NOTES ON SCHRÖDINGER MAPS

BENJAMIN DODSON

Abstract. These are the notes for a course in Senegal.

1. The differentiated Schrödinger maps equation

In this course we will consider the Schrödinger map initial value problem

$$\partial_t \phi = \phi \times \Delta \phi, \qquad \text{on} \qquad \mathbb{R} \times \mathbb{R}^d, \qquad \phi(0) = \phi_0, \tag{1.1}$$

where $\phi : \mathbb{R} \times \mathbb{R}^d \to \mathbb{S}^2 \hookrightarrow \mathbb{R}^3$. The Schrödinger map problem has a rich geometric structure and arises in several different ways. For instance, it arises in ferromagnetism as the Heisenberg model for the ferromagnetic spin system whose classical spin ϕ, which belongs to $\mathbb{S}^2 \hookrightarrow \mathbb{R}^3$ is given by (1.1) in dimensions $d = 1, 2, 3$. See [CSU00], [NSU03], [PT91], and [TH13] for more details. On the mathematical side, this problem may be thought of as a generalization of the usual free Schrödinger equation,

$$u_t = i\Delta u, \qquad u(0, x) = u_0. \tag{1.2}$$

This is because the manifold $\mathbb{S}^2$ is a Kähler manifold with the complex structure $v \in T_\phi \mathbb{S}^2 \mapsto \phi \times v$.

Remark 1. *To see why, suppose without loss of generality that $\phi = \overrightarrow{e_1}$ and $v = \overrightarrow{e_2}$. Then $\phi \times v = \overrightarrow{e_3}$ and $\phi \times (\phi \times v) = -\overrightarrow{e_2}$. Thus, if $\mathcal{J} v = \phi \times v$, $\mathcal{J}^2 = -I$.*

However, unlike (1.2), a solution to (1.1) is decidedly nonlinear. Indeed, if ϕ solves (1.1) then $|\phi| = 1$. Because of this, not only is it not true that if ϕ_1 and ϕ_2 are solutions to (1.1) then $\phi_1 + \phi_2$ is also a solution to (1.1), but it is unclear what $\phi_1 + \phi_2$ even means in the context of (1.1), since $\phi_1 + \phi_2$ will almost certainly not lie on $\mathbb{S}^2$. For this reason it is useful to consider the differentiated Schrödinger map equation. Such an equation will still be nonlinear, but the derivate of a solution to (1.1) will at least lie in the space $T_\phi \mathbb{S}^2$, which is a linear space.

Let $\phi : \mathbb{R}^d \times (-T, T) \to \mathbb{S}^2$ be a smooth function that satisfies (1.1). Then consider the derivatives

$$\partial_m \phi(x, t), \qquad \text{for} \qquad m = 1, ..., d + 1, \qquad \partial_{d+1} = \partial_t. \tag{1.3}$$

These are tangent vectors to the sphere at $\phi(x, t)$. Suppose we have a smooth orthonormal frame $(v(t, x), w(t, x))$ in $T_{\phi(x,t)} \mathbb{S}^2$. Then we can introduce the differentiated variables

$$\psi_m =^t v \cdot \partial_m \phi + i^t w \cdot \partial_m \phi. \tag{1.4}$$

Thus,

$$\partial_m \phi = vRe(\psi_m) + wIm(\psi_m). \tag{1.5}$$

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In order to write the equations for ψ_m we need to know how $v(x,t)$ and $w(x,t)$ vary as functions of (x,t). For this reason, we introduce the real coeffcients

$$A_m =^t w \cdot \partial_m v. \tag{1.6}$$

Since $|\phi| = |v| = |w| = 1$ for all (x,t), $\langle \partial_m \phi, \phi \rangle = \langle \partial_m v, v \rangle = \langle \partial_m w, w \rangle = 0$ and $\langle \partial_m v, \phi \rangle = -\langle \phi_m \phi, v \rangle$, $\langle \partial_m v, w \rangle = -\langle \partial_m w, v \rangle$, $\langle \partial_m w, \phi \rangle = -\langle \partial_m \phi, w \rangle$. Thus,

$$\partial_m v = -\phi Re(\psi_m) + w A_m, \qquad \partial_m w = -\phi Im(\psi_m) - v A_m. \tag{1.7}$$

Lemma 1. *The variables ψ_m satisfy the curl type relation,*

$$(\partial_l + iA_l)\psi_m = (\partial_m + iA_m)\psi_l. \tag{1.8}$$

Proof. By direct calculation,

$$\begin{aligned} (\partial_l + iA_l)\psi_m &= (\partial_l + iA_l)({}^t v \cdot \partial_m \phi + i {}^t w \cdot \partial_m \phi) \\ &= ({}^t v \cdot \partial_l \partial_m \phi + i {}^t w \cdot \partial_l \partial_m \phi) + ({}^t \partial_l v \cdot \partial_m \phi + i {}^t \partial_l w \cdot \partial_m \phi) \\ +i({}^t w \cdot \partial_l v)({}^t v \cdot \partial_m \phi) &- ({}^t w \cdot \partial_l v)({}^t w \cdot \partial_m \phi) = ({}^t v \cdot \partial_l \partial_m \phi + i {}^t w \cdot \partial_l \partial_m \phi). \end{aligned} \tag{1.9}$$

The last step uses the fact that ${}^t \partial_l v \cdot \partial_m \phi = ({}^t w \cdot \partial_l v)({}^t v \cdot \partial_m \phi)$ and $\partial_l w \cdot \partial_m \phi = -({}^t w \cdot \partial_l v)({}^t v \cdot \partial_m \phi)$. □

Thus, with the notation $\mathbf{D}_m = \partial_m + iA_m$, (1.8) is equivalent to

$$\mathbf{D}_l \psi_m = \mathbf{D}_m \psi_l. \tag{1.10}$$

Next, by direct computation and using the fact that $(\partial_l w \cdot \partial_m v = (w \cdot \partial_l \phi)(v \cdot \partial_m \phi) = Im(\psi_l) Re(\psi_m)$,

$$\partial_l A_m - \partial_m A_l = ({}^t \partial_l w \cdot \partial_m v -^t \partial_m w \cdot \partial_l v) = Im(\psi_l \overline{\psi_m}) = q_{lm}. \tag{1.11}$$

Thus, the curvature of the connection is given by

$$\mathbf{D}_l \mathbf{D}_m - \mathbf{D}_m \mathbf{D}_l = i q_{lm}. \tag{1.12}$$

Now suppose that the smooth function ϕ satisfies the Schrödinger map equation (1.1). By direct computation using (1.8), (1.9), (1.11), $\phi \times v = w$, and $\phi \times w = -v$,

$$\psi_{d+1} =^t v \cdot \partial_{d+1} \phi + i^t w \cdot \partial_{d+1} \phi =^t v \cdot (\phi \times \Delta \phi) + i^t w \cdot (\phi \times \Delta \phi) = -({}^t w \cdot \Delta \phi) + i({}^t v \cdot \Delta \phi) = i \sum_{l=1}^{d} \mathbf{D}_l \psi_l. \tag{1.13}$$

Using (1.10) and (1.12),

$$\mathbf{D}_{d+1} \psi_m = \mathbf{D}_m \psi_{d+1} = i \mathbf{D}_m \sum_{l=1}^{d} \mathbf{D}_l \psi_l = i \sum_{l=1}^{d} \mathbf{D}_l \mathbf{D}_l \psi_m + \sum_{l=1}^{d} q_{lm} \psi_l, \tag{1.14}$$

which is equivalent to

$$\partial_t \psi_m + i A_{d+1} \psi_m = i \Delta \psi_m - 2 \sum_{l=1}^{d} A_l \partial_l \psi_m - \sum_{l=1}^{d} (\partial_l A_l) \psi_m - i (\sum_{l=1}^{d} A_l^2) \psi_m + \sum_{l=1}^{d} Im(\psi_l \overline{\psi_m}) \psi_l. \tag{1.15}$$

Doing some algebra,

$$(i\partial_t + \Delta_x)\psi_m = -2i \sum_{l=1}^{n} A_l \partial_l \psi_m + (A_{n+1} + \sum_{l=1}^{n} (A_l^2 - i \partial_l A_l)) \psi_m - i \sum_{l=1}^{n} Im(\overline{\psi_l} \psi_m) \psi_l. \tag{1.16}$$

Remark 2. *The reason writing the last term in* (1.16) *in this way is to highlight the focusing nature the differentiated equation. Indeed,*

$$-iIm(\overline{\psi_l}\psi_m)\psi_l = -\frac{1}{2}|\psi_l|^2\psi_m + \frac{1}{2}\overline{\psi_m}\psi_l^2, \tag{1.17}$$

which is a negative definite operator on ψ_m. *If the terms with* A_l*'s in them could somehow be removed, it is likely that* (1.16) *could be analyzed in a manner analogous to* [Dod15].

