2(t)\int_0^t \frac{y_1(s)g(s)}{W(s)}ds. \tag{11.12}$$

Therefore, the general solution is given by

$$y(t) = c_1y_1(t) + c_2y_2(t) + Y(t). \tag{11.13}$$

12. Mechanical and electrical vibrations

We know from physics that $F = ma$. Under Hooke's law, the force is given by $-kx$, where x is the displacement. Therefore, our equation is given by

$$m\frac{d^2x}{dt^2} + kx = 0. \tag{12.1}$$

The characteristic polynomial of (12.1) is given by

$$mr^2 + k = 0, \tag{12.2}$$

and the general solution is given by

$$c_1 \cos(\sqrt{\frac{k}{m}}t) + c_2 \sin(\sqrt{\frac{k}{m}}t). \tag{12.3}$$

Now let us add the force of damping to our equation. In this case, it is reasonable to think that there will be a damping force opposing the direction of motion. So then,

$$F = ma + \gamma v, \qquad \gamma > 0. \tag{12.4}$$

In this case we have

$$m\frac{d^2x}{dt^2} + \gamma\frac{dx}{dt} + kx = 0. \tag{12.5}$$

For this, the discriminant is given by

$$\gamma^2 - 4mk. \tag{12.6}$$

When $\gamma^2 - 4mk < 0$, our solutions are of the form

$$c_1 e^{-\frac{\gamma}{2m}t}\cos(\mu t) + c_2 e^{-\frac{\gamma}{2m}t}\sin(\mu t). \tag{12.7}$$

When $\gamma^2 - 4mk = 0$, our solution is of the form

$$c_1 e^{-\frac{\gamma}{2m}t} + c_2 t e^{-\frac{\gamma}{2m}t}. \tag{12.8}$$

When $\gamma^2 - 4mk > 0$, observe that $\gamma - \sqrt{\gamma^2 - 4mk} > 0$, so we have a solution of the form

$$c_1 e^{-r_1 t} + c_2 e^{-r_2 t}, \qquad r_1, r_2 > 0. \tag{12.9}$$

We can also add a forcing term.

We have an identical calculation for an RLC circuit. The voltage drop across a resistor is $RI = R\frac{dQ}{dt}$, the voltage drop across a capacitor is $\frac{Q}{C}$, and the voltage drop across an inductor is $L\frac{dI}{dt} = L\frac{d^2Q}{dt^2}$. Then by Kirchoff's law,

$$L\frac{d^2Q}{dt^2} + R\frac{dQ}{dt} + \frac{1}{C}Q = E(t). \tag{12.10}$$

13. Vector spaces and linear transformations

Recall the notion of vectors in $\mathbb{R}^n$. If v is such a vector,

$$v = (v_1, ..., v_n). \tag{13.1}$$

We can add two vectors in $\mathbb{R}^n$.

$$v + w = (v_1 + w_1, ..., v_n + w_n), \tag{13.2}$$

or multiply a vector by a scalar,

$$av = (av_1, ..., av_n). \tag{13.3}$$

Remark 1. *We are interested in vectors on $\mathbb{R}^n$, but we could also take vectors on $\mathbb{C}^n$.*

We have laws for vector addition:

(1) Commutative law $u + v = v + u$,
(2) Associative law $(u + v) + w = u + (v + w)$,
(3) Zero vector, there exists $0 \in V$ such that $v + 0 = v$ for any $v \in V$.
(4) For any vector $v \in V$, there exists $-v \in V$ such that $v + (-v) = 0$.

We also have laws for multiplication by scalars.

(1) Associative law, $a(bv) = (ab)v$,
(2) Unit law. $1v = v$.

Finally, we have the distributive property.

(1) $a(u + v) = au + av$,
(2) $(a + b)u = au + bu$.

There are other vector spaces, other than $\mathbb{R}^n$. For example, a subset W of a vector space is a linear subspace provided $w_j \in W$ implies $a_1w_1 + a_2w_2 \in W$ for any $a_1, a_2 \in \mathbb{R}$.

Remark 2. *We can also generalize the notion of a vector space and consider, for example, the vector space of polynomials.*

If V and W are vector spaces, a map

$$T : V \to W, \tag{13.4}$$

is said to be a linear transformation provided

$$T(a_1v_1 + a_2v_2) = a_1Tv_1 + a_2Tv_2. \tag{13.5}$$

We say that $T \in \mathcal{L}(V, W)$.

The linear transformations also are a vector space. Indeed, linear transformations may be added,

$$T_1 + T_2 : V \to W, \qquad (T_1 + T_2)v = T_1v + T_2v, \tag{13.6}$$

or multiplied by a scalar,

$$aT : V \to W, \qquad (aT)v = a(Tv). \tag{13.7}$$

One important example of a linear transformation is a $n \times m$ matrix. Other examples include our differentiation and integration operators. Recall

$$L[\phi] = \phi''(t) + p(t)\phi'(t) + q(t)\phi(t). \tag{13.8}$$

We can also compose linear transformations using matrix multiplication. Suppose A and B are matrices, $A = (a_{ij})$, $B = (b_{ij})$, and let

$$AB = (d_{ij}), \qquad d_{ij} = \sum_{l=1}^{n} a_{il} b_{lj}. \tag{13.9}$$

14. Basis and dimension

For any linear transformation T there is the null space of T and the range of T,

$$\mathcal{N}(T) = \{v \in V : Tv = 0\}, \tag{14.1}$$

$$\mathcal{R}(T) = \{Tv : v \in V\}. \tag{14.2}$$

The null space is a subspace of V and the range is a subspace of W. If $\mathcal{N}(T) = \{0\}$, we say that T is an injection, or one-to-one. If $\mathcal{R}(T) = W$, we say that T is surjective or onto. If both are true, we say that T is an isomorphism. We also say that T is invertible.

Let $S = \{v_1, ..., v_k\}$ be a finite set in a vector space V. The span of S is the set of vectors in V that are of the form

$$c_1 v_1 + ... + c_k v_k, \qquad c_k \in \mathbb{R}. \tag{14.3}$$

This set, $Span(S)$ is a linear subspace of V.

Definition 2. *The set S is said to be linearly dependent if and only if there exist scalars $c_1, ..., c_k$, not all zero, such that* (14.3) $= 0$. *Otherwise, S is said to be linearly independent.*

Definition 3. *If $\{v_1, ..., v_k\}$ is linearly independent, we say that S is a basis of $span(S)$, and that k is the dimension of $span(S)$. In particular, if $span(S) = V$, $k = dim(V)$. Also, V has finite basis and is finite dimensional.*

It remains to show that any two bases of a finite dimensional vector space V must have the same number of elements, and thus $dim(V)$ is well-defined. Suppose V has a basis $S = \{v_1, ..., v_k\}$. Then define the linear transformation

$$A : \mathbb{R}^k \to V, \tag{14.4}$$

by

$$A(c_1 e_1 + ... + c_k e_k) = c_1 v_1 + ... + c_k v_k, \tag{14.5}$$

where $\{e_1, ..., e_k\}$ is the standard basis of $\mathbb{R}^k$.

Linear independence of S is equivalent to the injectivity of A. The statement that S spans V is equivalent to the surjectivity of A. The statement that S is a basis of V is equivalent to the statement that A is an isomorphism, with inverse specified by

$$A^{-1}(c_1 v_1 + ... + c_k v_k) = c_1 e_1 + ... + c_k e_k. \tag{14.6}$$

We can show that $dim(V)$ is well-defined.

Lemma 1. *If $v_1, ..., v_{k+1}$ are vectors in $\mathbb{R}^k$, then they are linearly dependent.*

Proof. This is clear for $k = 1$. Now we can suppose that the last component of some v_j is nonzero, since otherwise we are in $\mathbb{R}^{k-1}$. Reorder so that the last component of v_{k+1} is nonzero. We can assume it is equal to 1. Then take

$$w_j = v_j - v_{kj} v_{k+1}. \tag{14.7}$$

Then by induction, there exist $a_1, ..., a_k$, not all zero such that $a_1 w_1 + ... + a_k w_k = 0$. Therefore,

$$a_1 v_1 + ... + a_k v_k = (a_1 v_{k1} + ... + a_k v_{kk}) v_{k+1}, \tag{14.8}$$

which gives linear dependence. □

Proposition 1. *If V has a basis $\{v_1, ..., v_k\}$ with k elements and $\{w_1, ..., w_l\} \subset V$ is linearly independent, then $l \leq k$.*

Proof. Take the isomorphism $A : \mathbb{R}^k \to V$. Then, $\{A^{-1} w_1, ..., A^{-1} w_l\}$ is linearly independent in $\mathbb{R}^k$, so $l \leq k$. □

Corollary 1. *If V is finite dimensional, then any two bases of V have the same number of elements. If V is isomorphic to W, these two spaces have the same dimension.*

Proposition 2. *Suppose V and W are finite dimensional vector spaces, and*

$$A : V \to W, \tag{14.9}$$

is a linear map. Then,

$$dim \mathcal{N}(A) + dim \mathcal{R}(A) = dim(V). \tag{14.10}$$

Proof. Let $\{w_1, ..., w_l\}$ be a basis of $\mathcal{N}(A) \subset V$, and complete it to a basis of V,

$$\{w_1, ..., w_l, u_1, ..., u_m\}. \tag{14.11}$$

Let $L = span\{u_1, ..., u_m\}$ and let $A_0 = A|_L$. Then,

$$\mathcal{R}(A_0) = \mathcal{R}(A), \tag{14.12}$$

and

$$\mathcal{N}(A_0) = \mathcal{N}(A) \cap L = 0. \tag{14.13}$$

Therefore, $dim \mathcal{R}(A) = dim \mathcal{R}(A_0) = dim(L) = m$. □

Corollary 2. *Let V be finite dimensional and let $A : V \to V$ be linear. Then A is injective if and only if A is surjective if and only if A is an isomorphism.*

Proposition 3. *Let A be an $n \times n$ matrix defining $A : \mathbb{R}^n \to \mathbb{R}^n$. Then the following are equivalent. A is invertible, the columns of A are linearly independent, the columns of A span $\mathbb{R}^n$.*

15. Eigenvalues and eigenvectors

Let $T : V \to V$ be linear. If there exists a nonzero $v \in V$ such that

$$T v = \lambda_j v, \tag{15.1}$$

for some $\lambda \in \mathbb{F}$, then λ_j is an eigenvalue of T and v is an eigenvector.

