Applications of the Endoscopic Classification to Statistics of Cohomological Automorphic Representations on Unitary Groups

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• Anything in gray is a technical detail not relevant to this particular topic
• Anything in orange I will only explain intuitively and imprecisely due to time constraints.
Outline

- Motivation: Understanding $AR_{\text{disc}}$.
- Statement of Results
- Background: Arthur’s Classification
- Background: Taïbi’s Inductive Analysis
- Tricks for computation

See ArXiv for details.

WARNING: This work depends on Arthur’s classification for non-quasisplit unitary groups! This uses unpublished/unwritten references.
What is an Automorphic Representation?

Modular Forms:
- Functions on upper-half plane $\text{symmetric space } \text{GL}_2^\mathbb{R}/\text{O}_2^\mathbb{R}$
- w/ symmetries translation by “arithmetic” lattice in $\text{GL}_2^\mathbb{R}$

Automorphic Representations: generalize beyond $\text{GL}_2$
- Exact generalization very non-obvious: black box for this talk
- Representations: notion of newform doesn’t generalize, analog of space generated by newform
Why do we care?

Just like modular forms:

- They have a lot of handles to grab onto when studying
  - representation theory of reductive groups
  - harmonic analysis
- They mysteriously encode information about so much else:
  - **Number Theory**: Galois representations (Langlands conjectures)
  - **Computer Science**: expander graphs/higher-dimensional expanders
  - **Differential Geometry**: spectra of Laplacians on locally symmetric spaces
  - **Combinatorics**: identities for the partition function
  - **Finite Groups**: representation theory of large sporadic simple groups (moonshine)
  - **Mathematical Physics**: representations of infinite-dimensional Lie algebras, certain scattering amplitudes in string theory
Black-Box Definition

Definition

Let $G$ be a reductive group over a number field $F$. A discrete automorphic representation for $G$ is an irreducible subrepresentation of $L^2(G(F) \backslash G(\mathbb{A}_F), \chi)$.

- **Reductive group**: algebraic group with nice representation theory (root and weight theory works).
  - ex. $\text{GL}_n, \text{SL}_n, \text{U}_n, \text{SO}_n, \text{Sp}_n$.
  - Non ex. Upper triangular matrices.

- $L^2$: square-integrable functions as a unitary representation of $G(\mathbb{A}_F)$ under right-translation.

- $\mathbb{A}_F = \prod_v F_v$ ($\mathbb{A}_Q = \mathbb{R} \times \prod_p \mathbb{Q}_p$)

- **Intuition**: $\mathbb{Z}$ is to $\mathbb{R}$ as $F$ is to $\mathbb{A}_F$.

- **Subrepresentation**: analysis issue—infinite-dimensional representations can be direct integrals instead of direct sums.
Perspective on Automorphic Representations

- What does $G$ do?
  - $G_\infty$: determines symmetric space $G_\infty/K_\infty$
  - $G^\infty$: determines possible lattices $\Gamma$: “Levels”

- Factor into local components:

$$\pi = \bigotimes'_{v} \pi_v,$$

- $\pi_\infty$: “qualitative type” of the representation: modular vs. Maass, holomorphic, algebraic, cohomological.
- $\pi^\infty$: information analogous to level and Hecke eigenvalues
Key Problem: Which combinations of $\pi_v$ actually produce an automorphic representation?

- e.g. which combinations of Hecke eigenvalues do the modular forms of weight $k$ and level $N$ have?

Most Basic Version: counts/statistics w/ local restrictions

- e.g. what fraction modular forms of weight $k$ have Hecke eigenvalue at $p$ with norm bigger than something as level $N \to \infty$?
Complexity Ranking

Informal ranking of complexity based on qualitative type $\pi_\infty$:

- **Discrete-at-$\infty$**: $\pi_\infty$ discrete inside $L^2(G(F_\infty))$.
- **Cohomological**: $\pi_\infty$ regular, integral infinitesimal character
- **Algebraic**: $\pi_\infty$ integral infinitesimal character
- **General**: all $\pi_\infty$

Different application need different generality:

- Cohomology of locally symmetric spaces
- Galois Representations
Example: Modular Forms

Fix $G = \text{GL}_2/\mathbb{Q}$

- Automorphic Representations on $G \approx$ classical modular and Maass forms
- Discrete-at-$\infty$: modular forms of weight $\geq 2$
- Cohomological: add in the trivial rep, (there is more to add on other groups)
- Algebraic: add in weight 1 modular and Maass forms
- General: add in other Maass forms
Answering Key Question