The solution ψ_m for the above system (1.10), (1.11), and (1.16) cannot be uniquely determined as it depends on the choice of the orthonormal frame $(v(x,t), w(x,t))$. If $w = \phi \times v$, the system is invariant with respect to a coordinate rotation in $T_\phi \mathbb{S}^2$. Under such a rotation,

(1.18)
$$\psi_m = \cos\theta({}^t v\cdot\partial_m\phi)+\sin\theta({}^t w\cdot\partial_m\phi)+i\cos\theta({}^t w\cdot\partial_m\phi)-i\sin\theta({}^t v\cdot\partial_m\phi) = (\cos\theta - i\sin\theta)\psi_m = e^{-i\theta}\psi_m.$$

Plugging (1.18) into (1.6),

(1.19)
$$((\cos\theta)w - (\sin\theta)v)^t \cdot \partial_m((\cos\theta)v + (\sin\theta)w) = w^t \cdot \partial_m v - \sin^2\theta(\partial_m\theta) - \cos^2\theta(\partial_m\theta) = A_m - \partial_m\theta.$$

In order to obtain a well–posed system, one needs to make a choice which uniquely determines the gauge. Ideally, one may hope that this choice uniquely determines the A_m' in terms of the ψ_m's so that the nonlinearity is perturbative.

1.1. Homework.

(1) Prove the dispersive estimates for a solution to (1.2).

2. The Coulomb gauge

One natural choice is the Coulomb gauge, where one adds the equation

$$\sum_{m=1}^{d} \partial_m A_m = 0. \tag{2.1}$$

In view of (1.11), (2.1) leads to

$$A_m = \Delta^{-1} \sum_{l=1}^{d} \partial_l Im(\overline{\psi_l}\psi_m). \tag{2.2}$$

Thus, for a given gauge $\tilde{A}_m$, let θ solve the elliptic partial differential equation

$$\Delta\theta = -\sum_{m=1}^{d} \partial_m \tilde{A}_m, \tag{2.3}$$

and then if $A_m = \tilde{A}_m + \partial_m\theta$, (2.1) holds. We only need to find a gauge $\tilde{A}_m$ with sufficient regularity such that it is reasonable to discuss a solution to (2.3).

To that end, assume $n \in [1,\infty) \cap \mathbb{Z}$, $a_1, ..., a_n \in [0,\infty)$, and let

$$\mathcal{D}^n = [-a_1, a_1] \times ... \times [-a_n, a_n]. \tag{2.4}$$

For $n = 0$ let $\mathcal{D}^0 = \{0\}$.

Lemma 2. *Assume* $n \geq 0$ *and* $\phi : \mathcal{D}^n \to \mathbb{S}^2$ *is a continuous function. Then there is a continuous function* $v : \mathcal{D}^n \to \mathbb{S}^2$ *with the property that*

$$\phi(x)\cdot v(x) = 0, \qquad \textit{for any} \qquad x \in \mathcal{D}^n. \tag{2.5}$$

Proof. We argue by induction over n, the case when $n = 0$ is trivial. Since ϕ is continuous, there is $\epsilon > 0$ with the property that

$$|\phi(x) - \phi(y)| \leq 2^{-10}, \qquad \text{for any} \qquad x, y \in \mathcal{D}^n, \qquad \text{with} \qquad |x - y| \leq \epsilon. \tag{2.6}$$

For $x \in \mathcal{D}_n$ we can write $x = (x', x_n) \in \mathcal{D}^{n-1} \times [-a_n, a_n]$. For any $b \in [-a_n, a_n]$ let $\mathcal{D}^n_b = \mathcal{D}^{n-1} \times [-a_n, b] = \{x = (x', x_n) \in \mathcal{D}^n : x_n \in [-a_n, b]\}$. By the induction hypothesis we can define $v : \mathcal{D}^n_{-a_n} \to \mathbb{S}^2$ continuous such that

$$\phi(x) \cdot v(x) = 0, \qquad \text{for any} \qquad x \in \mathcal{D}^n_{-a_n}. \tag{2.7}$$

Now extend the function ϕ to $\mathcal{D}^n$. For ϵ as in (2.6) it suffices to prove that if $b, b' \in [-a_n, a_n]$, $0 \leq b' - b \leq \epsilon$, $v : \mathcal{D}^n_b \to \mathbb{S}^2$ is continuous, and $\phi(x) \cdot v(x) = 0$ for any $x \in \mathcal{D}^n_b$, then v can be extended to a continuous function $\tilde{v} : \mathcal{D}^n_{b'} \to \mathbb{S}^2$ such that $\phi(x) \cdot \tilde{v}(x) = 0$ for any $x \in \mathcal{D}^n_{b'}$.

Now let

$$\mathcal{R} = \{(u_1, u_2) \in \mathbb{R}^3 \times \mathbb{R}^3 : |u_1|, |u_2| \in (\frac{1}{2}, 2) \qquad \text{and} \qquad |u_1 \cdot u_2| < 2^{-5}\}, \tag{2.8}$$

and let $N : \mathcal{R} \to \mathbb{S}^2$ denote the smooth function lying in the plane generated by u_1 and u_2 and orthogonal to u_2. We do this via the Gram–Schmidt orthogonalization process,

$$N[u_1, u_2] = \frac{u_1 - \frac{(u_1 \cdot u_2)}{|u_2|^2} u_2}{|u_1 - \frac{(u_1 \cdot u_2)}{|u_2|^2} u_2|}. \tag{2.9}$$

Now construct the extension $\tilde{v} : \mathcal{D}^n_{b'} \to \mathbb{S}^2$. For $x' \in \mathcal{D}^{n-1}$ and $x_n \in [-a_n, b']$, let

$$\tilde{v}(x', x_n) = \begin{cases} N[v(x', b), \phi(x', x_n)] & \text{if } x_n \in [b, b'], \\ v(x', x_n) & \text{if } x_n \in [-a_n, b]. \end{cases} \tag{2.10}$$

In view of (2.6), the function $\tilde{v} : \mathcal{D}^n_{b'} \to \mathbb{S}^2$ is well–defined, continuous, and $\phi(x) \cdot \tilde{v}(x) = 0$ for any $x \in \mathcal{D}^n_{b'}$. □

It is possible to extend this argument to all of $\mathbb{R}^n$ on a compact time interval $[-T, T]$ if $\phi(x, t)$ converges to some $Q \in \mathbb{S}^2$ as $|x| \to \infty$ for any fixed $t \in [-T, T]$.

Lemma 3. *Assume $T \in [0, 2]$, $Q, Q' \in \mathbb{S}^2$, $Q \cdot Q' = 0$, and $\phi : \mathbb{R}^d \times [-T, T] \to \mathbb{S}^2$ is a continuous function with the property that*

$$\lim_{x \to \infty} \phi(x, t) = Q, \qquad \textit{uniformly in} \qquad t \in [-T, T]. \tag{2.11}$$

Then there is a continuous function $v : \mathbb{R}^d \times [-T, T] \to \mathbb{S}^2$ with the property that

$$\begin{cases} \phi(x, t) \cdot v(x, t) = 0, & \textit{for any } (x, t) \in \mathbb{R}^d \times [-T, T], \\ \lim_{x \to \infty} v(x, t) = Q' & \textit{uniformly in } t \in [-T, T]. \end{cases} \tag{2.12}$$

Proof. Fix $R > 0$ such that

$$|\phi(x, t) - Q| \leq 2^{-10}, \qquad \text{if} \qquad |x| \geq R, \qquad \text{and} \qquad t \in [-T, T]. \tag{2.13}$$

Using Lemma 2, we can define a continuous function $v_0 : B_R \times [-T, T] \to \mathbb{S}^2$ such that $\phi(x, t) \cdot v_0(x, t) = 0$ for $(x, t) \in B_R \times [-T, T]$, where $B_R = \{x \in \mathbb{R}^d : |x| \leq R\}$.

Let $S_R = \{y \in \mathbb{R}^n : |y| = R\}$ and $\mathbb{S}^1_Q = \{x \in \mathbb{S}^2 : x \cdot Q = 0\}$. Then define the continuous function

$$w : S_R \times [-T, T] \to \mathbb{S}^1_Q, \qquad w(y, t) = \frac{(\phi(y, t) \cdot Q) v_0(y, t) - (v_0(y, t) \cdot Q) \phi(y, t)}{|(\phi(y, t) \cdot Q) v_0(y, t) - (v_0(y, t) \cdot Q) \phi(y, t)|}, \tag{2.14}$$

so $w(y,t)$ is a vector in $\mathbb{S}^1_Q$ and in the plane generated by $\phi(y,t)$ and $v_0(y,t)$. Since $n \geq 3$, the space $S_R \times [-T,T]$ is simply connected and compact, so the function w is homotopic to a constant function. Thus, there is a continuous function $\tilde{w} : S_R \times [-T,T] \times [1,2] \to \mathbb{S}^1_Q$ such that $\tilde{w}(y,t,1) = w(y,t)$ and $\tilde{w}(y,t,2) \equiv Q'$. Then define

$$v_1(x,t) = N[\tilde{w}(R\frac{x}{|x|}, t, \frac{|x|}{R}), \phi(x,t)], \tag{2.15}$$

for $|x| \in [R, 2R]$, and

$$v_2(x,t) = N[Q', \phi(x,t)], \tag{2.16}$$

for $|x| \geq 2R$. The function v in Lemma 3 is obtained by gluing the functions v_0, v_1, and v_2 together. $\square$