Let $\mathcal{E}(T, \lambda_j)$ denote the set of vectors $v \in V$ such that (15.1) holds. Then $\mathcal{E}(T, \lambda_j)$ is a vector subspace of V and

$$T : \mathcal{E}(T, \lambda_j) \to \mathcal{E}(T, \lambda_j). \tag{15.2}$$

Definition 4. *The set of $\lambda_j \in \mathbb{F}$ such that $\mathcal{E}(T, \lambda_j) \neq 0$ is denoted $Spec(T)$.*

If V is finite dimensional, then $\lambda_j \in Spec(T)$ if and only if

$$det(\lambda_j I - T) = 0. \tag{15.3}$$

Then, $K_T(\lambda) = det(\lambda I - T)$ is called the characteristic polynomial of T.

Proposition 4. *If V is a finite dimensional vector space and $T \in \mathcal{L}(V)$, then T has at least one eigenvector in V.*

Proof. Fundamental theorem of algebra. □

A linear transformation might have only one eigenvector, up to scalar multiple. Consider

$$\begin{pmatrix} 2 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{pmatrix}. \tag{15.4}$$

In this case the characteristic polynomial is given by $(\lambda - 2)^3$. Now then, if

$$\begin{pmatrix} 2 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = 2 \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix}, \tag{15.5}$$

then $v_2 = v_3 = 0$.

Proposition 5. *Suppose that the characteristic polynomial of $T \in \mathcal{L}(V)$ has k distinct roots $\lambda_1, ..., \lambda_k$ with eigenvectors $v_j \in \mathcal{E}(T, \lambda_j)$, $1 \leq j \leq k$. Then $\{v_1, ..., v_k\}$ is linearly independent. In particular, if $k = dim(V)$, these vectors form a basis of V.*

Proof. Suppose $\{v_1, ..., v_k\}$ is a linearly dependent set. Then

$$c_1 v_1 + ... + c_k v_k = 0, \tag{15.6}$$

reordering so that $c_1 \neq 0$. Applying $T - \lambda_k I$ to (15.6) gives

$$c_1(\lambda_1 - \lambda_k) v_1 + ... + c_{k-1}(\lambda_{k-1} - \lambda_k) v_{k-1} = 0. \tag{15.7}$$

Thus, $\{v_1, ..., v_{k-1}\}$ is linearly dependent. Arguing by induction, we obtain a contradiction. □

Observe that in the case that we have k linearly independent eigenvectors, the eigenvectors $\{v_1, ..., v_k\}$ form a natural basis of $\mathbb{R}^k$. Indeed, for any vector $v \in \mathbb{R}^k$,

$$T(c_1 v_1 + ... + c_k v_k) = c_1 \lambda_1 v_1 + ... + c_k \lambda_k v_k. \tag{15.8}$$

16. The matrix exponential

Define the matrix exponential

$$e^A = \sum_{k=1}^{\infty} \frac{1}{k!} A^k. \tag{16.1}$$

We can define the norm of a matrix,

$$\|T\| = \sup\{|Tv| : |v| \leq 1\}. \tag{16.2}$$

Then, we can compute $\|A^k\| \leq \|A\|^k$, so the matrix exponential (16.1) converges. Therefore, by the ratio test, (16.1) converges. Similarly, we can define,

$$e^{tA} = \sum_{k=0}^{\infty} \frac{t^k}{k!} A^k, \tag{16.3}$$

which converges for any $t \in \mathbb{C}$.

Differentiating term by term,

$$\frac{d}{dt} e^{tA} = \sum_{k=1}^{\infty} k \frac{t^{k-1}}{k!} A^k = e^{tA} A = A e^{tA}. \tag{16.4}$$

Therefore, $v(t) = e^{tA} v_0$ solves the first order system

$$\frac{dv}{dt} = Av, \qquad v(0) = v_0. \tag{16.5}$$

This solution is unique. Indeed, let $u(t) = e^{-tA} v(t)$. Then $u(0) = v(0) = v_0$ and

$$\frac{d}{dt} u(t) = e^{-tA} A v(t) + e^{-tA} v'(t) = 0, \tag{16.6}$$

so $u(t) \equiv u(0) = v_0$. The same argument implies

$$\frac{d}{dt}(e^{tA} e^{-tA}) = 0, \qquad \text{hence} \qquad e^{tA} e^{-tA} = I, \tag{16.7}$$

so $v(t) = e^{tA} v_0$.

Proposition 6. *Given $A \in M(n, \mathbb{C})$, $s, t \in \mathbb{R}$,*

$$e^{(s+t)A} = e^{sA} e^{tA}. \tag{16.8}$$

Proof. Using the product rule,

$$\frac{d}{dt}(e^{(s+t)A} e^{-tA}) = e^{(s+t)A} A e^{-tA} - e^{(s+t)A} A e^{-tA} = 0. \tag{16.9}$$

Therefore, $e^{(s+t)A} e^{-tA}$ is independent of t, so (16.9) $= e^{sA}$. If we take $s = 0$, $e^{tA} e^{-tA} = I$, so multiplying the left and right hand sides by e^{tA} gives (16.8). □

On the other hand, in general, it is not true that $e^{A+B} = e^A e^B$. However, it is true if $AB = BA$.

Proposition 7. *Given $A, B \in M(n, \mathbb{C})$,*

$$e^{A+B} = e^A e^B. \tag{16.10}$$

Proof.
(16.11)
$$\frac{d}{dt}(e^{t(A+B)} e^{-tB} e^{-tA}) = e^{t(A+B)}(A+B) e^{-tB} e^{-tA} - e^{t(A+B)} B e^{-tB} e^{-tA} - e^{t(A+B)} e^{-tB} A e^{-tA}.$$

Since $AB^k = B^k A$ for any k, (16.11) $= 0$, which gives (16.10). □

Let's do some computations.

$$A = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}, \qquad B = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}. \tag{16.12}$$

Then,

$$e^{tA} = \begin{pmatrix} e^t & 0 \\ 0 & e^{2t} \end{pmatrix}, \qquad e^{tB} = \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}. \tag{16.13}$$

If we take

$$C = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \qquad e^{tC} = e^{tI}e^{tB} = \begin{pmatrix} e^t & te^t \\ 0 & e^t \end{pmatrix}. \tag{16.14}$$

Now suppose we have a basis of eigenvectors. Since $Av_j = \lambda_j v_j$,

$$e^{tA}v_j = \sum_{k=0}^{\infty} \frac{t^k}{k!} A^k v_j = e^{t\lambda_j} v_j. \tag{16.15}$$

For example, take

$$A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \lambda_1 = 1, \qquad \lambda_2 = -1, \qquad v_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \qquad v_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}. \tag{16.16}$$

Then, $e^{tA}v_1 = e^t v_1$ and $e^{tA}v_2 = e^{-t}v_2$. Now then,

$$e^{tA}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \frac{1}{2}e^t \begin{pmatrix} 1 \\ 1 \end{pmatrix} + \frac{1}{2}e^{-t}\begin{pmatrix} 1 \\ -1 \end{pmatrix}, \qquad e^{tA}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \frac{1}{2}e^t \begin{pmatrix} 1 \\ 1 \end{pmatrix} - \frac{1}{2}e^{-t}\begin{pmatrix} 1 \\ -1 \end{pmatrix}. \tag{16.17}$$

Therefore,

$$e^{tA} = \begin{pmatrix} \cosh(t) & \sinh t \\ \sinh t & \cosh t \end{pmatrix}. \tag{16.18}$$

Next, consider the matrix

$$A = \begin{pmatrix} 0 & -2 \\ 1 & 2 \end{pmatrix}. \tag{16.19}$$

The characteristic polynomial of A is

$$det(A - \lambda I) = \lambda^2 - 2\lambda + 2 = 0. \tag{16.20}$$

The eigenvalues of (16.20) are $\lambda_1 = 1 + i$ and $\lambda_2 = 1 - i$, with corresponding eigenvectors

$$v_1 = \begin{pmatrix} -2 \\ 1+i \end{pmatrix}, \qquad v_2 = \begin{pmatrix} -2 \\ 1-i \end{pmatrix}. \tag{16.21}$$

Then,

$$e^{tA}v_1 = e^{(1+i)t}v_1, \qquad e^{tA}v_2 = e^{(1-i)t}v_2. \tag{16.22}$$

Doing some algebra,

$$\begin{aligned} \begin{pmatrix} 1 \\ 0 \end{pmatrix} &= -\frac{i+1}{4}\begin{pmatrix} -2 \\ 1+i \end{pmatrix} + \frac{i-1}{4}\begin{pmatrix} -2 \\ 1-i \end{pmatrix}, \\ \begin{pmatrix} 0 \\ 1 \end{pmatrix} &= -\frac{i}{2}\begin{pmatrix} -2 \\ 1+i \end{pmatrix} + \frac{i}{2}\begin{pmatrix} -2 \\ 1-i \end{pmatrix}. \end{aligned} \tag{16.23}$$

Now then,
(16.24)

$$-\frac{i+1}{4}\begin{pmatrix}-2\\1+i\end{pmatrix}e^{(1+i)t}+\frac{i-1}{4}\begin{pmatrix}-2\\1-i\end{pmatrix}e^{(1-i)t}=\frac{e^t}{4}\begin{pmatrix}(2i+2)e^{it}+(2-2i)e^{-it}\\-2ie^{it}+2ie^{-it}\end{pmatrix}=e^t\begin{pmatrix}\cos t-\sin t\\\sin t\end{pmatrix},$$

and

$$-\frac{i}{2}e^{(1+i)t}\begin{pmatrix}-2\\1+i\end{pmatrix}+\frac{i}{2}e^{(1-i)t}\begin{pmatrix}-2\\1-i\end{pmatrix}=\frac{e^t}{2}\begin{pmatrix}2ie^{it}-2ie^{-it}\\(1-i)e^{it}+(1+i)e^{-it}\end{pmatrix}=e^t\begin{pmatrix}-2\sin t\\\cos t+\sin t\end{pmatrix}. \tag{16.25}$$

Therefore,

$$e^{tA}=e^t\begin{pmatrix}\cos t-\sin t & -2\sin t\\\sin t & \cos t+\sin t\end{pmatrix}. \tag{16.26}$$

17. Generalized eigenvectors and the minimal polynomial

Recall that the matrix

$$A=\begin{pmatrix}2&1&0\\0&2&1\\0&0&2\end{pmatrix}, \tag{17.1}$$

has just one eigenvalue 2 and one eigenvector e_1. However,

$$(A-2I)^2e_2=0, \qquad (A-2I)^3e_3=0. \tag{17.2}$$

Definition 5. *For $T\in\mathcal{L}(V)$, we say a nonzero $v\in V$ is a generalized λ_j eigenvector if there exists $k\in\mathbb{N}$ such that $(T-\lambda_jI)^kv=0$.*