How far can we go? **Basic Version:** use Arthur’s trace formula

- **Discrete-at-∞:** coarse info. [Art89], fine info. [Fer07].
  - Need: **orbital integrals, endoscopic transfers**
  - Exact counts: many, many results for low level on small rank
  - Statistics: most powerful/general [ST16] coarse, [Dal22] fine

- **Cohomological:** inductive arg. w/ endoscopic class. [Tai17]
  - Need: **orbital integrals, endoscopic transfers, stable transfers**
  - Exact counts: [Tai17] +Chenevier, Renard, Taïbi at level-1
  - Statistics: [MS19] + Marshall, Gerbelli-Gauthier upper bounds, *this work* many exact asymptotics and more upper bounds

- **Beyond:** very hard—asymptotic counts not known even for weight-1 modular forms :'( 
Consider:

- Symmetric space $X = U(p, q)/(U(p) \times U(q))$
- A specific type of tower of arithmetic lattices $\cdots \subseteq \Gamma_2 \subseteq \Gamma_1$
- $h_n^i := H^i(\Gamma_n \backslash X, V_\lambda) = H^i(g, K; C_\infty(\Gamma_n \backslash G(\mathbb{R})) \otimes V_\lambda)$ as reps of $U(p, q)$.

**Problem:** Given $\pi_0$ unirrep of $G(R)$, understand asymptotics of count of $\pi_0 \in h_n^i$ weighted by arbitrary moment of Satake parameters.

- Analogue: weight-2 modular forms in $H^1(\Gamma(N))$ weighted by power of Hecke eigenvalue
- **Matsushima’s formula:** translate to counting $\pi \in \mathcal{AR}_{\text{disc}}(G)$ with $\pi_\infty = \pi_0$. 
Main Result

Theorem

Let $E/F$ be an unram. CM-extension and $G$ an unram. inner form of $U_{E/F}(N)$. Fix $\pi_0$ cohom. on $G_{\infty}$. Let $\mathfrak{n}$ be an ideal of $\mathcal{O}_F$ only divisible by primes split in $E/F$ and $f_S$ an unram. test function at some set of places $S$ not dividing $\mathfrak{n}$. Then for good $\pi_0$

$$|\mathfrak{n}|^{-R(\pi_0)} L_{\pi_0}(\mathfrak{n})^{-1} \sum_{\pi \in \mathcal{AR}_{\text{disc}}(G)} \dim((\pi_{\infty})^{K(\mathfrak{n})}) \text{tr}_{\pi_S} f_S$$

$$= M(\pi_0) \mu_S^{pl}(\pi_0)(f_S) + O(|\mathfrak{n}|^{-C} q_S^{A+B\kappa(f_S)}).$$

- There are some strong conditions: $E/F$, level, and $\pi_0$
- Good $\pi_0$: Explicit: combinatorial data classifying $\pi_0$. 

Theorem
Main Result Cont.

\[ |n|^{-R(\pi_0)} L_{\pi_0}(n)^{-1} \sum_{\pi \in \mathcal{AR}_{\text{disc}}(G), \pi_{\infty} = \pi_0} \dim((\pi_{\infty})^K(n)) \text{tr}_{\pi_S} f_S \]

\[ = M(\pi_0) \mu_S^{\text{pl}(\pi_0)}(f_S) + O(|n|^{-C} q_S A + B \kappa(f_S)). \]

- Asymptotic in \( n, S, f_S \)
- \( n \): Counting fixed vectors in aut. reps with component \( \pi_{\infty} = \pi_0 \) (i.e. aut. forms of level \( n \))
- \( f_S \): averaging a Satake parameter over these forms (e.g. moment of Hecke eigenvalue)
- Constants: combo. param. of \( \pi_0 \), Plancherel equidistribution
- Constants: Inexplicit
Example: parallel $U(N - 1, 1)$

Assume $\deg F/\mathbb{Q} = d$, $G_{\infty} \cong U(N - 1, 1)^d$ (if possible) $\pi_0 \cong \pi^d$

- Cohomological Reps of $U(N - 1, 1)$ at inf. char of trivial:
  - ordered partitions $(a_1, \ldots, a_k)$ of $N$
  - one marked index $1 \leq m \leq k$, $a_i = 1$ for $i \neq m$.
  - Discrete series: all $a_i = 1$.