Now we define some Sobolev spaces for functions $\phi : \mathbb{R}^n \times \mathbb{R} \to \mathbb{S}^2 \hookrightarrow \mathbb{R}^3$. For $\sigma \geq 0$ and $d \in \{1, 2, ...\}$ let $H^\sigma = H^\sigma(\mathbb{R}^n; \mathbb{C}^d)$ denote the Banach space of $\mathbb{C}^d$–valued Sobolev functions on $\mathbb{R}^d$, i.e.

$$H^\sigma = \{f : \mathbb{R}^n \to \mathbb{C}^d : \|f\|_{H^\sigma} = [\sum_{l=1}^{d} \|\mathcal{F}_{(n)}(f_l) \cdot (|\xi|^2 + 1)^{\sigma/2}\|^2_{L^2}]^{1/2} < \infty\}, \tag{2.17}$$

where $\mathcal{F}_{(n)}$ denotes the Fourier transform on $L^2(\mathbb{R}^n)$. For $\sigma \geq 0$, $d \in \{1, 2, ...\}$, and $f \in H^\sigma(\mathbb{R}^n; \mathbb{C}^d)$, define

$$\|f\|_{\dot{H}^\sigma} = [\sum_{l=1}^{d} \|\mathcal{F}_{(n)}(f_l)(\xi) \cdot |\xi|^\sigma\|^2_{L^2}]^{1/2}. \tag{2.18}$$

For $\sigma \geq 0$ and $Q = (Q_1, Q_2, Q_3) \in \mathbb{S}^2$ we define the complete metric space

$$H^\sigma_Q = H^\sigma_Q(\mathbb{R}^n; \mathbb{S}^2 \hookrightarrow \mathbb{R}^3) = \{f : \mathbb{R}^n \to \mathbb{R}^3 : |f(x)| \equiv 1, \qquad \text{and} \qquad f - Q \in H^\sigma\}, \tag{2.19}$$

with the induced distance

$$d^\sigma_Q(f,g) = \|f - g\|_{H^\sigma}. \tag{2.20}$$

Let $\|f\|_{H^\sigma_Q} = d^\sigma_Q(f, Q)$ for $f \in H^\sigma_Q$. Let $\mathbb{Z}_+ = \{0, 1, ...\}$. For $d \in \{1, 2, ...\}$ and $Q \in \mathbb{S}^2$ define the complete metric spaces

$$H^\infty = H^\infty(\mathbb{R}^n; \mathbb{C}^d) = \cap_{\sigma \in \mathbb{Z}_+} H^\sigma \qquad \text{and} \qquad H^\infty_Q = \cap_{\sigma \in \mathbb{Z}_+} H^\sigma_Q, \tag{2.21}$$

with the induced distances.

2.1. **Derivation of the modified Schrödinger map equations.** Now suppose that $T \in [0,1]$, $Q, Q' \in \mathbb{S}^2$, and $Q \cdot Q' = 0$. Also suppose that

$$\begin{cases} \phi \in C([-T,T] : H^\infty_Q), \\ \partial_t \phi \in C([-T,T] : H^\infty). \end{cases} \tag{2.22}$$

Extend the function ϕ to a function $\tilde{\phi} \in C([-T-1, T+1] : H^\infty_Q)$ by setting $\tilde{\phi}(\cdot, t) = \phi(\cdot, T)$ if $t \in [T, T+1]$ and $\tilde{\phi}(\cdot, t) = \phi(\cdot, -T)$ if $t \in [-T-1, T]$. The function $\tilde{\phi} : \mathbb{R}^n \times [-T-1, T+1] \to \mathbb{S}^2$ is continuous and $\lim_{x \to \infty} \tilde{\phi}(x,t) = Q$, uniformly in t. Apply Lemma 3 to construct a continuous function $\tilde{v} : \mathbb{R}^n \times [-T-1, T+1] \to \mathbb{S}^2$ such that $\tilde{\phi} \cdot \tilde{v} \equiv 0$ and $\lim_{x \to \infty} \tilde{v}(x,t) = Q'$ uniformly in t.

Now regularize the function $\tilde{v}$. Let $\varphi : \mathbb{R}^n \times \mathbb{R} \to [0, \infty)$ denote a smooth function supported on the ball $\{(x,t) : |x|^2 + t^2 \leq 1\}$ with $\int_{\mathbb{R}^d \times \mathbb{R}} \varphi(x,t) dx dt = 1$. Since $\tilde{v}$ is a uniformly continuous function, there exists $\epsilon(\tilde{v})$ with the property that

$$|\tilde{v}(x,t) - (\tilde{v} * \varphi_\epsilon)(x,t)| \leq 2^{-20}, \qquad \text{for any} \qquad (x,t) \in \mathbb{R}^d \times [-T - 1/2, T + 1/2], \tag{2.23}$$

where $\varphi_\epsilon(x,t) = \epsilon^{-d-1} \varphi(\frac{x}{\epsilon}, \frac{t}{\epsilon})$. Using a partition of 1, we can smoothly replace $(\tilde{v} * \varphi_\epsilon)(x,t)$ with Q' for $|x|$ sufficiently large. Thus, we have constructed a smooth function $v' : \mathbb{R}^n \times (-T - 1/2, T + 1/2) \to \mathbb{R}^3$ with the properties

$$\begin{cases} |v'(x,t)| \in [1 - 2^{-10}, 1 + 2^{-10}] & \text{for any} \quad (x,t) \in \mathbb{R}^n \times [-T,T], \\ |v'(x,t) \cdot \phi(x,t)| \leq 2^{-10}, & \text{for any} \quad (x,t) \in \mathbb{R}^n \times [-T,T], \\ v'(x,t) = Q' & \text{for } |x| \text{ large enough and} \quad t \in [-T,T]. \end{cases} \tag{2.24}$$

Then define

$$v(x,t) = N[v'(x,t), \phi(x,t)], \tag{2.25}$$

with N as in (2.9). Therefore, the continuous function $v : \mathbb{R}^n \times [-T,T] \to \mathbb{S}^2$ is well–defined, $\phi(x,t) \cdot v(x,t) = 0$, and

$$\begin{cases} \partial_m v \in C([-T,T] : H^\infty) & \text{for} \qquad m = 1, ..., n, \\ \partial_t v \in C([-T,T] : H^\infty). \end{cases} \tag{2.26}$$

Given ϕ satisfying (2.22) and v satisfying (2.26), define

$$w(x,t) = \phi(x,t) \times v(x,t). \tag{2.27}$$

Since H^σ is an algebra for $\sigma > \frac{n}{2}$, we have

$$\begin{cases} \partial_m w \in C([-T,T] : H^\infty) & \text{for} \qquad m = 1, ..., n, \\ \partial_t w \in C([-T,T] : H^\infty). \end{cases} \tag{2.28}$$

Therefore, to summarize, we have constructed continuous functions $v, w : \mathbb{R}^d \times [-T,T] \to \mathbb{S}^2$ such that $\phi \cdot v = \phi \cdot w = v \cdot w \equiv 0$ and (2.26) and (2.28) hold.

Plugging (2.26) and (2.28) into (2.3), let

$$\theta(x,t) = c \int_{\mathbb{R}^n} e^{ix \cdot \xi} |\xi|^{-2} \sum_{m=1}^{n} (i\xi_m) \mathcal{F}_{(n)}(A_m)(\xi, t) d\xi, \tag{2.29}$$

where $A_m = (\partial_m v) \cdot w$. This integral converges absolutely when $n \geq 3$. Since

$$A_m = (\partial_m v) \cdot w, \tag{2.30}$$

$$\partial_m \chi, \partial_t \chi \in C([-T,T] : H^\infty). \tag{2.31}$$

Replacing A_m with $A_m + \partial_m \theta$, we have proved the following.