Consider for example the matrix

$$\begin{pmatrix}0&1\\-4&-4\end{pmatrix}. \tag{17.3}$$

This matrix has one eigenvalue, -2. Now then,

$$A=-2I+T, \qquad T=\begin{pmatrix}2&1\\-4&-2\end{pmatrix}. \tag{17.4}$$

In this case, $T\begin{pmatrix}1\\-2\end{pmatrix}=0$, $T\begin{pmatrix}2\\1\end{pmatrix}=5\begin{pmatrix}1\\-2\end{pmatrix}$. Therefore, $T^2=0$. Then,

$$e^{tA}=\exp(-2It+tT)=e^{-2t}\begin{pmatrix}1+2t & t\\-4t & 1-2t\end{pmatrix}. \tag{17.5}$$

Let $\mathcal{GE}(T,\lambda_j)$ be the set of vectors $v\in V$ such that $(T-\lambda_jI)^kv=0$ for some $k\in\mathbb{N}$. Then, $\mathcal{GE}(T,\lambda_j)$ is a linear subspace of V and

$$T:\mathcal{GE}(T,\lambda_j)\to\mathcal{GE}(T,\lambda_j). \tag{17.6}$$

Lemma 2. *For each $\lambda_j\in\mathbb{C}$ such that $\mathcal{GE}(T,\lambda_j)\neq 0$,*

$$T-\mu I:\mathcal{GE}(T,\lambda_j)\to\mathcal{E}(T,\lambda_j), \tag{17.7}$$

is an isomorphism for all $\mu\neq\lambda_j$.

Proof. If $T-\mu I$ is not an isomorphism then $Tv=\mu v$ for some $v\in\mathcal{GE}(T,\lambda_j)$. But then, $(T-\lambda_jI)^k=(\mu-\lambda_j)^kv$ for any $k\in\mathbb{N}$, which cannot be true unless $\mu=\lambda_j$. ☐

Lemma 3. *If V is finite dimensional and $T \in \mathcal{L}(V)$, then there exists a nonzero polynomial p such that $p(T) = 0$.*

Proof. If $dim(V) = n^2$ then $\{I, T, ..., T^{n^2}\}$ is linearly dependent. □

Now let

$$\mathcal{I}_T = \{p : p(T) = 0\}. \tag{17.8}$$

Certainly we can add such polynomials together and get new polynomials that satisfy (17.8) or multiply them.

Lemma 4. *Let p_1 be the polynomial with minimal degree among the nonzero polynomials in an ideal $\mathcal{I}$. Then any polynomial in $\mathcal{I}$ is of the form $p_1(\lambda)q(\lambda)$ for some polynomial q.*

Proof. Indeed, we can divide polynomials, so

$$p(\lambda) = p_1(\lambda)q(\lambda) + r(\lambda), \tag{17.9}$$

where $r(\lambda)$ has degree less than the degree of p_1. Since the degree of p_1 is minimal, $r(\lambda) = 0$. □

The minimal polynomial of T is of the form

$$m_T(\lambda) = \prod_{j=1}^{K} (\lambda - \lambda_j)^{k_j}. \tag{17.10}$$

Then let

$$p_l(\lambda) = \prod_{j \neq l} (\lambda - \lambda_j)^{k_j}. \tag{17.11}$$

Proposition 8. *If V is an n-dimensional complex vector space and $T \in \mathcal{L}(V)$, then for each $l \in \{1, ..., K\}$,*

$$\mathcal{GE}(T, \lambda_l) = \mathcal{R}(p_l(T)). \tag{17.12}$$

Proof. For any $v \in V$,

$$(T - \lambda_l)^{k_l} p_l(T) = 0, \tag{17.13}$$

so $p_l(T) : V \to \mathcal{GE}(T, \lambda_l)$. Also, each factor

$$(T - \lambda_j)^{k_j} : \mathcal{GE}(T, \lambda_l) \to \mathcal{GE}(T, \lambda_l), \tag{17.14}$$

for any $j \neq l$, is an isomorphism, so $p_l(T) : \mathcal{GE}(T, \lambda_l) \to \mathcal{GE}(T, \lambda_l)$ is an isomorphism. □

Proposition 9. *If V is an n-dimensional complex vector space and $T \in \mathcal{L}(V)$, then*

$$V = \mathcal{GE}(T, \lambda_1) + ... + \mathcal{GE}(T, \lambda_K). \tag{17.15}$$

Proof. We claim that the ideal generated by $p_1, ..., p_K$ is equal to all polynomials. Indeed, any ideal is generated by a minimal element, which must have a zero. But p_1, ..., p_K have no common zeros.

Therefore,

$$p_1(T)q_1(T) + ... + p_K(T)q_K(T) = I. \tag{17.16}$$

Therefore,

$$v = p_1(T)q_1(T)v + ... + p_K(T)q_K(T)v = v_1 + ... + v_K. \tag{17.17}$$

□

Proposition 10. *Let $\mathcal{GE}(T,\lambda_l)$ denote the generalized eigenspaces of T, and let $S_l = \{v_{l1},...,v_{l,d_l}\}$, with $d_l = dim\mathcal{GE}(T,\lambda_l)$ be a basis of $\mathcal{GE}(T,\lambda_l)$. Then,*

$$S = S_1 \cup ... \cup S_K \tag{17.18}$$

is a basis of V.

Proof. We know that S spans V. We need to show that S is linearly independent. Suppose w_l are nonzero elements of $\mathcal{GE}(T,\lambda_l)$. We can apply the same argument as in the case of distinct eigenvalues, only we replace $(T-\lambda I)$ with $(T-\lambda I)^k$. □

Definition 6. *We say that $T \in \mathcal{L}(V)$ is nilpotent provided $T^k = 0$ for some $k \in \mathbb{N}$.*

Proposition 11. *If T is nilpotent then there is a basis of V for which T is strictly upper triangular.*

Proof. Let $V_k = T^k(V)$, so $V = V_0 \supset V_1 \supset V_2 \supset ... \supset V_{k-1} \supset \{0\}$ with $V_{k-1} \neq 0$. Then, choose a basis for V_{k-1}, augment it to produce a basis for V_{k-2}, and so on. Then we have an upper triangular matrix. □

Now decompose $V = V_1 + ... + V_l$, where $V_l = \mathcal{GE}(T,\lambda_l)$. Then,

$$T_l : V_l \to V_l, \tag{17.19}$$

where $T_l = T|_{V_l}$. Then $Spec(T_l) = \{\lambda_l\}$, and we can take a basis of V_l for which T_l is strictly upper triangular. Now for any strictly upper triangular matrix T of dimension k, $T^k = 0$. Thus,

$$K_T(\lambda) = det(T-\lambda I) = \prod_{l=1}^{K}(\lambda-\lambda_l)^{d_l}, \qquad d_l = dim(V), \tag{17.20}$$

and $K_T(\lambda)$ is a polynomial multiple of $m_T(\lambda)$.

18. Systems of first order linear equations

A general system of n functions is given by

$$\begin{aligned} x_1'(t) &= p_{11}(t)x_1(t) + ... + p_{1n}(t)x_n(t) + g_1(t),\\ x_2'(t) &= p_{21}(t)x_1(t) + ... + p_{2n}(t)x_n(t) + g_2(t),\\ &\cdots\\ x_n'(t) &= p_{n1}(t)x_1(t) + ... + p_{nn}(t)x_n(t) + g_n(t). \end{aligned} \tag{18.1}$$

Theorem 9. *If the functions $p_{11}(t),...,p_{nn}(t)$ and $g_1(t)$, ..., $g_n(t)$ are continuous on an interval I, $\alpha < t < \beta$, then there exists a unique solution $x_1(t) = \phi_1(t)$, ..., $x_n(t) = \phi_n(t)$ of the equation (18.1) that also satisfies the initial conditions $x_1(0) = x_1^0$, ..., $x_n(0) = x_n^0$.*

Proof. Let $A(t)$ denote the matrix

$$A(t) = \begin{pmatrix} p_{11}(t) & p_{12}(t) & ... & p_{1n}(t)\\ p_{21}(t) & p_{22}(t) & ... & p_{2n}(t)\\ ... & ... & ... & ...\\ p_{n1}(t) & p_{n2}(t) & ... & p_{nn}(t) \end{pmatrix}. \tag{18.2}$$

Then,

$$\frac{d}{dt}\overrightarrow{x}(t) = A(t)\overrightarrow{x}(t) + \overrightarrow{g}(t). \tag{18.3}$$

Let $S(t, 0)$ be the solution operator to

$$\frac{d}{dt}\overrightarrow{x}(t) = A(t)\overrightarrow{x}(t). \tag{18.4}$$

That is, if $\overrightarrow{x}(t) = S(t, 0)\overrightarrow{x}(0)$, then $\overrightarrow{x}(t)$ solves (18.4) with initial data $\overrightarrow{x}(0)$. Then,

$$\overrightarrow{x}(t) = S(t, 0)\overrightarrow{x}(0) + \int_0^t S(t, s)\overrightarrow{g}(s)ds. \tag{18.5}$$

□

For example, let us consider the equation

$$\frac{d}{dt}\overrightarrow{x}(t) = \begin{pmatrix} 1 & 1 \\ 4 & 1 \end{pmatrix} \overrightarrow{x}(t). \tag{18.6}$$

Then the solution has the form

$$\overrightarrow{x}(t) = c_1 \begin{pmatrix} 1 \\ 2 \end{pmatrix} e^{3t} + c_2 \begin{pmatrix} 1 \\ -2 \end{pmatrix} e^{-t}. \tag{18.7}$$

Theorem 10. *If the vector functions $\overrightarrow{x}^{(1)}(t)$, ..., $\overrightarrow{x}^{(n)}(t)$ are linearly independent solutions of the system* (18.1) *for each point in the interval $\alpha < t < \beta$, then each solution $\overrightarrow{x}(t)$ can be expressed as a linear combination of $\overrightarrow{x}^{(1)}(t)$, ..., $\overrightarrow{x}^{(n)}(t)$ in exactly one way.*

Theorem 11 (Abel's theorem)**.** *If $x^{(1)}(t)$, ..., $x^{(n)}(t)$ are solutions to* (18.1) *on the interval $\alpha < t < \beta$, then in this interval, $W[x^{(1)}(t), ..., x^{(n)}(t)]$ is either identically zero or never vanishes.*

Proof. Choose a basis such that $W(t)$ is an upper triangular matrix. Now then, in any basis, $Tr(A(t)) = p_{11}(t) + ... + p_{nn}(t)$. Then by direct computation,

$$\frac{dW}{dt} = (p_{11}(t) + ... + p_{nn}(t))W(t). \tag{18.8}$$

Another way to prove this is to remember that if we have one row that is the multiple of another, $W(t) = 0$. The same is true of two columns. This means that $det(A) \neq 0$ if and only if the rows and columns are linearly independent.