- “good” class: $a_m$ is odd

- If $\pi_0$ d.s. $R(\pi_0) = N^2$, $M(\pi_0) = 1$. Otherwise:

$$R(\pi_0) = \frac{1}{2}(N^2 + (N - a_m)^2 - a_m^2) + 1$$

$$M(\pi_0) = \begin{cases} 
N^{-d} \dim(\pi_{a_m\lambda_{m-1}})\tau'(G) & d \text{ even or } m \text{ correct parity} \\
0 & d \text{ odd and } m \text{ wrong parity}
\end{cases}$$

$(\pi_{a_m\lambda_{m-1}}$; f.d. rep. of $GL_{N-a_m}$, $\lambda_i$: $i$th fundamental weight)

- Vary $m$: different masses, growth rates
Main Result: other $\pi_0$

Remove conditions $\implies$ upper bound instead of exact asymptotic:

**Theorem**

*Recall the setup for the main result except $E/F$ can be ramified.* Let $S_0$ be a set of places containing all the ramified ones and disjoint from $S$ and $\mathfrak{n}$. Let $\varphi_{S_0}$ be a test function on $G_{S_0}$. Then for all $\pi_0$:

$$
\sum_{\pi \in \mathcal{AR}_{\text{disc}}(G), \pi_{\infty} = \pi_0} \dim((\pi^\infty)^K(n_i)) \operatorname{tr}_{\pi_S} f_S \operatorname{tr}_{\pi_{S_0}} \varphi_{S_0} = O(|n_i|^{R(\pi_0)} q_{S_1}^{A+B\kappa(f_S)}).
$$
Corollaries

This gives us many corollaries:

- **Sato-Tate equidistribution in families**
  - GL$_2$ version: Hecke eigenvalues over all primes over all of $S_k(N)$ follow semicircle rule
  - Prove: expectation from interpreting $\pi$ with $\pi_\infty = \pi_0$ as *non-endoscopic* functorial transfers from smaller group depending on $\pi_0$

- **Sarnak density**
  - $R(\pi_0)$ achieves a certain bound depending on matrix coefficient decay of $\pi_0$, useful in analytic number theory applications
  - Prove: for all cohomological $\pi_0$ except a single rep. on $U(2, 2)$

- **Growth rates of** $H^{p,q}$ **of towers of locally symmetric spaces**
  - *Exact asymptotics*: e.g. every other degree for $U(N, 1)$ with certain towers of lattices
Overview

Goal: Parametrize discrete automorphic representations for $G$ in terms of all automorphic representations on $\GL_n$.

$\implies$ Known info on $\GL_n$ gives info on $G$

- Moeglin-Waldspurger classification in terms of cuspidals
- Local Langlands

Stated in terms of two key concepts:

- **Parameters**: $\psi$: reps on $\GL_n$ encoded in a way to emphasize known info
- **Packets**: $\psi \mapsto \Pi_\psi$: subsets of $\mathcal{AR}_{\text{disc}}(G)$ with determined structure of local components

$G$ can be: $\SO_n$ or $\Sp_{2n}$ (Arthur), q-split $U_{E/F}(N)$ (Mok), General unitary groups [KMSW14].
Some details:

**Definition**

An elliptic $A$-parameter for $U_{E/F,+}(N)$ is a formal sum

$$\psi = \bigoplus_i \tau_i[d_i]$$

where each $\tau_i$ is a conjugate self-dual cuspidal automorphic representation of $\text{GL}_{t_i}/E$ and $\sum_i t_i d_i = N$ and each $\tau_i$ has the appropriate parity.

- $\psi$ determines local parameters $\psi_v$ by LL + lots of work

$$\psi_v : L_{F_v} \times \text{SL}_2 \to L_{U_{E/F}}(N) : \bigoplus_i LL(\tau_{i,v}) \boxtimes [d_i]$$
Some details:

**Theorem (KMSW classification)**

Let $G$ be an extended pure inner form of $G^* = U_{E/F}(N)$. To each elliptic parameter $\psi$ of $U_{E/F}(N)$, there is an associated packet $\Pi^G_\psi \subseteq \mathcal{AR}_{\text{disc}}(G)$ such that for any test function $f$ on $G(\mathbb{A})$:

$$\text{tr}_{\mathcal{AR}_{\text{disc}}(G)}(f) = \sum_{\psi \in \Psi_{\text{ell}}(G^*)} l_\psi(f) := \sum_{\psi \in \Psi_{\text{ell}}(G^*)} \sum_{\pi \in \Pi^G_\psi} \text{tr}_\pi(f)$$

- $\Pi_\psi$ is a subset of a restricted product of local packets $\Pi_{\psi_v}$ determined by a multiplicity formula.
Stable Multiplicity