Proposition 1. *Assume $T \in [0,1]$, $Q \in \mathbb{S}^2$, and*

$$\begin{cases} \phi \in C([-T,T] : H_Q^\infty), \\ \partial_t \phi \in C([-T,T] : H^\infty). \end{cases} \tag{2.32}$$

Then there are continuous functions $v, w : \mathbb{R}^d \times [-T,T] \to \mathbb{S}^2$, $\phi \cdot v \equiv 0$, $w = \phi \times v$, such that

$$\partial_m v, \partial_m w \in C([-T,T] : H^\infty), \qquad \textit{for} \qquad m = 0, 1, ..., d, \tag{2.33}$$

where $\partial_0 = \partial_t$. *In addition, if*

$$A_m = (\partial_m v) \cdot w, \qquad \textit{for} \qquad m = 1, ..., d, \qquad \textit{then} \qquad \sum_{m=1}^{n} \partial_m A_m \equiv 0. \tag{2.34}$$

Then by (2.2),

$$A_m = \nabla^{-1}[\sum_{l=1}^{n} R_l[Im(\psi_m \overline{\psi_l})], \tag{2.35}$$

where R_l denotes the Riesz transform defined by the Fourier multiplier $\xi \mapsto \frac{i\xi_l}{|\xi|}$ and ∇^{-1} is the operator defined by the Fourier multiplier $\xi \mapsto |\xi|^{-1}$. By direct computation, using (1.11), (2.34), (1.13),

$$\begin{aligned} \Delta A_{n+1} &= \sum_{l=1}^{n} \partial_l \partial_l A_{n+1} = \sum_{l=1}^{n} \partial_l (\partial_{n+1} A_l + Im(\psi_l \overline{\psi_{n+1}})) = \sum_{l=1}^{n} \partial_l Im(\psi_l \overline{\psi_{n+1}}) \\ &= -\sum_{l,m=1}^{n} \partial_l Re(\psi_l \overline{\mathbf{D}_m \psi_m}) = -\sum_{l,m=1}^{n} \partial_l \partial_m Re(\psi_l \overline{\psi_m}) + \sum_{l,m=1}^{n} \partial_l Re(\mathbf{D}_m \psi_l \overline{\psi_m}) \\ &= -\sum_{l,m=1}^{n} \partial_l \partial_m Re(\psi_l \overline{\psi_m}) + \frac{1}{2} \sum_{l,m=1}^{n} \partial_l^2 |\psi_m|^2. \end{aligned} \tag{2.36}$$

Therefore,

$$A_{n+1} = \sum_{l,m=1}^{n} R_l R_m (Re(\overline{\psi_l} \psi_m)) + \frac{1}{2} \sum_{m=1}^{n} \psi_m \overline{\psi_m}. \tag{2.37}$$

2.2. **Homework.**

(1) Prove the paraproduct

$$\|\nabla^{1/2}(fg)\|_{L^2} \leq C \|\nabla^{1/2} f\|_{L^{p_1}} \|g\|_{L^{p_2}} + \|f\|_{L^{q_1}} \|\nabla^{1/2} g\|_{L^{q_2}}, \tag{2.38}$$

where $\frac{1}{p_1} + \frac{1}{p_2} = \frac{1}{2}$, $\frac{1}{q_1} + \frac{1}{q_2} = \frac{1}{2}$, and $p_1, p_2, q_1, q_2 < \infty$.

3. A quantitative estimate on the initial data when $n \geq 3$

Now we are ready to reprove the small data result of [BIK07]. Observe that a solution $\phi(x,t)$ to (1.1) has the scaling symmetry

$$\phi(x,t) \mapsto \phi(\frac{x}{\lambda}, \frac{t}{\lambda^2}). \tag{3.1}$$

That is, for any $\lambda > 0$, if $\phi(x,t)$ solves (1.1) then so does (3.1). Then the norm $\|\phi_0 - Q\|_{\dot{H}^{n/2}(\mathbb{R}^n)}$ is preserved under the scaling transformation (3.1).

Theorem 1. *Assume* $n \geq 3$ *and* $Q \in \mathbb{S}^2$. *Then there exists* $\epsilon_0(n) > 0$ *such that for any* $\phi_0 \in H_Q^\infty$ *with* $\|\phi_0 - Q\|_{\dot{H}^{n/2}} \leq \epsilon_0$ *there is a unique solution*

$$\phi = S_Q(\phi_0) \in C(\mathbb{R} : H_Q^\infty), \tag{3.2}$$

of the initial value problem (1.1). *Moreover,*

$$\sup_{t \in \mathbb{R}} \|\phi(t) - Q\|_{\dot{H}^{n/2}} \leq C \|\phi_0 - Q\|_{\dot{H}^{n/2}}, \tag{3.3}$$

and

$$\sup_{t \in [-T,T]} \|\phi(t)\|_{H^{\sigma}_{Q}} \leq C(\sigma, T, \|\phi_0\|_{H^{\sigma}_{Q}}), \tag{3.4}$$

for any $T \in [0, \infty)$ *and* $\sigma \in \mathbb{Z}_+$.

The proof of Theorem 1 may be divided into three parts: a quantitative estimate on the initial data, linear and bilinear estimates using the interaction Morawetz estimate, and a proof of the main result.

Lemma 4. *With the notation in the previous section, if* $\phi_0(x) = \phi(x, 0)$ *has the additional property* $\|\phi_0 - Q\|_{\dot{H}^{n/2}} \leq 1$ *and* $\sigma_0 = d + 10$ *then, for* $m = 1, ..., n$,

$$\begin{cases} \|\psi_m(\cdot, 0)\|_{\dot{H}^{\frac{n-2}{2}}} \leq C\|\phi_0 - Q\|_{\dot{H}^{n/2}}, \\ \|\psi_m(\cdot, 0)\|_{H^{\sigma'-1}} \leq C(\|\phi_0\|_{H^{\sigma'}_{Q}}), \qquad \text{for any} \qquad \sigma' \in [1, \sigma_0] \cap \mathbb{Z}. \end{cases} \tag{3.5}$$

Proof of Lemma 4. The main difficulty is that the construction does not give effective control of the Sobolev norms of v and w in terms of ϕ. For $\sigma \in [-1, \infty)$, let ∇^{σ} denote the operator defined by the Fourier multiplier $\xi \mapsto |\xi|^{\sigma}$. For $\sigma \in [-\frac{1}{2}, \frac{n}{2}]$ let $p_{\sigma} = \frac{n}{\sigma+1}$. Then, by the Sobolev embedding theorem,

$$\|\nabla^{\sigma} f\|_{L^{p_{\sigma}}} \leq C\|\nabla^{\sigma'} f\|_{L^{p_{\sigma'}}} \qquad \text{if} \qquad -\frac{1}{2} \leq \sigma \leq \sigma' \leq \frac{n}{2}, \qquad \text{and} \qquad f \in H^{\infty}. \tag{3.6}$$

Let $\phi_0(x) = \phi(x, 0)$, $v_0(x) = v(x, 0)$, $w_0(x) = w(x, 0)$, $\psi_{m,0}(x) = \psi_m(x, 0)$, $A_{m,0}(x) = A_m(x, 0)$, and let $\epsilon_0 = \|\phi_0 - Q\|_{\dot{H}^{n/2}} \leq 1$. Now then, by (1.4), (3.6), and the fact that $|v_0| = |w_0| = 1$,

$$\|\psi_{m,0}\|_{L^{p_0}} \leq C\epsilon_0, \qquad \text{for} \qquad m = 1, ..., n. \tag{3.7}$$

Then by (2.35),

$$\|\nabla^1 A_{m,0}\|_{L^{p_1}} \leq C\epsilon_0, \qquad \text{for} \qquad m = 1, ..., n, \tag{3.8}$$

which by (3.6) implies that $\|A_{m,0}\|_{L^{p_0}} \leq C\epsilon_0$ for $m = 1, ..., n$. Combining (1.5), (1.7), and the fact that for $f \in H^{\infty}$,

$$\|\nabla^k f\|_{L^p} \equiv \sum_{k_1 + ... + k_n = k} \|\partial_1^{n_1} \cdots \partial_n^{k_n} f\|_{L^p}, \qquad \text{if} \qquad k \in \mathbb{Z}_+ \qquad \text{and} \qquad p \in [p_{n/2}, p_{-1/2}]. \tag{3.9}$$

Thus,

$$\|\nabla^1 v_0\|_{L^{p_0}} + \|\nabla^1 w_0\|_{L^{p_0}} \leq C\epsilon_0, \tag{3.10}$$

and

$$\sum_{m=1}^{n} \|\psi_{m,0}\|_{L^{p_0}} + \sum_{m=1}^{n} \|\nabla^1 A_{m,0}\|_{L^{p_1}} + \|\nabla^1 v_0\|_{L^{p_0}} + \|\nabla^1 w_0\|_{L^{p_0}} \leq C\epsilon_0. \tag{3.11}$$

Now prove by induction that

$$\sum_{m=1}^{n} \|\nabla^k \psi_{m,0}\|_{L^{p_k}} + \sum_{m=1}^{n} \|\nabla^{k+1} A_{m,0}\|_{L^{p_{k+1}}} + \|\nabla^{k+1} v_0\|_{L^{p_k}} + \|\nabla^{k+1} w_0\|_{L^{p_k}} \leq C\epsilon_0, \tag{3.12}$$

for any $k \in \mathbb{Z} \cap [0, \frac{n-2}{2}]$. The base case $k = 0$ was proved in (3.11). Now suppose $k \geq 1$ and that (3.12) holds for any $k' \in [0, k-1] \cap \mathbb{Z}$. Using (1.4), (3.11), and the induction hypothesis,

$$\begin{aligned}\|\nabla^k \psi_{m,0}\|_{L^{p_k}} \leq C \|\nabla^{k+1}\phi_0\|_{L^{p_k}} \|v_0\|_{L^\infty} + C \sum_{k'=0}^{k-1} \|\nabla^{k-k'}\phi_0\|_{L^{p_{k-k'-1}}} \cdot \|\nabla^{k'+1} v_0\|_{L^{p_{k'}}} \\ + C \|\nabla^{k+1}\phi_0\|_{L^{p_k}} \|w_0\|_{L^\infty} + C \sum_{k'=0}^{k-1} \|\nabla^{k-k'}\phi_0\|_{L^{p_{k-k'-1}}} \cdot \|\nabla^{k'+1} w_0\|_{L^{p_{k'}}},\end{aligned} \tag{3.13}$$

which suffices to control the first term in (3.12).