The only way to avoid that is if we have p_{11}, ..., p_{nn}. □

19. Nonhomogeneous linear systems

Now let us consider the nonhomogeneous linear system

$$\frac{d}{dt}\overrightarrow{x}(t) = P(t)\overrightarrow{x}(t) + \overrightarrow{g}(t). \tag{19.1}$$

Then recall (18.5),

$$\begin{aligned} \overrightarrow{x}(t) &= S(t, 0)\overrightarrow{x}(0) + \int_0^t S(t, s)\overrightarrow{g}(s)ds \\ &= S(t, 0)\overrightarrow{x}(0) + \int_0^t S(t, 0)S(0, s)\overrightarrow{g}(s)ds. \end{aligned} \tag{19.2}$$

Remark 3. *Note that $S(0, s) = S(s, 0)^{-1}$.*

Consider, for example, the system

$$\frac{d}{dt}\overrightarrow{x}(t) = \begin{pmatrix} -2 & 1 \\ 1 & -2 \end{pmatrix}\overrightarrow{x}(t) + \begin{pmatrix} 2e^{-t} \\ 3t \end{pmatrix}. \tag{19.3}$$

In this case, since $A(t)$ is constant,

$$\overrightarrow{x}(t) = e^{tA}\overrightarrow{x}(0) + \int_0^t e^{(t-s)A}\begin{pmatrix} 2e^{-s} \\ 3s \end{pmatrix} ds. \tag{19.4}$$

In this case, the eigenvalues are given by $\lambda = -1, -3$ with eigenvectors $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ and $\begin{pmatrix} 1 \\ -1 \end{pmatrix}$. Then,

$$e^{tA}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} \frac{1}{2}e^{-t} + \frac{1}{2}e^{-3t} \\ \frac{1}{2}e^{-t} - \frac{1}{2}e^{-3t} \end{pmatrix}, \tag{19.5}$$

and

$$e^{tA}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} \frac{1}{2}e^{-t} - \frac{1}{2}e^{-3t} \\ \frac{1}{2}e^{-t} + \frac{1}{2}e^{-3t} \end{pmatrix}. \tag{19.6}$$

Therefore,

$$e^{tA} = \begin{pmatrix} \frac{1}{2}e^{-t} + \frac{1}{2}e^{-3t} & \frac{1}{2}e^{-t} - \frac{1}{2}e^{-3t} \\ \frac{1}{2}e^{-t} - \frac{1}{2}e^{-3t} & \frac{1}{2}e^{-t} + \frac{1}{2}e^{-3t} \end{pmatrix}. \tag{19.7}$$

Then,

$$\overrightarrow{x}(t) = e^{tA}\overrightarrow{x}(0) + \int_0^t \begin{pmatrix} \frac{1}{2}e^{-(t-s)} + \frac{1}{2}e^{-3(t-s)} & \frac{1}{2}e^{-(t-s)} - \frac{1}{2}e^{-3(t-s)} \\ \frac{1}{2}e^{-(t-s)} - \frac{1}{2}e^{-3(t-s)} & \frac{1}{2}e^{-(t-s)} + \frac{1}{2}e^{-3(t-s)} \end{pmatrix}\begin{pmatrix} 2e^{-s} \\ 3s \end{pmatrix} ds. \tag{19.8}$$

We can convert an n-th order differential equation into a system of first order equations. Indeed, consider the n-th order differential equation

$$\frac{d^n y}{dt^n} + a_{n-1}\frac{d^{n-1}y}{dt^{n-1}} + ... + a_1\frac{dy}{dt} + a_0 y = 0. \tag{19.9}$$

Then $\overrightarrow{x}(t) = (x_0(t), ..., x_{n-1}(t))$ will satisfy

$$\begin{aligned} \frac{d}{dt}x_0(t) &= x_1(t), \\ &\cdots \\ \frac{d}{dt}x_{n-2}(t) &= x_{n-1}(t), \\ \frac{d}{dt}x_{n-1}(t) &= -a_{n-1}x_{n-1}(t) - ... - a_0 x_0(t). \end{aligned} \tag{19.10}$$

Equivalently,

$$\frac{d}{dt}\overrightarrow{x}(t) = A\overrightarrow{x}(t), \tag{19.11}$$

with

$$A = \begin{pmatrix} 0 & 1 & \cdots & 0 & 0 \\ 0 & 0 & \cdots & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & 0 & 1 \\ -a_1 & -a_2 & \cdots & -a_{n-1} & -a_n \end{pmatrix}. \tag{19.12}$$

Definition 7. *The matrix A given by* (19.12) *is called the companion matrix of the polynomial*

$$p(\lambda) = \lambda^n + a_{n-1}\lambda^{n-1} + ... + a_1\lambda + a_0. \tag{19.13}$$

Proposition 12. *If $p(\lambda)$ is a polynomial of the form* (19.13)*, with companion matrix A given by* (19.12)*, then*

$$p(\lambda) = det(\lambda I - A). \tag{19.14}$$

Proof. The determinant of a matrix is equal to the determinant of the transpose. Then,

$$det(\lambda I - A) = \lambda det \begin{pmatrix} \lambda & -1 & \cdots 0 & 0 & \\ 0 & \lambda & \cdots & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & \lambda & -1 \\ a_1 & a_2 & \cdots & a_{n-2} & a_{n-1} \end{pmatrix} + (-1)^{n-1} a_0 (-1)^{n-1}. \tag{19.15}$$

Therefore,

$$det(\lambda I - A) = \lambda(\lambda^{n-1} + a_{n-1}\lambda^{n-1} + ... + a_1) + a_0. \tag{19.16}$$

□

20. Variable coefficient systems

Consider a variable coefficient $n \times n$ first order system,

$$\frac{dx}{dt} = A(t)x, \qquad x(t_0) = x_0. \tag{20.1}$$

Then,

$$\vec{x}(t) = S(t, t_0)\vec{x}(t_0). \tag{20.2}$$

Now suppose that $\vec{x}_1(t_0)$, ..., $\vec{x}_n(t_0)$ are linearly independent. Then, by Abel's theorem, if $\vec{x}_j(t)$ is a solution to (20.1) with initial data $\vec{x}_j(t_0)$. Then let $M(t)$ denote the matrix,

$$M(t) = (x_1(t), ..., x_n(t)). \tag{20.3}$$

Then,

$$M(t) = S(t, t_0)(x_1(t_0), ..., x_n(t_0)). \tag{20.4}$$

Therefore,

$$S(t, t_0) = M(t)(x_1(t_0), ..., x_n(t_0))^{-1}, \tag{20.5}$$

and

$$S(t_0, t) = (x_1(t_0), ..., x_n(t_0))M(t)^{-1}, \tag{20.6}$$

and

$$S(t, t_0)S(t_0, s) = M(t)M(s)^{-1}. \tag{20.7}$$

Therefore, the solution to

$$\frac{dx}{dt} = A(t)x + g(t), \qquad x(t_0) = 0, \tag{20.8}$$

so the solution to (20.8) is given by

$$x(t) = \int_{t_0}^{t} M(t)M(s)^{-1}g(s)ds. \tag{20.9}$$

For the ordinary differential equation,

$$\frac{d^n y}{dt^n} + a_{n-1}\frac{d^{n-1}y}{dt^{n-1}} + ... + a_1\frac{dy}{dt} + a_0 y = g(t), \tag{20.10}$$

We can use the variation of parameters and Cramer's rule to compute
(20.11)

$$Y(t) = \int_0^t M(t)M(s)^{-1}\begin{pmatrix} 0 \\ 0 \\ \cdots \\ 0 \\ g(s) \end{pmatrix} ds = \sum_{i=1}^n y_i(t)\int_0^t (M(s)^{-1}\begin{pmatrix} 0 \\ 0 \\ \cdots \\ 0 \\ g(s) \end{pmatrix})_i ds = \sum_{i=1}^n y_i(t)\int_0^t \frac{1}{W(s)}(detM(s))_{ni} ds.$$

Lemma 5 (Cramer's rule)**.** *If A is a square matrix, then the inverse of A is given by the matrix M, where*

$$M_{ij} = \frac{1}{det(M)}det(M)_{ij}, \tag{20.12}$$

where $det(M)_{ij}$ is the determinant of the matrix with the j-th row replaced by the vector $(0, ..., 0, 1, 0, ..., 0)$, with 1 in the i-th column and 0 everywhere else.

Case 1: In this case, we assume that (19.13) has n distinct real roots. Then $y_1(t) = e^{r_1 t}$, ..., $y_n(t) = e^{r_n t}$ form a nonzero Wronskian.

Case 2: If r is a complex root to (19.13) and (19.13) has only real coefficients, then $\bar{r}$ is also a complex root. Thus, if (19.13) has n distinct complex roots, then

$$e^{r_1 t}, e^{r_2 t}, ..., e^{r_m t}, e^{r_{m+1} t}\sin(a_m t), e^{r_{m+1} t}\cos(a_m t), ..., e^{r_{m+j} t}\sin(a_{m+j} t), e^{r_{m+j} t}\cos(a_{m+j} t). \tag{20.13}$$

Case 3: If (19.13) has m repeated roots r, then we can choose a basis for A that is in Jordan canonical form. Then, e^{rt}, te^{rt}, t^2e^{rt}, ..., $t^{m-1}e^{rt}$ form m linearly independent solutions to (19.13). Indeed, if

$$N = \begin{pmatrix} 0 & 1 & 0 & 0 & ... & 0 \\ 0 & 0 & 1 & 0 & ... & 0 \\ 0 & 0 & 0 & 1 & ... & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & 0 & \cdots & 1 \\ 0 & 0 & 0 & 0 & ... & 0 \end{pmatrix}, \tag{20.14}$$

then

$$e^{tN} = I + tN + \frac{1}{2}t^2N^2 + ... + \frac{1}{(m-1)!}t^{m-1}N^{m-1}. \tag{20.15}$$

21. Laplace Transform

The computations in the previous section can be quite cumbersome, depending on $g(t)$. In many cases, the Laplace transform is often useful.