\( I_\psi \): summands of Arthur’s \( I_{\text{disc}} \rightarrow S_\psi \): summands of \( S_{\text{disc}} \)

- **Stabilization**: \( I_\psi^G = \sum_{H, \psi^H} S_{\psi^H}^H \), \( H \) smaller endoscopic groups

**Formula:**

\[
S_{\psi}^H(f) = \epsilon_\psi C_\psi \text{tr}_\psi(f)
\]

- very difficult sign attached to \( \psi \)
- easy constant attached to \( \psi \)
- Stable trace \( \sum_{\pi \in \Pi_\psi} \pm \text{tr}_\pi(f) \).
  - related to trace of a rep \( \pi_\psi \) on some twisted \( \text{GL}_n \)
  - \( \pi_\psi \) explicitly described as Langlands quotient of \( \pi_{\tau_i} \) with very complicated twist
AJ-packets

We care about a special kind of packet at $\infty$:

- Parameters $\psi_{\infty}$ at $\infty$ have associated infinitesimal characters.
- If the infinitesimal character is regular integral, then $\Pi_{\pi_{\infty}}$ is an Adams-Johnson packet $\implies$ explicit combinatorial description of elements.
- Exactly that packets that contain cohomological representations.
- **Key property**: for cohom. $\pi_0$, there exists pseudocoefficient $\varphi$ such that among the $\pi$ that share an $A$-packet with $\pi_0$:

$$\text{tr}_{\pi} \varphi = 1_{\pi=\pi_0}$$
The inductive analysis depends on a key definition:

**Definition**

The refined shape $\Delta$ of $A$-parameter

$$\psi = \bigoplus_i \tau_i[d_i]$$

is $\Delta = (T_i, d_i, \lambda_i, \eta_i)_i$ where

- $T_i$ is the dimension of $\tau_i$
- $\lambda_i$ is the infinitesimal character of $\tau_{i,\infty}$.

**Key Property:** $\Delta$ determines $\psi_\infty$ among AJ-params if $\lambda_i$ regular integral
Step 1: Induction Setup

Let $\psi_{i,\infty}$ be list of AJ-parameters such that $\pi_0 \in \prod_{\psi_{i,\infty}}$. Let $\Delta(\pi_0)$ be the set of $\Delta$ that determine $\psi_\infty$ to be one of the $\psi_{i,\infty}$:

$$\sum_{\pi \in \text{AR}_{\text{disc}}(G)} \text{tr}_{\pi_\infty}(f_\infty) = \sum_{\Delta \in \Delta(\pi_0)} I_\Delta(\varphi f_\infty)$$

where

$$I_\Delta(f) := \sum_{\psi \in \Delta} \sum_{\pi \in \prod_{\psi}} \text{tr}_{\pi}(f)$$

- **Stabilization + hyperendoscopy**: Can switch freely between $I_\Delta(\varphi f_\infty), S_\Delta(EP_\lambda f_\infty)$ by adding lower order terms in $n_i$.
- **Goal**: Understand $S_\Delta(EP_\lambda f_\infty)$ for shapes $\Delta$. 
Induction: Base Case

What is the base case at the bottom?

- Arthur’s simple trace formula: Euler-Poincaré function $\text{EP}_\lambda$

\[
I^H(\text{EP}_\lambda f^\infty) = \sum_{\pi \in AR_{\text{disc}}(H) \atop \text{inf. char. } \pi^\infty = \lambda} \mathcal{L}(\pi^\infty) \text{tr}_{\pi^\infty}(f^\infty)
\]

(similar result holds for pseudocoefficient $\varphi$).

- Shin-Templier’s analysis: geometric expression for $I^H(\text{EP}_\lambda f^\infty)$ can be bounded very explicitly (error terms as in main theorem)

\[
f^\infty = 1_{K(n_i)} f_{S_1} \implies \text{tr}_{\pi^\infty}(f^\infty) = \dim((\pi^\infty)^K(n_i)) \text{tr}_{\pi_{S_1}} f_{S_1}.
\]

- Recall: we don’t care $S^H$ vs. $I^H$
The Induction: Heuristic Dream

Trivial Shape: $\Sigma_{\lambda, \eta} = (T, 1, \lambda, \eta)$, cuspidal parameters on $GL_n$:

\[
S^H_{\Sigma, \lambda} (EP_{\lambda} f^\infty) = S^H(EP_{\lambda} f^\infty) - \sum_{\Delta \neq \Sigma, \text{inf. char. } \Delta = \lambda} S^H_{\Delta}(EP_{\lambda} f^\infty)
\]