For the second term, using (2.35) and (3.11),

$$\|\nabla^{k+1} A_{m,0}\|_{L^{p_{k+1}}} \leq C \sum_{l,l'=1}^{n} \sum_{k'=0}^{k} \|\nabla^{k'}\psi_{l,0}\|_{L^{p_{k'}}} \cdot \|\nabla^{k-k'}\psi_{l',0}\|_{L^{p_{k-k'}}}, \tag{3.14}$$

which suffices in view of the induction hypothesis and the bound on the first term. The bound on the other two terms in (3.13) is similar.

If n is even then Lemma 4 follows by taking $k = \frac{n-2}{2}$. If n is odd, the bounds (3.12) hold with $k = \frac{n-3}{2}$,

$$\|\nabla^{\sigma+1} v_0\|_{L^{p_\sigma}} + \|\nabla^{\sigma+1} w_0\|_{L^{p_\sigma}} \leq C\epsilon_0, \qquad \text{for} \qquad \sigma \in [-\frac{1}{2}, \frac{n-3}{2}]. \tag{3.15}$$

In view of the hypothesis and (3.6), we also have the bound

$$\|\nabla^{\sigma+1}\phi_0\|_{L^{p_\sigma}} \leq C\epsilon_0, \qquad \text{for} \qquad \sigma \in [-\frac{1}{2}, \frac{n-2}{2}]. \tag{3.16}$$

Now utilize the fractional Leibniz rule of [KPV93] (see also [Tay00]),

$$\|\nabla^{1/2}(fg) - g\nabla^{1/2} f\|_{L^2} \leq C \|\nabla^{1/2} g\|_{L^{q_1}} \|f\|_{L^{q_2}}, \tag{3.17}$$

if $\frac{1}{q_1} + \frac{1}{q_2} = \frac{1}{2}$ and $q_1, q_2 \in [p_{n/2}, p_{-1/2}]$. Using (3.11) and (1.4),

$$\|\nabla^{\frac{n-2}{2}}\psi_{m,0}\|_{L^2} \leq C \sum_{u_0 \in \{v_0, w_0\}} \sum_{k=0}^{\frac{n-3}{2}} \|\nabla^{1/2}(\partial_m D^k \phi_0 \cdot D^{\frac{n-3}{2}-k} u_0)\|_{L^2}, \tag{3.18}$$

where D^k denotes any derivatives of the form $\partial_1^{k_1} \cdots \partial_n^{k_n}$, with $k_1 + ... + k_n = k$. Then the first inequality in (3.5) follows from (3.15)–(3.17) and the fact that $|u_0| \equiv 1$.

For the second inequality in (3.5) observe that $\|\psi_{m,0}\|_{L^2} \leq C\|\phi_0\|_{H^1_Q}$, since $|v_0| \equiv |w_0| \equiv 1$. Now then, suppose $\sigma' \geq \frac{n+1}{2}$. Now then,

$$\sum_{m=1}^{n} \|\nabla^k \psi_{m,0}\|_{L^2 \cap L^{p_{k-\sigma'+n/2}}} + \sum_{m=1}^{n} \|\nabla^k A_{m,0}\|_{L^2 \cap L^{p_{k-\sigma'+n/2}}} + \sum_{u_0 \in \{v_0, w_0\}} \|\nabla^{k+1} u_0\|_{L^2 \cap L^{p_{k-\sigma'+n/2}}} \leq C(\|\phi_0\|_{H^{\sigma'}_Q}), \tag{3.19}$$

for any $k \in [0, \sigma'-1] \cap \mathbb{Z}$, where $p_\sigma = p_{-1/2} = 2n$ if $\sigma \leq -\frac{1}{2}$. The bound (3.19) follows by induction on k, using (1.5), (1.7), and (2.35), along with the inequalities (3.6), (3.11), and

$$\sum_{k_1 + ... + k_n \leq \sigma' - \frac{n+1}{2}} \|\partial_1^{k_1} \cdots \partial_n^{k_n} \phi_0\|_{L^\infty} \leq C(\|\phi_0\|_{H^{\sigma'}_Q}). \tag{3.20}$$

Thus, we have shown that for initial data small in the critical Sobolev norm $\dot{H}^{n/2}$, the Coulomb gauged initial data is small in $\dot{H}^{\frac{n-2}{2}}$. $\square$

The main technical difficulty to using small data arguments to prove a global well–posedness result to (1.16) lies in the quasilinear term $-2i\sum_{l=1}^{d} A_l \partial_l \psi_m$. Indeed, suppose for a moment that this term were not there, and that

$$(i\partial_t + \Delta_x)\psi_m = (A_{n+1} + \sum_{l=1}^{n}(A_l^2 - i\partial_l A_l))\psi_m - i\sum_{l=1}^{n} Im(\overline{\psi_l}\psi_m)\psi_l. \tag{3.21}$$

Remark 3. *In the Coulomb gauge,* $\sum_{l=1}^{n}\partial_l A_l = 0$. *However, we will not use that fact here, since this term could also be handled perturbatively using* (2.2)*, and in any case, it is instructive to perform the analysis even if we do not need to.*

Since $n \geq 3$, recall the usual Strichartz spaces

$$S^0(\mathbb{R}^n \times I) = L_t^\infty L_x^2(\mathbb{R}^n \times I) \cap L_t^2 L_x^{\frac{2n}{n-2}}(I \times \mathbb{R}^n), \tag{3.22}$$

and for any $\sigma \geq 0$,

$$S^\sigma(\mathbb{R}^n \times I) = \{f : \nabla^\sigma f \in S^0\}. \tag{3.23}$$

Let N^0 be the dual to S^0 and let N^σ be the space of functions such that $\nabla^\sigma f \in N^0$. Using the endpoint Strichartz estimates of [KT98] (since $n \geq 3$), global well–posedness for (3.21) would follow from

$$\|e^{it\Delta}\psi_{m,0}\|_{S^{\frac{n-2}{2}}(I\times\mathbb{R}^n)} \lesssim \|\psi_{m,0}\|_{\dot{H}^{\frac{n-2}{2}}} \leq C\epsilon_0, \tag{3.24}$$

and the bound

$$\|\int_0^t e^{i(t-\tau)\Delta}F(\psi_x)d\tau\|_{S^{\frac{n-2}{2}}(I\times\mathbb{R}^n)} \lesssim \|\psi_x\|^3_{S^{\frac{n-2}{2}}(I\times\mathbb{R}^n)}, \tag{3.25}$$

where ψ_x is the vector $\psi_x = (\psi_1, ..., \psi_n)^t$ and $F(\psi_x)$ is a vector with components $m = 1, ..., n$ given by

$$(A_{n+1} + \sum_{l=1}^{n}(A_l^2 - i\partial_l A_l))\psi_m - i\sum_{l=1}^{n} Im(\overline{\psi_l}\psi_m)\psi_l. \tag{3.26}$$

For any $m = 1, ..., n$, by the product rule and the fractional Leibniz rule,

$$\|i\sum_{l=1}^{n} Im(\overline{\psi_l}\psi_m)\psi_l\|_{N^{\frac{n-2}{2}}} \lesssim \|\psi_x\|^3_{S^{\frac{n-2}{2}}}. \tag{3.27}$$

By a similar argument using the Littlewood–Paley theorem, (2.2), and (2.37),

$$\| - i\psi_m \sum_{l=1}^{n}\partial_l A_l\|_{N^{\frac{n-2}{2}}} + \|A_{n+1}\psi_m\|_{N^{\frac{n-2}{2}}} \lesssim \|\psi_x\|^3_{S^{\frac{n-2}{2}}}. \tag{3.28}$$

For the term $\sum_{l=1}^{n} A_l^2\psi_m$, it is enough to prove $\|A_l\|_{S^{\frac{n-2}{2}}} \lesssim \|\psi_x\|^2_{S^{\frac{n-2}{2}}}$. Indeed, by the fractional Leibniz rule, (2.35), and the Sobolev embedding theorem,

$$\|\nabla^{\frac{n-2}{2}}A_l\|_{S^0} \lesssim \|\nabla^{\frac{n-2}{2}}\psi_x\|_{L_t^2 L_x^{\frac{2n}{n-2}}} \|\psi_x\|_{L_t^\infty L_x^n}, \tag{3.29}$$

which implies that

$$\|\psi_m \sum_{l=1}^{n} A_l^2\|_{N^{\frac{n-2}{2}}} \lesssim \|\psi_x\|_{S^{\frac{n-2}{2}}}^5 \lesssim \|\psi_x\|_{S^{\frac{n-2}{2}}}^3. \tag{3.30}$$

Remark 4. *The last inequality follows from the fact that the solution is small.*

Therefore, by Picard iteration, the proof of global well–posedness of (3.21) is complete.

3.1. **Homework problem.**

(1) Prove (3.27)–(3.30).

4. Conservation Laws

The quasilinear term $-2\sum_{l=1}^{n} A_l \partial_l \psi_m$ is more difficult in the case when A_l is at a low frequency and ψ_m is at a high frequency. In that case, the $\frac{1}{|\nabla|}$ in A_l will not cancel out the contribution of ∂_l to ψ_m. We would like to move the derivative from ψ_m to A_l. The way to do that is to use integration by parts and conservation laws.