Definition 8 (Laplace transform)**.** *The Laplace transform of a function $f(t)$ is given by*

$$\mathcal{L}\{f(t)\} = F(s) = \int_0^\infty e^{-st}f(t)dt. \tag{21.1}$$

Theorem 12. *Suppose that f is piecewise continuous on the interval $0 \leq t \leq A$ for any positive $A > 0$. Also suppose that there exist constants $K > 0$, a, and $M > 0$, such that*

$$|f(t)| \leq Ke^{at}, \qquad \text{when} \qquad t \geq M. \tag{21.2}$$

Then the Laplace transform $\mathcal{L}\{f(t)\} = F(s)$ exists for $s > a$.

Proof. We can compute

$$\int_M^\infty |f(t)| e^{-st} dt \leq \int_M^\infty Ke^{(a-s)t} \leq \frac{K}{s-a}. \tag{21.3}$$

□

We can compute Laplace transforms of some important functions.

$$\mathcal{L}\{e^{at}\} = \int_0^\infty e^{-st} e^{at} dt = \frac{1}{s-a}. \tag{21.4}$$

One of the important aspects of the Laplace transform is that we can take a Laplace transform of a function that is not continuous. Suppose $f(t) = 1$ for $0 \leq t < 1$, $f(t) = k$ for $t = 1$, and $f(t) = 0$ for $t > 1$. Then,

$$\int_0^\infty e^{-st} f(t) dt = \int_0^1 e^{-st} dt = -\frac{e^{-st}}{s}|_0^1 = \frac{1-e^{-s}}{s}, \qquad s > 0. \tag{21.5}$$

In general, $\mathcal{L}$ is a linear functional. Indeed,

$$\mathcal{L}\{c_1 f_1(t) + c_2 f_2(t)\} = c_1 \mathcal{L}\{f_1(t)\} + c_2 \mathcal{L}\{f_2(t)\}. \tag{21.6}$$

We can use this to compute the Laplace transform of $\sin(at)$.

$$\mathcal{L}\{\sin(at)\} = \frac{1}{2i}\mathcal{L}\{e^{iat}\} - \frac{1}{2i}\mathcal{L}\{e^{-iat}\} = \frac{1}{2i}\frac{1}{s-ia} - \frac{1}{2i}\frac{1}{s+ia} = \frac{2ia}{2i(s^2+a^2)} = \frac{a}{s^2+a^2}. \tag{21.7}$$

$$\mathcal{L}\{\cos(at)\} = \frac{1}{2}\mathcal{L}\{e^{iat}\} + \frac{1}{2}\mathcal{L}\{e^{-iat}\} = \frac{1}{2}\frac{1}{s-ia} + \frac{1}{2}\frac{1}{s+ia} = \frac{s}{s^2+a^2}. \tag{21.8}$$

Next,

$$\int_0^\infty t^n e^{-st} dt = (-1)^n \frac{d^n}{d^n s} \int_0^\infty e^{-st} dt = (-1)^n \frac{d^n}{d^n s}(\frac{1}{s}) = \frac{n!}{s^{n+1}}. \tag{21.9}$$

More generally,

$$\int_0^\infty t^n e^{at} e^{-st} dt = \frac{n!}{(s-a)^{n+1}}. \tag{21.10}$$

Indeed,

Theorem 13. *If $F(s) = \mathcal{L}\{f(t)\}$ exists for $s > a \geq 0$ and if c is a constant,*

$$\mathcal{L}\{e^{ct} f(t)\} = F(s-c), \qquad s > a + c. \tag{21.11}$$

Conversely, if $f(t) = \mathcal{L}^{-1}\{F(s)\}$, then

$$e^{ct} f(t) = \mathcal{L}^{-1}\{F(s-c)\}. \tag{21.12}$$

Proof.

$$\mathcal{L}\{e^{ct}f(t)\} = \int_0^\infty e^{-st}e^{ct}f(t)dt = \int_0^\infty e^{-(s-c)t}f(t)dt = F(s-c). \tag{21.13}$$

□

Now we examine the Laplace transform of a derivative.

Theorem 14. *Suppose f is continuous and f' is piecewise continuous on any interval $0 \le t \le A$. Also suppose that there exist constants K, a, M such that $|f(t)| \le Ke^{at}$ for $t \ge M$. Then $\mathcal{L}\{f'(t)\}$ exists for $s > a$, and*

$$\mathcal{L}\{f'(t)\} = s\mathcal{L}\{f(t)\} - f(0). \tag{21.14}$$

Proof. Integrating by parts,

$$\int_0^A e^{-st}f'(t)dt = e^{-st}f(t)|_0^A + s\int_0^A e^{-st}f(t)dt. \tag{21.15}$$

Taking the limit as $A \to \infty$,

$$\mathcal{L}\{f'(t)\} = -f(0) + s\mathcal{L}\{f(t)\}. \tag{21.16}$$

□

Corollary 3. *Suppose that f, f', ..., $f^{(n-1)}$ are continuous and that $f^{(n)}$ is piecewise continuous on an interval $0 \le t \le A$. Also suppose that there exists constants K, a, and M such that $|f(t)| \le Ke^{at}$, and all the derivatives of f are bounded by Ke^{at} for $t \ge M$. Then,*

$$\mathcal{L}\{f^{(n)}(t)\} = s^n\mathcal{L}\{f(t)\} - s^{n-1}f(0) - ... - sf^{(n-2)}(0) - f^{(n-1)}(0). \tag{21.17}$$

Now, solve the differential equation,

$$y'' - y' - 2y = 0, \qquad y(0) = 1, \qquad y'(0) = 0. \tag{21.18}$$

Doing the Laplace transform,

$$(s^2 - s - 2)\mathcal{L}\{y(t)\} - (s-1)y(0) = 0. \tag{21.19}$$

Therefore, doing partial fractions,

$$\mathcal{L}\{y(t)\} = \frac{s-1}{(s-2)(s+1)} = \frac{1}{3(s-2)} + \frac{2}{3(s+1)}. \tag{21.20}$$

Therefore,

$$y(t) = \frac{1}{3}e^{2t} + \frac{2}{3}e^{-t}. \tag{21.21}$$

22. Initial value problems

Now consider the initial value problem

$$\frac{d^ny}{dt^n} + a_{n-1}\frac{d^{n-1}y}{dt^{n-1}} + ... + a_0y(t) = 0, \qquad y(0) = c_0, ..., y^{(n-1)}(0) = c_{n-1}. \tag{22.1}$$

Taking the Laplace transform of both sides,

(22.2)

$$\mathcal{L}\{y(t)\}\cdot\{s^n + a_{n-1}s^{n-1} + ... + a_0\} = s^{n-1}y(0) + ... + y^{(n-1)}(0) + a_{n-1}\{s^{n-2}y(0) + ... + y^{(n-2)}(0)\} + ... + a_1y(0).$$

Therefore, doing some algebra,

$$\mathcal{L}\{y(t)\} = \frac{b_{n-1}s^{n-1} + ... + b_0}{s^n + a_{n-1}s^{n-1} + ... + a_0}. \tag{22.3}$$

By the fundamental theorem of algebra,

$$s^n + a_{n-1}s^{n-1} + ... + a_0 = (s - m_1) \cdots (s - m_n) = \prod_{n=k_1+...+k_l} (s - m_j)^{k_j}. \tag{22.4}$$

Then by partial fractions,

$$\mathcal{L}\{y(t)\} = \sum_{k_1+...+k_l=n} \frac{p_j(s)}{(s - m_j)^{k_j}} = \sum_{n=k_1+...+k_l} \sum_{1 \le i \le k_j} \frac{a_{ij}}{(s - m_j)^i}. \tag{22.5}$$

If (22.2) is real valued then $a_{ij} = \bar{a}_{ij'}$ when $m_j = \bar{m}_{j'}$. Then, doing the inverse Laplace transform,

$$\mathcal{L}^{-1}(\frac{a_{ij}}{(s - m_j)^i}) = \frac{a_{ij}}{(i-1)!} t^{i-1} e^{tm_j}. \tag{22.6}$$

23. Convolution

We define the convolution,

$$h(t) = (f * g)(t) = \int_0^t f(t - \tau)g(\tau)d\tau = \int_0^t f(\tau)g(t - \tau)d\tau = (g * f)(t). \tag{23.1}$$

Theorem 15. *If $F(s) = \mathcal{L}\{f(t)\}$ and $G(s) = \mathcal{L}\{g(t)\}$ both exist for $s > a \ge 0$, then*

$$H(s) = F(s)G(s) = \mathcal{L}\{h(t)\}, \qquad s > a, \tag{23.2}$$

where

$$h(t) = (f * g)(t) = (g * f)(t). \tag{23.3}$$

Proof. By direct computation,

$$F(s)G(s) = \int_0^\infty e^{-s\tau} f(\tau) \cdot \int_0^\infty e^{-s\xi} g(\xi)d\xi = \int_0^\infty f(\tau) \int_0^\infty e^{-s(\tau+\xi)} g(\xi)d\xi d\tau. \tag{23.4}$$

Setting $t = \tau + \xi$, $\xi = t - \tau$, so by a change of variables, since $\tau \le t$,

$$F(s)G(s) = \int_0^\infty e^{-ts} \int_0^t f(\tau)g(t - \tau)d\tau dt = H(s). \tag{23.5}$$

□

We can use this computation to compute the inverse Laplace transform. Indeed, let

$$H(s) = \frac{a}{s^2(s + a)} = \frac{1}{s^2} \cdot \frac{a}{s^2 + a^2}. \tag{23.6}$$

We know that

$$\mathcal{L}^{-1}\{\frac{1}{s^2}\} = t, \qquad \mathcal{L}^{-1}\{\frac{a}{s^2 + a^2}\} = \sin(at). \tag{23.7}$$

Therefore,

$$h(t) = \int_0^t (t - \tau)\sin(a\tau)d\tau = -\frac{t}{a}\cos(at) + \frac{t}{a} + \frac{\tau}{a}\cos(a\tau)|_0^t - \frac{1}{a}\int_0^t \cos(a\tau)d\tau = \frac{t}{a} - \frac{\sin(at)}{a^2}. \tag{23.8}$$

Now find the solution of the initial value problem

$$y'' + 4y = g(t), \qquad y(0) = 3, \qquad y'(0) = -1. \tag{23.9}$$

Taking the Laplace transform of both sides,

$$(s^2 + 4)Y(s) - 3s + 1 = G(s). \tag{23.10}$$

Doing some algebra,

$$Y(s) = \frac{3s - 1}{s^2 + 4} + \frac{G(s)}{s^2 + 4}. \tag{23.11}$$

Decomposing $Y(s)$,

$$Y(s) = 3\frac{s}{s^2 + 4} - \frac{1}{2}\frac{2}{s^2 + 4} + \frac{1}{2}\frac{2}{s^2 + 4}G(s). \tag{23.12}$$