- “Just” need to reduce $S^H_{\Delta}$ to $S^H_{\Sigma}$ for smaller $H_i$.
- Step 1: “Stable transfer” $\epsilon \text{tr} \bigoplus_i \tau_i[d_i] f = \prod_i \text{tr} \tau_i[d_i] f_i$
- Step 2: “Speh transfer” $\text{tr} \tau_i[d_i] f_i = \text{tr} \tau_i f_i'$

Total:

\[
S^H_{(T_i, d_i, \lambda_i)} (EP_{\lambda} f^\infty) = \prod_i S^H_{(T_i, 1, \lambda_i)} (EP_{\lambda_i} (f^\infty)')
\]
The Induction: Reality

Stable transfer and Speh transfer are hard, open problems in general :(  

- **Main work in analysis**: Find an easy special case where you can compute them!
- General idea: use relation to twisted representations on $GL_n$ and Langlands quotients
- $\Delta^{\text{max}}(\pi_0)$: shapes with dominant-in-$|n_i|$ contribution, need transfers computed exactly here
- The rest of $\Delta(\pi_0)$: error term, only need upper bounds here.
- Rest of talk: explaining which easy special case we use
The $\epsilon$-sign: $\epsilon_\psi C_\psi \text{tr}_\psi f$

For upper bounds:

- If $\psi$ has one summand, then $\epsilon_\psi = 1$ and the signs in $\text{tr}_\psi$ are all $+1$.
- $\implies$ if $\text{tr}_{\pi_\infty}(f^\infty) \geq 0$ always, can take absolute value and get upper bound

$$\text{tr}_{\bigoplus_i \tau_i[d_i]} f = \prod_i \text{tr}_{\tau_i[d_i]} f_i \implies S^H_{\bigoplus_i \tau_i[d_i]}(f) \leq \prod_i S^H_{\tau_i[d_i]}(f_i)$$

For exact computation:

- If all the $d_i$ are odd, then $\epsilon_\psi = 1$.
- **Restriction**: $\Delta^\text{max}(\pi_0)$ can only have shapes with all $d_i$ odd.
Unramified Places: $\epsilon_\psi C_\psi \tr_\psi f$

At places $\nu$ where $f_\nu$ unramified:

- $\Pi_{\psi_\nu}$ has at most one unramified member $\pi^\ur_{\psi_\nu}$. This always has coefficient $+1$ in $\tr_{\psi_\nu}$.
- $\tr_{\psi_\nu} f_\nu = \tr_{\pi^{\ur}_{\psi_\nu}} f_\nu$
- Its Satake parameters are determined explicitly by those of the unramified members in $\Pi_{\tau_i,\nu}$.

$\Rightarrow$ can compute stable and Speh transfers of $f_\nu$ dual to transfer of Satake parameters through Satake isomorphism (analogy—full fundamental lemma).
Motivation

Results

End. Class.

Ind. Analysis

Computation

Split Places: $\epsilon_{\psi} C_{\psi} \text{tr}_{\psi} f$

At split $v$, $G_v \cong \text{GL}_N(F)$

- Check: stable transfer $=$ constant term ($=$end. trans.)
- Check: $\prod_{\psi_v} \text{singleton}$: from $\pi_{\psi_v}$ from before on $\text{GL}_N(E)$.

Speh transfer upper bounds: If $\text{tr}_{\pi_v}(f_v) \geq 0$:  

- Can bound trace against Langlands quotient $\text{tr}_{\pi_\tau}[d] f_v$ by trace against parabolic induction
- $\implies$ constant term integral upper bounds

Speh transfer exact computation

- If $T_i = 1$, then $\pi_\tau[d]$ is a character $\implies$ Speh transfer is integration against $G^{\text{der}}$.
- Restriction: $\Delta^{\text{max}}(\pi_0)$ can only have shapes where all summands have either $T_i = 1$ or $d_i = 1$. 
These are the only cases we needed with our setup:

- $f^\infty$ is only ramified at split places
- The “good” class of $\pi_0$ becomes $\pi_0$ such that for $\Delta \in \Delta^{\max}(\pi_0)$
- All summands have $d_i$ odd
- All summands have $T_i = 1$ or $d_i = 1$
- There is a relatively simple equivalent combinatorial condition

**Last Technicality:** Need slightly stronger upper bounds of Marshall-Shin for $d_i = 2, 3$ to get that those terms are truly errors
Papers Mentioned


Rahul Dalal, *Sato–Tate equidistribution for families of automorphic representations through the stable trace formula*, Algebra Number Theory 16 (2022), no. 1, 59–137. MR 4384564


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