A solution to (1.1) has two conserved quantities, the mass,

$$E_0(t) = \int |\phi(t,x) - Q|^2 dx, \tag{4.1}$$

and the energy

$$E_1(t) = \int \sum_{m=1}^{n} |\partial_m \phi(x,t)|^2 dx. \tag{4.2}$$

Indeed,

$$\frac{d}{dt} E_0(t) = 2\int \langle \phi - Q, \phi \times \Delta\phi \rangle = -2\int \langle Q, \phi \times \Delta\phi \rangle = -2\int \langle Q, \nabla(\phi \times \nabla\phi) \rangle = 0, \tag{4.3}$$

and

$$\frac{d}{dt} E_1(t) = 2\int \sum_{m=1}^{n} \langle \partial_m \phi, \partial_m(\phi \times \Delta\phi) \rangle = -2\int \langle \Delta\phi, \phi \times \Delta\phi \rangle = 0. \tag{4.4}$$

In particular, (4.4) implies that

$$\frac{d}{dt} \int \sum_{m=1}^{n} |\psi_m(x,t)|^2 dx = 0. \tag{4.5}$$

Now let P_k be the Littlewood–Paley projection to frequencies $|\xi| \sim 2^k$. Plugging (1.16) into (4.4) with $P_k \psi_m$ replaced by ψ_m,

(4.6)

$$\begin{aligned} \frac{d}{dt}\|P_k \psi_m\|_{L^2}^2 &= 2\langle P_k \psi_m, P_k(i\Delta\psi_m - 2\sum_{l=1}^{n} A_l \partial_l \psi_m - i(A_{n+1} + \sum_{l=1}^{n}(A_l^2 - i\partial_l A_l))\psi_m - \sum_{l=1}^{n} Im(\overline{\psi_l}\psi_m)\psi_l) \rangle \\ &= 2\langle P_k \psi_m, P_k(-2\sum_{l=1}^{n} A_l \partial_l \psi_m - i(A_{n+1} + \sum_{l=1}^{n}(A_l^2 - i\partial_l A_l))\psi_m - \sum_{l=1}^{n} Im(\overline{\psi_l}\psi_m)\psi_l) \rangle. \end{aligned}$$

Next, decompose

$$P_k(-2\sum_{l=1}^{n} A_l \partial_l \psi_m - i(A_{n+1} + \sum_{l=1}^{n}(A_l^2 - i\partial_l A_l))\psi_m - \sum_{l=1}^{n} Im(\overline{\psi_l}\psi_m)\psi_l) = \mathcal{N}_{k,1} + \mathcal{N}_{k,2} + \mathcal{N}_{k,3}, \tag{4.7}$$

where

$$\begin{aligned} \mathcal{N}_{k,1} &= P_k(i(A_{n+1} + \sum_{l=1}^{n}(A_l^2 - i\partial_l A_l))\psi_m - \sum_{l=1}^{n} Im(\overline{\psi_l}\psi_m)\psi_l), \\ \mathcal{N}_{k,2} &= P_k(-2\sum_{l=1}^{n} A_l \partial_l \psi_m) + 2\sum_{l=1}^{n} A_l \partial_l P_k \psi_m, \\ \mathcal{N}_{k,3} &= -2\sum_{l=1}^{n} A_l \partial_l P_k \psi_m. \end{aligned} \tag{4.8}$$

To prove Theorem 1 it suffices to prove a bound on the terms in the right hand side of (4.8).

Theorem 2. *Let α_k denote the frequency envelope*

$$\alpha_k = \sup_j 2^{-\frac{1}{10}|j-k|}(2^{j\frac{n-2}{2}}\|P_j\psi_x\|_{L^2}) = \sup_j 2^{-\frac{1}{10}|j-k|}\|P_j\psi_x\|_{\dot{H}^{\frac{n-2}{2}}}. \tag{4.9}$$

Then if I is an interval for which (1.1) is locally well–posed and $\|\psi_x\|_{L_t^\infty \dot{H}^{\frac{n-2}{2}}} \leq 2\|\psi_x(0)\|_{\dot{H}^{\frac{n-2}{2}}}$, we have the bound,

$$\|\mathcal{N}_{k,1}P_k\psi_x\|_{L^1_{t,x}(I)} + \|\mathcal{N}_{k,2}P_k\psi_x\|_{L^1_{t,x}(I)} \lesssim \epsilon^2 2^{-k(n-2)}\alpha_k^2. \tag{4.10}$$

Furthermore, for any $\sigma \in \mathbb{Z}_+$, $\sigma > \frac{n-2}{2}$, let

$$\alpha_k(\sigma) = \sup_j 2^{-\frac{1}{10}|j-k|}(2^{j\sigma}\|P_j\psi_x\|_{L^2}) = \sup_j 2^{-\frac{1}{10}|j-k|}\|P_j\psi_x\|_{\dot{H}^\sigma}. \tag{4.11}$$

Then we have the bound,

$$\|\mathcal{N}_{k,1}P_k\psi_x\|_{L^1_{t,x}(I)} + \|\mathcal{N}_{k,2}P_k\psi_x\|_{L^1_{t,x}(I)} \lesssim \epsilon^2 2^{-2k\sigma}\alpha_k(\sigma)^2. \tag{4.12}$$

Proof of Theorem 1. Using the result of [McG07], it is known that equation (1.1) is locally well–posed for sufficiently regular initial data. Let $I = [a,b]$ be the interval upon which local well–posedness holds and $\|\psi_x\|_{L_t^\infty \dot{H}^{\frac{n-2}{2}}} \leq 2\|\psi_x(0)\|_{\dot{H}^{\frac{n-2}{2}}}$. Plugging (4.10) into (4.6) and integrating by parts,

$$\|P_k\psi_x(t)\|^2_{\dot{H}^{\frac{n-2}{2}}} \leq \|P_k\psi_x(0)\|^2_{\dot{H}^{\frac{n-2}{2}}} + O(\epsilon^2\alpha_k^2). \tag{4.13}$$

Indeed, integrating by parts, the Coulomb gauge implies

$$\langle P_k\psi_x, -2\sum_l A_l\partial_l P_k\psi_x\rangle = 0. \tag{4.14}$$

Remark 5. *Even if we ignore for a moment the Coulomb gauge, after integrating by parts, a term like $(\sum_l \partial_l A_l)(P_k\psi_x)^2$ can be handled using the analysis of $\mathcal{N}_{k,1}$.*

Furthermore, by Young's inequality and (4.9), $\sum_k \alpha_k^2 \lesssim \|\psi_x(0)\|^2_{\dot{H}^{\frac{n-2}{2}}}$. Therefore, (4.13) implies that

$$\|\psi_x(t)\|_{L_t^\infty \dot{H}^{\frac{n-2}{2}}(I\times\mathbb{R}^n)} \leq \frac{3}{2}\|\psi_x(0)\|_{\dot{H}^{\frac{n-2}{2}}}. \tag{4.15}$$

Then by the results of [McG07], the interval I must be both open and closed in $\mathbb{R}$, and therefore $I = \mathbb{R}$. $\square$

The estimates (4.13) rely heavily on bilinear estimates. Moreover, due to the presence of the quasilinear term $-\sum_l A_l \partial_l \psi_x$, the proof will utilize the bilinear interaction Morawetz estimates of [PV09], to take advantage of the conservation laws to integrate by parts.

To see this, define the Morawetz potential

(4.16)
$$M(t) = \int\int |u(t,y)|^2 \frac{(x-y)}{|x-y|} \cdot Im[\bar{v}\nabla v](t,x)dxdy + \int\int |v(t,y)|^2 \frac{(x-y)}{|x-y|} \cdot Im[\bar{u}\nabla u](t,x)dxdy,$$

where $u = P_j \psi_x$ and $v = P_k \psi_x$, j and k could be, but need not be, equal. By direct computation, if $u = v$,

$$\begin{aligned} \int_0^T \frac{d}{dt} M(t) &= 2\| |\nabla|^{\frac{3-n}{2}} |u(t,x)|^2 \|_{L^2_{t,x}([0,T]\times\mathbb{R}^n)}^2 \\ &+ \int_0^T \int |v(t,y)|^2 \frac{(x-y)}{|x-y|} \cdot Im[\bar{v}\nabla(\mathcal{N}_{k,1} + \mathcal{N}_{k,2} + \mathcal{N}_{k,3})](t,x)dxdydt \\ &+ \int_0^T \int |v(t,y)|^2 \frac{(x-y)}{|x-y|} \cdot Im[\overline{\mathcal{N}_{k,1} + \mathcal{N}_{k,2} + \mathcal{N}_{k,3}}\nabla v](t,x)dxdydt \\ &+ 2\int_0^T \int Re[\bar{v}(\mathcal{N}_{k,1} + \mathcal{N}_{k,2} + \mathcal{N}_{k,3})](t,y) \frac{(x-y)}{|x-y|} \cdot Im[\bar{v}\nabla v](t,x)dxdydt. \end{aligned} \tag{4.17}$$