Therefore,

$$y(t) = 3\cos(2t) - \frac{1}{2}\sin(2t) + \frac{1}{2}\int_0^t \sin(2(t - \tau))g(\tau)d\tau. \tag{23.13}$$

In general, suppose we have the initial value problem with the forcing function,

$$y^{(n)}(t) + a_{n-1}y^{(n-1)}(t) + ... + a_0 y(t) = g(t), \qquad y(0) = c_0, \qquad y^{(n-1)}(0) = c_{n-1}. \tag{23.14}$$

Then if we let

$$H(s) = \frac{1}{s^n + a_{n-1}s^{n-1} + ... + a_0}, \tag{23.15}$$

$$Y(s) = (b_{n-1}s^{n-1} + ... + b_0)H(s) + G(s)H(s). \tag{23.16}$$

Then if we let $h(t) = \mathcal{L}^{-1}\{H(s)\}$, then

$$\mathcal{L}^{-1}\{G(s)H(s)\} = \int_0^t h(t - \tau)g(\tau)d\tau. \tag{23.17}$$

Definition 9. *The function H is called the transfer function.*

We can use this formula to obtain the variation of parameters formula. Suppose we have a second order equation,

$$y'' + a_1 y' + a_0 y = g(t), \qquad y(0) = c_0, \qquad y'(0) = c_1. \tag{23.18}$$

Taking the Laplace transform of both sides,

$$(s^2 + a_1 s + a_0)Y(s) = b_1 s + b_0 + G(s). \tag{23.19}$$

Therefore,

$$y(t) = \mathcal{L}^{-1}\{\frac{b_1 s + b_0}{s^2 + a_1 s + a_0}\} + \mathcal{L}^{-1}\{\frac{G(s)}{s^2 + a_1 s + a_0}\}. \tag{23.20}$$

Case 1, no real roots: In this case, $s^2 + a_0 s + a_1 = (s - a)^2 + b^2$ for some real a and b. Then,

$$\mathcal{L}^{-1}\{\frac{1}{(s - a)^2 + b^2}\} = \frac{e^{at}}{b}\sin(bt). \tag{23.21}$$

Therefore, if $c_0 = c_1 = 0$,

$$y(t) = \int_0^t e^{a(t-\tau)}\sin(b(t - \tau))g(\tau)d\tau. \tag{23.22}$$

Meanwhile, doing the variation of parameters calculation,

$$y(t) = -e^{at}\cos(bt)\int_0^t \frac{e^{a\tau}\sin(b\tau)g(\tau)}{be^{2a\tau}}d\tau + e^{at}\sin(bt)\int_0^t \frac{e^{a\tau}\cos(b\tau)g(\tau)}{be^{2a\tau}}d\tau. \tag{23.23}$$

Case 2, one real root: In this case, $s^2 + a_0 s + a_1 = (s-a)^2$. In this case,

$$\mathcal{L}^{-1}\{\frac{1}{(s-a)^2}\} = te^{at}, \tag{23.24}$$

so if $c_0 = c_1 = 0$,

$$y(t) = \int_0^t (t-\tau)e^{a(t-\tau)}g(\tau)d\tau. \tag{23.25}$$

Doing the variation of parameters calculation,

$$y(t) = -e^{at}\int_0^t \frac{e^{a\tau}\tau g(\tau)}{e^{2a\tau}}d\tau + te^{at}\int_0^t \frac{e^{a\tau}g(\tau)}{e^{2a\tau}}d\tau. \tag{23.26}$$

Case 3, two real roots: In this case, $s^2 + a_0 s + a_1 = (s-r_1)(s-r_2)$. Doing the variation of parameters formula,

(23.27)

$$y(t) = -e^{r_1 t}\int_0^t \frac{e^{r_2\tau}}{(r_2-r_1)e^{(r_1+r_2)\tau}}g(\tau)d\tau + e^{r_2 t}\int_0^t \frac{g(\tau)e^{r_1\tau}}{(r_2-r_1)e^{(r_1+r_2)\tau}}d\tau = \int_0^t \frac{g(\tau)}{r_2-r_1}(e^{r_2(t-\tau)} - e^{r_1(t-\tau)})d\tau.$$

Meanwhile, doing partial fractions,

$$\mathcal{L}^{-1}\{\frac{1}{(s-r_1)(s-r_2)}\} = \frac{1}{r_2-r_1}e^{r_2 t} - \frac{1}{r_2-r_1}e^{r_1 t}. \tag{23.28}$$

We can use the Laplace transform and convolution to solve a system of equations,

$$\frac{d}{dt}\vec{x}(t) = \begin{pmatrix} -2 & 1 \\ 1 & -2 \end{pmatrix}\vec{x}(t) + \begin{pmatrix} 2e^{-t} \\ 3t \end{pmatrix}. \tag{23.29}$$

Taking the Laplace transform of both sides,

$$sX(s) - \vec{x}(0) = AX(s) + G(s), \tag{23.30}$$

where

$$G(s) = \begin{pmatrix} \frac{2}{s+1} \\ \frac{3}{s^2} \end{pmatrix}. \tag{23.31}$$

If $\vec{x}(0) = 0$, doing some algebra,

$$(sI - A)X(s) = G(s). \tag{23.32}$$

Doing some algebra,

$$X(s) = (sI-A)^{-1}G(s), \tag{23.33}$$

where

$$sI - A = \begin{pmatrix} s+2 & -1 \\ -1 & s+2 \end{pmatrix}, \qquad (sI-A)^{-1} = \frac{1}{(s+1)(s+3)}\begin{pmatrix} s+2 & 1 \\ 1 & s+2 \end{pmatrix}. \tag{23.34}$$

Therefore,

$$X(s) = \begin{pmatrix} \frac{2(s+2)}{(s+1)^2(s+3)} + \frac{3}{s^2(s+1)(s+3)} \\ \frac{2}{(s+1)^2(s+3)} + \frac{3(s+2)}{s^2(s+1)(s+3)} \end{pmatrix}. \tag{23.35}$$

Therefore,

$$\vec{x}(t) = \begin{pmatrix}2\\1\end{pmatrix} e^{-t} - \frac{2}{3}\begin{pmatrix}1\\-1\end{pmatrix} e^{-3t} + \begin{pmatrix}1\\1\end{pmatrix} te^{-t} + \begin{pmatrix}1\\2\end{pmatrix} t - \frac{1}{3}\begin{pmatrix}4\\5\end{pmatrix}. \tag{23.36}$$

24. Heaviside function

Now consider the problem where the right hand side of (22.1) is not equal to zero and need not be continuous.

Definition 10 (Heaviside function)**.** *Let $u_c(t) = 0$ for $t < c$ and $u_c(t) = 1$ for $t \geq c$.*

$$\mathcal{L}\{u_c(t)\} = \int_0^\infty e^{-st} u_c(t) dt = \int_c^\infty e^{-st} = \frac{e^{-cs}}{s}. \tag{24.1}$$

The Laplace transform intertwines multiplication by an exponential and translation.

Theorem 16. *If the Laplace transform of $f(t)$, $F(s) = \mathcal{L}\{f(t)\}$ exists for $s > a \geq 0$, and if c is a positive constant, then*

$$\mathcal{L}\{u_c(t) f(t-c)\} = e^{-cs}\mathcal{L}\{f(t)\} = e^{-cs}F(s), \qquad s > a. \tag{24.2}$$

Conversely, if $f(t)$ is the inverse Laplace transform of $F(s)$, $f(t) = \mathcal{L}^{-1}\{F(s)\}$, then

$$u_c(t) f(t-c) = \mathcal{L}^{-1}\{e^{-cs}F(s)\}. \tag{24.3}$$

Proof. By direct computation and a change of variables,

$$\mathcal{L}\{u_c(t) f(t-c)\} = \int_c^\infty e^{-st} f(t-c) dt = e^{-cs}F(s). \tag{24.4}$$

□

We can apply Theorem 16 to obtain (24.1). Also, if $f(t) = \sin(t) + u_{\frac{\pi}{4}}(t)\cos(t - \frac{\pi}{4})$, then

$$F(s) = \mathcal{L}\{\sin t\} + e^{-\pi s/4}\mathcal{L}\{\cos t\}. \tag{24.5}$$

On the other hand, if

$$F(s) = \frac{1 - e^{-2s}}{s^2}, \qquad f(t) = t - u_2(t)(t-2). \tag{24.6}$$

Consider the ordinary differential equation

$$2y'' + y + 2y = g(t), \qquad g(t) = u_5(t) - u_{20}(t), \qquad y(0) = y'(0) = 0. \tag{24.7}$$

Taking the Laplace transform of both sides, let $Y(s)$ be the Laplace transform of $y(t)$.

$$(2s^2 + s + 2)Y(s) = \frac{1}{s}(e^{-5s} - e^{-20s}). \tag{24.8}$$

Doing some algebra,

$$Y(s) = \frac{e^{-5s} - e^{-20s}}{s(2s^2 + s + 2)}. \tag{24.9}$$

Doing some partial fractions,

$$\frac{1}{s(2s^2 + s + 2)} = \frac{a}{s} + \frac{bs + c}{2s^2 + s + 2}, \qquad a = \frac{1}{2}, \qquad b = -1, \qquad c = -\frac{1}{2}. \tag{24.10}$$

Then if

$$H(s) = \frac{1}{s(2s^2 + s + 2)}, \qquad h(t) = \frac{1}{2} - \mathcal{L}^{-1}\{\frac{(s + \frac{1}{4}) + \frac{1}{4}}{2((s + \frac{1}{4})^2 + \frac{15}{16})}\}. \tag{24.11}$$

Next,

$$-\mathcal{L}^{-1}\{\frac{(s + \frac{1}{4})}{2((s + \frac{1}{4})^2 + \frac{15}{16})}\} = -\frac{1}{2}e^{-t/4}\cos(\frac{\sqrt{15}}{4}t). \tag{24.12}$$

$$-\mathcal{L}^{-1}\{\frac{\frac{1}{4}}{2((s + \frac{1}{4})^2 + \frac{15}{16})}\} = -\frac{1}{\sqrt{15}}\mathcal{L}^{-1}\{\frac{\frac{\sqrt{15}}{4}}{2((s + \frac{1}{4})^2 + \frac{15}{16})}\} = -\frac{e^{-t/4}}{2\sqrt{15}}\sin(\frac{\sqrt{15}}{4}t). \tag{24.13}$$

Next, using Theorem 16,

$$y(t) = u_5(t)h(t) - u_{20}(t)h(t). \tag{24.14}$$

Next, consider the problem

$$y''(t) + 4y(t) = g(t), \qquad g(t) = 0, \qquad 0 \leq t < 5, \qquad g(t) = \frac{1}{5}(t - 5), \qquad 5 \leq t < 10, \\ g(t) = 1, \qquad t \geq 10, \qquad y(0) = y'(0) = 0. \tag{24.15}$$

Taking the Laplace transform of both sides,

$$(s^2 + 4)Y(s) = \mathcal{L}\{g(t)\} = G(s). \tag{24.16}$$

Rewriting,

$$g(t) = \frac{1}{5}(u_5(t)(t - 5) - u_{10}(t)(t - 10)), \tag{24.17}$$

so

$$G(s) = \frac{e^{-5s} - e^{-10s}}{5s^2}. \tag{24.18}$$

Doing some algebra,

$$Y(s) = \frac{e^{-5s} - e^{-10s}}{5} \cdot \frac{1}{s^2(s^2 + 4)}. \tag{24.19}$$

Now then,

$$\frac{1}{s^2}\frac{1}{s^2 + 4} = \frac{1/4}{s^2} - \frac{1/4}{s^2 + 4}, \tag{24.20}$$

Now then,

$$\mathcal{L}^{-1}\{\frac{1/4}{s^2} - \frac{1/4}{s^2 + 4}\} = \frac{1}{4}t - \frac{1}{8}\sin(2t). \tag{24.21}$$

Therefore,

$$y(t) = \frac{1}{5}(\frac{1}{4}u_5(t)(t - 5) - \frac{1}{8}u_5(t)\sin(2(t - 5)) - \frac{1}{4}u_{10}(t)(t - 10) + \frac{1}{8}u_{10}(t)\sin(2(t - 10))). \tag{24.22}$$

25. Impulse functions

Consider the differential equation

$$ay'' + by' + cy = g(t), \tag{25.1}$$

where $g(t)$ is large during a short interval $t_0 - \tau < t < t_0 + \tau$ for some $\tau > 0$, and zero otherwise. Now then, define the intergral

$$I(\tau) = \int_{t_0-\tau}^{t_0+\tau} g(t)dt, \tag{25.2}$$

and since $g(t) = 0$ outside the interval $(t_0 - \tau, t_0 + \tau)$, then

$$I(\tau) = \int_{-\infty}^{\infty} g(t)dt. \tag{25.3}$$

For example, define

$$g(t) = d_\tau(t) = \frac{1}{2\tau}, \qquad -\tau < t < \tau, \qquad g(t) = 0, \qquad \text{otherwise}. \tag{25.4}$$

Then,

$$\lim_{\tau \searrow 0} d_\tau(t) = 0, \qquad t \neq 0, \qquad \lim_{\tau \searrow 0} I(\tau) = 1. \tag{25.5}$$

Define the unit impulse function, $\delta(t)$, $\delta(t) = 0$, $t \neq 0$ and $\int_{-\infty}^{\infty} \delta(t)dt = 1$. This is called the Dirac delta function. Now then,

$$\mathcal{L}\{\delta(t - t_0)\} = \lim_{\tau \searrow 0} \mathcal{L}\{d_\tau(t - t_0)\} = \int_0^{\infty} e^{-st} d_\tau(t - t_0)dt = e^{-st_0}. \tag{25.6}$$

In general, for any continuous function $f(t)$,

$$\int_{-\infty}^{\infty} f(t)\delta(t - t_0)dt = f(t_0). \tag{25.7}$$

For example, solve the differential equation,

$$2y'' + y' + 2y = \delta(t - 5), \qquad y(0) = y'(0) = 0. \tag{25.8}$$

Taking the Laplace transform of both sides,

$$(2s^2 + s + 2)Y(s) = e^{-5s}. \tag{25.9}$$

Taking the inverse Laplace transform of both sides,

$$\mathcal{L}^{-1}\{\frac{1}{2s^2 + s + 2}\} = \mathcal{L}^{-1}\{\frac{2}{\sqrt{15}} \frac{\frac{\sqrt{15}}{4}}{(s + \frac{1}{4})^2 + \frac{15}{16}}\} = \frac{2}{\sqrt{15}} e^{-t/4} \sin(\frac{\sqrt{15}}{4}t). \tag{25.10}$$

Therefore,

$$y(t) = \frac{2}{\sqrt{15}} u_5(t) e^{-(t-5)/4} \sin(\frac{\sqrt{15}}{4}(t - 5)). \tag{25.11}$$

26. Existence and uniqueness of solutions

Consider the equation,

$$\frac{dx}{dt} = F(t, x), \qquad x(t_0) = x_0. \tag{26.1}$$

Also suppose that $F(t, x)$ satisfies the Lipschitz condition,

$$\|F(t, x) - F(t, y)\| \leq L\|x - y\|. \tag{26.2}$$

We can achieve this bound if

$$\|D_x F(t, x)\| \leq L. \tag{26.3}$$

Proposition 13. *Suppose $F : I \times \Omega \to \mathbb{R}^n$ is bounded and continuous and satisfies the Lipschitz condition and that $x_0 \in \Omega$. Then, there exists $T_0 > 0$ and a unique C^1 solution to* (26.1) *for $|t - t_0| < T_0$.*

Proof. We prove this using Picard iteration. Indeed, a solution to (26.1) satisfies

$$x(t) = x_0 + \int_{t_0}^{t} F(s, x(s)) ds. \tag{26.4}$$

Then by Picard iteration, define

$$x_{n+1}(t) = x_0 + \int_{t_0}^{t} F(s, x_n(s)) ds. \tag{26.5}$$

We assume that there exists $R > 0$ such that $B_R(x_0) \subset \Omega$ and

$$\|F(s, x)\| \leq M, \tag{26.6}$$

for any $x \in \overline{B_R(x_0)}$. Clearly, $x_0(t) = x_0$ for all t. Also,

$$\|x_{n+1}(t) - x_0\| \leq M|t - t_0|, \tag{26.7}$$

so for $|t - t_0| < T_0$, T_0 sufficiently small, implies that $x_{n+1}(t)$ also takes values in $\overline{B_R(x_0)}$.

Now then, by the Lipschitz continuity,

$$\|x_{n+1}(t) - x_n(t)\| \leq LT_0 \max_{|s-t_0| \leq T_0} \|x_n(s) - x_{n-1}(s)\|. \tag{26.8}$$

Thus, for $T_0 \leq \frac{1}{2L}$,

$$\max_{|t-t_0| \leq T_0} \|x_{n+1}(t) - x_n(t)\| \leq 2^{-n} R, \tag{26.9}$$

so then, the infinite sequence,

$$x(t) = x_0 + \sum_{n=0}^{\infty} (x_{n+1}(t) - x_n(t)). \tag{26.10}$$

□

For each closed, bounded subset K of Ω, (26.2) and (26.6) hold. If a solution stays in K, then we can extend a solution.

Proposition 14. *Let F be as in Proposition 13 but with Lipschitz and boundedness conditions only holding on a closed, bounded set K. Assume that $[a,b]$ is contained in the open interval I and that $x(t)$ solves (26.4) for $t \in (a,b)$. Assume that there exists a closed, bounded set $K \subset \Omega$ such that $x(t) \in K$ for all $t \in (a,b)$. Then there exist $a_1 < a$ and $b_1 > b$ such that $x(t)$ solves (26.1) for $t \in (a_1, b_1)$.*

We can use this result to prove global existence. For example, consider the 2×2 system,

$$\frac{dy}{dt} = v, \qquad \frac{dv}{dt} = -y^3. \tag{26.11}$$

In this case,

$$\frac{d}{dt}(\frac{v^2}{2} + \frac{y^4}{4}) = 0. \tag{26.12}$$

Therefore, the solution $x(t) = (y(t), v(t))$ lies on a level curve

$$\frac{y^4}{4} + \frac{v^2}{2} = C. \tag{26.13}$$

27. Nonlinear ODEs : The phase plane

We turn now to nonlinear ordinary differential equations of the form

$$\frac{dy}{dt} = f(y). \tag{27.1}$$

Such equations usually do not have a solution constructed of elementary functions.

Of particular importance are the critical points of (27.1). These are points y_0 such that $f(y_0) = 0$. In this case, of course, $y(t) = y_0$ is a solution to (27.1). Then, by Taylor's formula,

$$\frac{d(y - y_0)}{dt} = f(y) - f(y_0) = Df(y_0) \cdot (y - y_0) + o(y - y_0). \tag{27.2}$$

Then we study the eigenvalues and eigenvectors of $Df(y_0) = A$.

Definition 11. *We say that a critical point is stable if, given any $\epsilon > 0$, there exists $\delta > 0$ such that if $\|x(0) - x^0\| < \delta$, then the solution exists for all positive t and satisfies $\|x(t) - x^0\| < \epsilon$. A point that is not stable is unstable. A solution is said to be asymptotically stable if, in addition to being stable,*

$$\lim_{t\to\infty} x(t) = x_0. \tag{27.3}$$

Case 1: Real, unequal eigenvalues of the same sign. In this case, the solution to the linearized equation is

$$\vec{x}(t) = c_1 \xi^{(1)} e^{r_1 t} + c_2 \xi^{(2)} e^{r_2 t}. \tag{27.4}$$

Stable if negative, unstable if both positive.

Case 2: Real, unequal eigenvalues of opposite sign. In this case, we have a stable direction and an unstable direction, see (27.4).

Saddle point.

Case 3: Equal eigenvalues. In this case, we could have $Df(y_0) = \lambda I$, so then in that case we can again use (27.4). This is called a proper node. Otherwise, if we have one eigenvalue and a generalized eigenvalue, which is called an improper node,

$$\vec{x}(t) = c_1 \xi e^{rt} + c_2(\xi t e^{rt} + \eta e^{rt}). \tag{27.5}$$

This is unstable if positive and stable if negative.

Case 4: Complex eigenvalues with nonzero real part. In this case, we may have either a spiral sink or a spiral source. In this case, the linearized equation is

$$\frac{dx}{dt} = \begin{pmatrix} \lambda & \mu \\ -\mu & \lambda \end{pmatrix} x, \tag{27.6}$$

and the matrix exponential is given by

$$e^{tA} = e^{\lambda t} \begin{pmatrix} \cos(\mu t) & \sin(\mu t) \\ -\sin(\mu t) & \cos(\mu t) \end{pmatrix}. \tag{27.7}$$

This is an unstable spiral if $\lambda > 0$ and a stable spiral if $\lambda < 0$.