Now then, by (4.10), the contribution of $\mathcal{N}_{k,1}$ and $\mathcal{N}_{k,2}$ to (4.17) can be bounded by $2^k \epsilon^2 \alpha_k^4$. Meanwhile, integrating by parts and using the Hardy–Littlewood–Sobolev inequality and Hardy's inequality, the contribution of $\mathcal{N}_{k,3}$ is given by

(4.18)
$$\begin{aligned} &-\int |v(t,y)|^2 \frac{(x-y)}{|x-y|} \cdot Im[\bar{v}\nabla \sum_l A_l \partial_l v]dxdydt + \int |v(t,y)|^2 \frac{(x-y)}{|x-y|} \cdot Im[\overline{-\sum_l A_l \partial_l v}\nabla v]dxdydt \\ &+2\int Re[\bar{v}(-\sum_l A_l \partial_l v)] \frac{(x-y)}{|x-y|} \cdot Im[\bar{v}\nabla v]dxdydt \\ &= O(\int |v(t,y)|^2 \frac{1}{|x-y|}(A(t,x) + A(t,y))|v(t,x)|^2 |\nabla v(t,x)|dxdydt) \\ &+O(\int |v(t,y)|^2 (\partial A(t,y) + \partial A(t,x))|v(t,x)||\nabla v(t,x)|dxdydt) \lesssim 2^k \epsilon^2 \alpha_k^4. \end{aligned}$$

Therefore, we have proved

$$\|P_k \psi_x\|_{L^4_{t,x}}^4 \lesssim \alpha_k^4 2^{k(n-2)}. \tag{4.19}$$

It is possible to wring even more information out of the interaction Morawetz estimates when j is not equal to k. Specifically, using the arguments in [PV09] it is possible to replace $2\| |\nabla|^{\frac{3-n}{2}} |u(t,x)|^2 \|_{L^2_{t,x}([0,T]\times\mathbb{R}^n)}^2$ with

$$\int_\omega \int_{x_\omega = y_\omega} |\partial_\omega (\overline{v(t,y)} u(t,x))|^2 dxdydtd\omega. \tag{4.20}$$

For any fixed ω, using the Littlewood–Paley potential, (4.20) combined with (4.18) implies that if $j \leq k-5$,

$$\int_{\omega} \int |\partial_{\omega}(\overline{v(t,x)}u(t,x))|^2 dx dt d\omega = \int |\nabla(\overline{v(t,x)}u(t,x))|^2 dx dy \lesssim 2^k 2^{j(n-1)} \alpha_j^2 \alpha_j^2. \tag{4.21}$$

The proof of (4.10) will occupy the last lecture.

4.1. **Homework problem.**

(1) Prove the Littlewood–Paley theorem.

5. Proof of Theorem 2

First compute

$$\begin{aligned} \|(P_k\psi_x)P_k(\psi_x^3)\|_{L^1_{t,x}} &\lesssim \|P_k\psi_x\|_{L^4_{t,x}} \|P_{\geq k-5}\psi_x\|^3_{L^4_{t,x}} + \|P_{\geq k-5}\psi_x\|^2_{L^4_{t,x}} \|(P_k\psi_x)(P_{\leq k-5}\psi_x)\|_{L^2_{t,x}} \\ &+ \|(P_k\psi_x)(P_{\leq k-5}\psi_x)\|_{L^2_{t,x}} \|(P_{k-5\leq \cdot \leq k+5}\psi_x)(P_{\leq k-5}\psi_x)\|_{L^2_{t,x}} \\ &\lesssim \|P_{\geq k-10}\psi_x\|^4_{L^4_{t,x}} + \sum_{k-5\leq k_1\leq k+5} \|(P_{k_1}\psi_x)(P_{\leq k_1-5}\psi_x)\|^2_{L^2_{t,x}}. \end{aligned} \tag{5.1}$$

Replacing $\mathcal{N}_{k,1} + \mathcal{N}_{k,2} + \mathcal{N}_{k,3}$ with $(P_k\psi_x)P_k(\psi_x^3)$, and plugging into (4.17)–(4.20), we have proved that for any $j \leq k$,

$$\begin{aligned} \|(P_j\psi_x)(P_k\psi_x)\|_{L^2_{t,x}} &\lesssim 2^{j\frac{n-1}{2}} 2^{-\frac{k}{2}} \|P_j\psi_x\|_{L^\infty_t L^2_x} \|P_k\psi_x\|_{L^\infty_t L^2_x} \\ +2^{j\frac{n-1}{2}} 2^{-\frac{k}{2}} \|P_j\psi_x\|_{L^\infty_t L^2_x} \|P_{\geq k-10}\psi_x\|^2_{L^4_{t,x}} &+ 2^{j\frac{n-1}{2}} 2^{-\frac{k}{2}} \|P_k\psi_x\|_{L^\infty_t L^2_x} \|P_{\geq j-10}\psi_x\|^2_{L^4_{t,x}} \\ &+2^{j\frac{n-1}{2}} 2^{-\frac{k}{2}} \|P_j\psi_x\|_{L^\infty_t L^2_x} \sum_{k-5\leq k_1\leq k+5} \|(P_{k_1}\psi_x)(P_{\leq k_1-5}\psi_x)\|_{L^2_{t,x}} \\ &+2^{j\frac{n-1}{2}} 2^{-\frac{k}{2}} \|P_k\psi_x\|_{L^\infty_t L^2_x} \sum_{j-5\leq j_1\leq j+5} \|(P_{\geq j_1}\psi_x)(P_{\leq j_1-5}\psi_x)\|_{L^2_{t,x}}. \end{aligned} \tag{5.2}$$

Now then, if I is an interval upon which the local well–posedness result of [McG07] holds,

$$\|P_k\psi_x\|_{\dot{H}^\sigma} \lesssim \alpha_k(\sigma). \tag{5.3}$$

Furthermore, since $\psi_x(0) \in H^\infty$, if $|I|$ is finite,

$$\sum_k 2^{2k\sigma} \|(P_k\psi_x)(P_{\leq k-5}\psi_x)\|^2_{L^2_{t,x}(I)} < \infty. \tag{5.4}$$

Summing (5.2) in j and using (4.9) and the trivial bound $\alpha_k \lesssim \epsilon$,

$$\begin{aligned} \|P_k\psi_x\|^2_{L^4_{t,x}} + \|(P_k\psi_x)(P_{\leq k-5}\psi_x)\|_{L^2_{t,x}} &\lesssim 2^{-k\frac{n-2}{2}} \epsilon\alpha_k + \alpha_k \|P_{\geq k-10}\psi_x\|^2_{L^4_{t,x}} \\ +\alpha_k \sum_{k-5\leq k_1\leq k+5} \|(P_{k_1}\psi_x)(P_{\leq k_1-5}\psi_x)\|_{L^2_{t,x}} &+ 2^{-k\frac{n-1}{2}} \alpha_k \sum_{j\leq k-5} 2^{j\frac{n-1}{2}} \|P_{\geq j-10}\psi_x\|^2_{L^4_{t,x}} \\ &+2^{-k\frac{n-1}{2}} \alpha_k \sum_{j\leq k-5} 2^{j\frac{n-1}{2}} \sum_{j-5\leq j_1\leq j+5} \|(P_{j_1}\psi_x)(P_{\leq j_1-5}\psi_x)\|_{L^2_{t,x}}. \end{aligned} \tag{5.5}$$

Squaring both sides, multiplying by $2^{k(n-2)}$, and summing using Young's inequality,

$$\begin{aligned}&\sum_k 2^{k(n-2)}\|P_k\psi_x\|_{L^4_{t,x}}^4+2^{k(n-2)}\|(P_k\psi_x)(P_{\leq k-5}\psi_x)\|_{L^2_{t,x}}^2\\ \lesssim\ &\epsilon^4+\epsilon^2\sum_k 2^{k(n-2)}\|P_{\geq k}\psi_x\|_{L^4_{t,x}}^4+\epsilon^2\sum_k 2^{k(n-2)}\|(P_k\psi_x)(P_{\leq k-5}\psi_x)\|_{L^2_{t,x}}^2.\end{aligned}\tag{5.6}$$

Therefore, for $\epsilon>0$ sufficiently small,

$$\sum_k 2^{k(n-2)}\|P_k\psi_x\|_{L^4_{t,x}}^4+2^{k(n-2)}\|(P_k\psi_x)(P_{\leq k-5}\psi_x)\|_{L^2_{t,x}}^2\lesssim\epsilon^4.\tag{5.7}$$

Plugging (5.7) into (5.2),

$$\|(P_k\psi_x)(P_{\leq k-5}\psi_x)\|_{L^2_{t,x}}\lesssim 2^{-k\frac{n-2}{2}}\epsilon\alpha_k,\tag{5.8}$$

and therefore,

$$(5.1)\lesssim\epsilon^2\alpha_k^2.\tag{5.9}$$

Now turn to (2.37). The contribution of $\sum_{m=1}^n\psi_m\overline{\psi_m}$ to $\|(P_k\psi_x)P_k(A_{n+1}\psi_x)\|_{L^1_{t,x}}$ is identical to the contribution of (5.1). Estimating the contribution of

$$\|(P_k\psi_x)P_k(\sum_{l,m=1}^n R_lR_m(Re(\overline{\psi_l}\psi_m))\psi_x)\|_{L^1_{t,x}}\tag{5.10}$$

is complicated by the Riesz transforms. Decompose

$$\overline{\psi_l}\psi_m=(P_{\leq k-5}\psi_x)^2+2(P_{\leq k-5}\psi_x)(P_{\geq k-5}\psi_x)+(P_{\geq k-5}\psi_x)^2.\tag{5.11}$$