Case 5: In this case, $\lambda = 0$, so we have a center. In this case,

$$\frac{dx}{dt} = \begin{pmatrix} 0 & \mu \\ -\mu & 0 \end{pmatrix} x, \tag{27.8}$$

and the matrix exponential is given by

$$e^{tA} = \begin{pmatrix} \cos(\mu t) & \sin(\mu t) \\ -\sin(\mu t) & \cos(\mu t) \end{pmatrix}. \tag{27.9}$$

This is a center, which is stable.

28. Predator-prey equations

The simplest model for a species population growth is

$$\frac{dx}{dt} = ax. \tag{28.1}$$

Remark 4. *Of course, a population should be an integer.*

The solution to this equation is

$$x(t) = e^{at} x(0). \tag{28.2}$$

Of course, resources are not unlimited. Instead, we consider the logistic population growth equation,

$$\frac{dx}{dt} = ax(1 - bx), \qquad b = \frac{1}{K}. \tag{28.3}$$

This is a separable equation,

$$\frac{dx}{x(1 - bx)} = a dt. \tag{28.4}$$

By partial fractions,

$$\frac{1}{x(1 - bx)} = \frac{1}{x} + \frac{K}{(1 - bx)}. \tag{28.5}$$

Integrating both sides,

$$\ln(x) - K^2 \ln(1 - bx) = at + C. \tag{28.6}$$

This equation has two critical points, $x = 0$ and $x = \frac{1}{b} = K$.

Now we turn to a 2×2 system of equations, the predator-prey equations. Let $x(t)$ be the population of prey, $y(t)$ the population of predator, and α the rate at which the predator eats the prey. Then we have the system of equations

$$\begin{aligned} \frac{dx}{dt} &= ax - \alpha xy = x(a - \alpha y), \\ \frac{dy}{dt} &= -cy + \gamma xy = y(-c + \gamma x). \end{aligned} \tag{28.7}$$

In this case, if $y = 0$, then we have exponential growth of the prey. If $x = 0$, the population of the predator goes to zero.

This equation has two critical points, $(x, y) = (0, 0)$, and $(x, y) = (\frac{c}{\gamma}, \frac{a}{\alpha})$.

The origin: $(x, y) = (0, 0)$

In this case, the linearization of (28.7) is given by

$$\frac{d}{dt}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} a & 0 \\ 0 & -c \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} + O(x^2 + y^2). \tag{28.8}$$

In this case, the critical point is a saddle point.

The point $(x, y) = (\frac{c}{\gamma}, \frac{a}{\alpha})$*:*

To simplify the computations, consider the equation

$$\begin{aligned} \frac{dx}{dt} &= x(1 - 0.5y), \\ \frac{dy}{dt} &= y(-0.75 + 0.25x). \end{aligned} \tag{28.9}$$

In this case, the critical point is $(x, y) = (3, 2)$. Expanding around $(3, 2)$,

(28.10)

$$\frac{dx}{dt} = x(1 - 0.5y) = (3 + (x - 3))(1 - (0.5) * 2 - 0.5(y - 2)) = -1.5(y - 2) - 0.5(x - 3)(y - 2),$$

$$\frac{dy}{dt} = y(-0.75 + 0.25x) = (2 + (y - 2))(-0.75 + 0.25 * 3 + 0.25(x - 3)) = 0.5(x - 3) + 0.25(x - 3)(y - 2).$$

Linearizing the matrix, we have

$$\frac{d}{dt}\begin{pmatrix} x - 3 \\ y - 2 \end{pmatrix} = \begin{pmatrix} 0 & -1.5 \\ 0.5 & 0 \end{pmatrix}\begin{pmatrix} x - 3 \\ y - 2 \end{pmatrix} + O((x - 3)^2 + (y - 2)^2). \tag{28.11}$$

The matrix $\begin{pmatrix} 0 & -1.5 \\ 0.5 & 0 \end{pmatrix}$ has two imaginary eigenvalues, so the linearization is periodic solution.

For the nonlinear solution, observe that

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{y(-0.75 + 0.25x)}{x(1 - 0.5y)}. \tag{28.12}$$

This equation is separable, obtaining

$$\frac{dy}{y}(1 - 0.5y) = \frac{dx}{x}(-0.75 + 0.25x). \tag{28.13}$$

Integrating both sides,

$$\ln(y) - 0.5y + 0.75\ln(x) - 0.25x = C. \tag{28.14}$$

Therefore, we have a trajectory that circles the critical point.

29. Competing species equation

Let $x(t)$ and $y(t)$ be two competing species,

$$\begin{aligned} \frac{dx}{dt} &= ax(1 - bx) - cxy, \\ \frac{dy}{dt} &= \alpha y(1 - \beta y) - \gamma xy. \end{aligned} \tag{29.1}$$

In this case, each population is governed by the logistic equation in the absence of the other species.

Let us consider the specific equation,

$$\begin{aligned} \frac{dx}{dt} &= x(1 - x - y), \\ \frac{dy}{dt} &= \frac{y}{4}(3 - 4y - 2x). \end{aligned} \tag{29.2}$$

There are two critical points of the second equation when $x = 0$: $y = 0$ and $y = \frac{3}{4}$. There are two critical points of the first equation when $y = 0$: $x = 1$ and $x = 0$. Finally, we have the critical point:

$$1 - x - y = 0, \qquad 3 - 4y - 2x = 0, \tag{29.3}$$

which has the fourth critical point $(\frac{1}{2}, \frac{1}{2})$.

In this case, we have the Jacobian

$$\begin{pmatrix} 1 - 2x - y & -x \\ -0.5y & 0.75 - 2y - 0.5x \end{pmatrix}. \tag{29.4}$$

At $(x, y) = (0, 0)$,

$$\frac{d}{dt}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 0.75 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} + O(x^2 + y^2). \tag{29.5}$$

This is an unstable equilibrium.

At $(x, y) = (0, \frac{3}{4})$,

$$\frac{d}{dt}\begin{pmatrix} x \\ y - \frac{3}{4} \end{pmatrix} = \begin{pmatrix} 0.25 & 0 \\ -0.375 & -0.75 \end{pmatrix}\begin{pmatrix} x \\ y - \frac{3}{4} \end{pmatrix} + O(x^2 + (y - \frac{3}{4})^2). \tag{29.6}$$

This is a saddle point. The eigenvalues and eigenvectors are

$$r_1 = \frac{1}{4}, \qquad e_1 = \begin{pmatrix} 8 \\ -3 \end{pmatrix}, \qquad r_2 = -\frac{3}{4}, \qquad e_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}. \tag{29.7}$$

At $(x, y) = (1, 0)$,

$$\frac{d}{dt}\begin{pmatrix} x - 1 \\ y \end{pmatrix} = \begin{pmatrix} -1 & -1 \\ 0 & 0.25 \end{pmatrix}\begin{pmatrix} x - 1 \\ y \end{pmatrix} + O((x - 1)^2 + y^2). \tag{29.8}$$

The eigenvalues and eigenvectors are

$$r_1 = -1, \qquad e_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \qquad r_2 = \frac{1}{4}, \qquad \begin{pmatrix} 4 \\ -5 \end{pmatrix}. \tag{29.9}$$

This is also a saddle point.

At $(x, y) = (\frac{1}{2}, \frac{1}{2})$,

$$\frac{d}{dt}\begin{pmatrix} x - \frac{1}{2} \\ y - \frac{1}{2} \end{pmatrix} = \begin{pmatrix} -0.5 & -0.5 \\ -0.25 & -0.5 \end{pmatrix}\begin{pmatrix} x - \frac{1}{2} \\ y - \frac{1}{2} \end{pmatrix} + O((x - \frac{1}{2})^2 + (y - \frac{1}{2})^2). \tag{29.10}$$

The eigenvalues and eigenvectors are

$$r_1 = \frac{1}{4}(-2 + \sqrt{2}), \qquad e_1 = \begin{pmatrix} \sqrt{2} \\ -1 \end{pmatrix}, \qquad r_2 = \frac{1}{4}(-2 - \sqrt{2}), \qquad e_2 = \begin{pmatrix} \sqrt{2} \\ 1 \end{pmatrix}. \tag{29.11}$$

This is a stable critical point.

Now consider the system of equations

$$\begin{aligned} \frac{dx}{dt} &= x(1 - x - y), \\ \frac{dy}{dt} &= y(0.5 - 0.25y - 0.75x). \end{aligned} \tag{29.12}$$

This has the critical points $(0, 0)$, $(0, 2)$, $(1, 0)$, and $(\frac{1}{2}, \frac{1}{2})$. We have the Jacobian

$$\begin{pmatrix} 1 - 2x - y & -x \\ -0.75y & 0.5 - 0.5y - 0.75x \end{pmatrix}. \tag{29.13}$$

At $(x, y) = (0, 0)$, we have the Jacobian

$$\begin{pmatrix} 1 & 0 \\ 0 & 0.5 \end{pmatrix}, \tag{29.14}$$

which is an unstable equilibrium.

At $(x, y) = (0, 2)$, we have the Jacobian

$$\begin{pmatrix} -1 & 0 \\ -1.5 & -0.5 \end{pmatrix}, \tag{29.15}$$

which has the eigenvalues $r_1 = -1$, $r_2 = -0.5$ and the eigenvectors $e_1 = \begin{pmatrix} 1 \\ 3 \end{pmatrix}$ and $e_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$. Thus, $(x, y) = (0, 2)$ is a stable equilibrium.

At $(x, y) = (1, 0)$, we have the Jacobian

$$\begin{pmatrix} -1 & -1 \\ 0 & -0.25 \end{pmatrix}, \tag{29.16}$$

which has the eigenvalues $r_1 = -1$ and $r_2 = -\frac{1}{4}$ and eigenvectors $e_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $e_2 = \begin{pmatrix} 4 \\ -3 \end{pmatrix}$. This is also a stable equilibrium.

At $(x, y) = (\frac{1}{2}, \frac{1}{2})$, the Jacobian is

$$\begin{pmatrix} -0.5 & -0.5 \\ -0.375 & -0.125 \end{pmatrix}. \tag{29.17}$$

The eigenvalues and eigenvectors are
(29.18)
$$r_1 = \frac{1}{16}(-5+\sqrt{57}), \qquad e_1 = \begin{pmatrix} 1 \\ \frac{1}{8}(-3-\sqrt{57}) \end{pmatrix}, \qquad r_2 = \frac{1}{16}(-5-\sqrt{57}), \qquad e_2 = \begin{pmatrix} 1 \\ \frac{1}{8}(-3+\sqrt{57}) \end{pmatrix}.$$
This critical point is a saddle point. This forms a separatrix.

References

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