The analysis of the second two terms is straightforward. Indeed,

$$\begin{aligned}&\|(P_k\psi_x)P_k(R_lR_m((P_{\leq k-5}\psi_x)(P_{\geq k-5}\psi_x))\cdot\psi_x)\|_{L^1_{t,x}}\\ \lesssim&\sum_{k-5\leq k_1\leq k+5}\|(P_{k_1}\psi_x)(P_{\leq k_1-5}\psi_x)\|_{L^2_{t,x}}\|(P_k\psi_x)(P_{\leq k-5}\psi_x)\|_{L^2_{t,x}}\\ &+\|(P_k\psi_x)(P_{\leq k-5}\psi_x)\|_{L^2_{t,x}}\|(P_{\geq k-10}\psi_x)\|_{L^4_{t,x}}^2,\end{aligned}\tag{5.12}$$

and

$$\|(P_k\psi_x)P_k(R_lR_m((P_{\geq k-5}\psi_x)^2)\psi_x)\|_{L^1_{t,x}}\lesssim\|P_k\psi_x\|_{L^4_{t,x}}\|P_{\geq k-5}\psi_x\|_{L^4_{t,x}}^3+\|(P_k\psi_x)(P_{\leq k-5}\psi_x)\|_{L^2_{t,x}}\|P_{\geq k-5}\psi_x\|_{L^4_{t,x}}^2.\tag{5.13}$$

For the term

$$\|(P_k\psi_x)P_k(R_lR_m((P_{\leq k-5}\psi_x)^2)\psi_x)\|_{L^1_{t,x}},\tag{5.14}$$

make a paraproduct of $(P_{\leq k-5}\psi_x)^2$,

$$(P_{\leq k-5}\psi_x)^2=2\sum_{j\leq k-5}(P_j\psi_x)(P_{\leq j-5}\psi_x)+\sum_{j\leq k-5}\ \sum_{j-5\leq j_1\leq\min\{j+5,k-5\}}(P_j\psi_x)(P_{j_1}\psi_x).\tag{5.15}$$

Since the term $(P_j\psi_x)(P_{\leq j-5}\psi_x)$ is localized to frequency $|\xi|\sim 2^j$, we can treat the Riesz projections as L^1 convolution kernels. Since the bilinear estimates are translation invariant, the same bounds hold for the first term in the paraproduct. For the second term, we may use the fact that bounds in (4.21) are smaller at lower frequencies. That is,

$$\|(P_k\psi_x)^2P_j((P_{j_1}\psi_x)^2)\|_{L^1_{t,x}}\lesssim 2^{j(n-1)}2^{-k}\alpha_k^2\alpha_{j_1}^2.\tag{5.16}$$

Thus, similar bounds hold for

$$\|(P_k\psi_x)P_k(A_{n+1}\psi_x)\|_{L^1_{t,x}}, \tag{5.17}$$

as hold for (5.1). The contribution of $P_k(-\sum_l A_l\partial_l\psi_x)$ is identical to the contribution of $A_{n+1}\psi_x$.

For the contribution of $\sum_{l=1}^n A_l^2\psi_x$, decompose

$$A_x \sim \frac{1}{|\nabla|}((P_{\leq k-5}\psi_x)^2 + 2(P_{\leq k-5}\psi_x)(P_{\geq k-5}\psi_x) + (P_{\geq k-5}\psi_x)^2). \tag{5.18}$$

By the Sobolev embedding theorem,

$$\|\frac{1}{|\nabla|}(2(P_{\leq k-5}\psi_x)(P_{\geq k-5}\psi_x)+(P_{\geq k-5}\psi_x)^2)\|_{L^4_{t,x}} \lesssim \|P_{\geq k-5}\psi_x\|_{L^4_{t,x}}\|\psi_x\|_{L_t^\infty \dot{H}^{\frac{n-2}{2}}} \lesssim \epsilon\|P_{\geq k-5}\psi_x\|_{L^4_{t,x}}. \tag{5.19}$$

On the other hand, using analysis similar to (5.16),

$$\|(P_k\psi_x)\frac{1}{|\nabla|}(P_{\leq k-5}\psi_x)^2\|_{L^2_{t,x}} \lesssim 2^{-k\frac{n-2}{2}}\epsilon^2\alpha_k, \tag{5.20}$$

and

$$\|\frac{1}{|\nabla|}((P_{\geq k-5}\psi_x)\psi_x)\cdot\frac{1}{|\nabla|}((P_{\leq k-5}\psi_x)^2)\|_{L^2_{t,x}} \lesssim 2^{-k\frac{n-2}{2}}\epsilon^3\alpha_k. \tag{5.21}$$

This takes care of the contribution of $\mathcal{N}_{k,1}$ to (5.10).

For $\mathcal{N}_{k,2}$, observe that

$$P_k(P_{\geq k-5}A_l\cdot\partial_l\psi_x) - P_{\geq k-5}A_l\cdot\partial_l P_k\psi_x \sim P_k(P_{\geq k-5}\nabla A_l\cdot\psi_x) - P_{\geq k-5}\nabla A_l\cdot P_k\psi_k, \tag{5.22}$$

which can be handled in a manner similar to $\mathcal{N}_{k,1}$. Finally, to handle

$$P_k(P_{\leq k-5}A_l\cdot\partial_l\psi_x) - P_{\leq k-5}A_l\cdot\partial_l P_k\psi_x, \tag{5.23}$$

observe that if $\phi(2^{-k}\xi)$ is the Fourier multiplier for P_k, the Fourier transform of (5.23) is given by

$$\begin{aligned}&\int_{\eta_1+\eta_2=\xi}\phi(2^{-k}\xi)\widehat{A_l}(\eta_1)\eta_{2,l}\widehat{\psi_x}(\eta_2) - \int_{\eta_1+\eta_2=\xi}\phi(2^{-k}\eta_2)\widehat{A_l(\eta_1)}\eta_{2,l}\widehat{\psi_x}(\eta_2)\\ &\lesssim \int 2^{-k}|\eta_1|\widehat{A_l}(\eta_1)\eta_{2,l}\widehat{\psi_x}(\eta_2) \sim (\nabla P_{\leq k-5}A)\widetilde{P_k}\psi_x,\end{aligned} \tag{5.24}$$

where $\widetilde{P_k} = P_{k-1} + P_k + P_{k+1}$. Thus, the contribution of $\mathcal{N}_{k,2}$ has a similar bound. The proof is therefore complete.

References

[BIK07] Ioan Bejenaru, Alexandru D Ionescu, and Carlos E Kenig. Global existence and uniqueness of schrödinger maps in dimensions $d \geq 4$. *Advances in Mathematics*, 215(1):263–291, 2007.

[CSU00] Nai-Heng Chang, Jalal Shatah, and Karen Uhlenbeck. Schrödinger maps. *Communications on Pure and Applied Mathematics: A Journal Issued by the Courant Institute of Mathematical Sciences*, 53(5):590–602, 2000.

[Dod15] Benjamin Dodson. Global well-posedness and scattering for the mass critical nonlinear Schrödinger equation with mass below the mass of the ground state. *Advances in mathematics*, 285:1589–1618, 2015.

[KPV93] Carlos E Kenig, Gustavo Ponce, and Luis Vega. Well-posedness and scattering results for the generalized Korteweg-de Vries equation via the contraction principle. *Communications on Pure and Applied Mathematics*, 46(4):527–620, 1993.

[KT98] Markus Keel and Terence Tao. Endpoint Strichartz estimates. *American Journal of Mathematics*, 120(5):955–980, 1998.

[McG07] Helena McGahagan. An approximation scheme for Schrödinger maps. *Communications in Partial Differential Equations*, 32(3):375–400, 2007.

[NSU03] Andrea Nahmod, Atanas Stefanov, and Karen Uhlenbeck. On Schrödinger maps. *Communications on Pure and Applied Mathematics: A Journal Issued by the Courant Institute of Mathematical Sciences*, 56(1):114–151, 2003.

[PT91] N Papanicolaou and TN Tomaras. Dynamics of magnetic vortices. *Nuclear Physics B*, 360(2-3):425–462, 1991.

[PV09] Fabrice Planchon and Luis Vega. Bilinear virial identities and applications. In *Annales scientifiques de l'Ecole normale supérieure*, volume 42, pages 261–290, 2009.

[Tay00] Michael E Taylor. *Tools for PDE: pseudodifferential operators, paradifferential operators, and layer potentials*. Number 81. American Mathematical Soc., 2000.

[TH13] Dirk Ter Haar. *Collected papers of LD Landau*. Elsevier, 2013.