A Variational Construction of the Teichmüller Map

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Abstract. Gerstenhaber and Rauch proposed the problem of constructing the Teichmüller map by a maximum-minimum approach involving harmonic maps. In this paper, we show that the Teichmüller map can be constructed by this variational characterization. The key idea is to consider a class of metrics on the target which include singular metrics and use the harmonic map theory in this setting.

1. Introduction

Let Σ_1 and Σ_2 be Riemann surfaces of the same genus ≥ 2 and $f : \Sigma_1 \rightarrow \Sigma_2$ be an orientation preserving homeomorphism. For a sufficiently smooth f, we measure the deviation of the map from conformality at each point $z \in \Sigma_1$ by the dilatation $K_f(z)$, defined by the ratio of the axes of the infinitesimal ellipse into which f takes an infinitesimal circle around z. (See [**Ah1**] for more details.) Let $K[f] = \sup_{z \in \Sigma_1} K_f(z)$.

Given an orientation preserving homeomorphism $h : \Sigma_1 \rightarrow \Sigma_2$, let K^* be the infimum of $K[f]$ amongst all quasiconformal maps homotopic to h. The Teichmüller's Theorem asserts the existence of a unique map f_0 with the property that $K_{f_0}(z) \equiv K^*$ everywhere except at isolated points. The extremal map f_0 can be described analytically by two holomorphic quadratic differentials Φ and Ψ defined on Σ_1 and Σ_2 respectively; for local parameters $z = x + iy$ and $w = u + iv$ so that $\Phi = dz^2$ and $\Psi = dw^2$, $f_0(x, y) = (u(x, y), v(x, y))$ is expressed by $u = Kx$ and $v = y$. The Teichmüller distance between Σ_1 and Σ_2, relative to the homotopy class of h, is defined by $\log K^*$. This distance function makes Teichmüller space (an equivalence class of conformal structures on a compact surface where two conformal structures are considered to be equivalent if there exists a conformal diffeomorphism between them which is homotopic to the identity map) into a metric space. The Riemann-Roch theorem and the fundamental relation between the Teichmüller space and the space of holomorphic quadratic differentials show that

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the Teichmüller space is homeomorphic to the Euclean space of dimension $6g-6$. The topological structure of the Teichmüller space was already known [**FK**] since the early 20th century, but Teichmüller realized its connection with holomorphic quadratic differentials in the 1940's. Complete details of Teichmüller's claims were worked out by L.V. Ahlfors [**Ah2**] and L. Bers [**Be**] in the 1950's and 60's.

In their 1954 paper [**GR**], Gerstenhaber and Rauch proposed an alternative approach; they attempted to characterize the Teichmüller map via a variational characterization using harmonic maps. Reich [**Re**] and Reich-Strebel [**RS**] have conducted a careful investigation of this principle when the two Riemann surfaces Σ_1 and Σ_2 are unit disks. This paper is a completion of the Gerstenhaber-Rauch program for maps between closed, compact Riemann surfaces.

Gerstenhaber and Rauch consider the following: Let $g = \rho|dw|^2$ be a conformal metric on Σ_2 so that

$$\int_{\Sigma_2} \rho(w)dudv = 1 \qquad (w = u + iv).$$

Let $\mathcal{M}$ be a family of such metrics and $\mathcal{F}_h$ be a family of maps from Σ_1 to Σ_2 homotopic to a given homeomorphism h. We assume that $g \in \mathcal{M}$ and $f \in \mathcal{F}_h$ are sufficiently nice so that the Dirichlet energy of f with respect to $g = \rho|dw|^2$,

$${}^gE^f = \int_{\Sigma_1} \rho(f(z))(|f_z|^2 + |f_{\bar{z}}|^2)dxdy \qquad (z = x + iy),$$

makes sense. Gerstenhaber and Rauch conjectured that

$$\sup_{g\in\mathcal{M}} \inf_{f\in\mathcal{F}_h} {}^gE^f = \frac{1}{2}\left(K^* + \frac{1}{K^*}\right)$$

and proposed constructing the Teichmüller map via this variational characterization. The above equality was later proved by E. Kuwert [**Ku**] assuming the existence of the Teichmüller map. In this paper, we prove the existence of the Teichmüller map using the theory of harmonic maps as suggested by [**GR**] and [**Ku**]. This problem is also mentioned in on harmonic maps by Eells and Lemaire [**EL**].

The idea to make this variational method work is to enlarge the class of target metrics by allowing singular surfaces. The study of harmonic maps from a smooth domain to singular targets, particularly Alexandrov spaces of curvature bounded from above, was initiated by the work of Gromov and Schoen [**GS**] who developed the general existence theory and regularity theory for harmonic maps into non-positively curved Riemannian simplicial complexes. Korevaar and Schoen [**KS1**] and Jost [**Jo**] have further generalized the setting in which we consider harmonic map theory. The method of this paper is a natural application of this theory.

A key to any variational construction is a compactness property; more precisely, a limit of a maximizing or a minimizing sequence of a given functional in a chosen set must also belongs to that set. We take a maximizing sequence of metrics for the functional

$$g \mapsto \inf_{f\in\mathcal{F}_h} {}^gE^f$$

in the set $\mathcal{M}_{a,\kappa}$ of (possibly singular and degenerate) metrics with a normalized area of $a > 0$ and an upper curvature bound of $\kappa > 0$. A compactness theorem for these metrics and a compactness theorem for energy minimizing maps were proved in [**Me2**] and [**Me3**]. Using these compactness results, we can obtain a metric g_0 and a map f_0 which satisfy

$$^{g_0}E^{f_0} = \sup_{g\in\mathcal{M}_{a,\kappa}} \inf_{f\in\mathcal{F}_h} {}^{g}E^{f}.$$

The latter part of this paper is devoted to proving that the map f_0 is a Teichmüller map if $\kappa > 0$. To accomplish this, we show that a first variation type argument remains valid in this singular setting. We outline this argument below: If $K_{f_0}(z_0) > K^*$ for some $z_0 \in \Sigma_1$, then $K_{f_0} > K^*$ in a neighborhood of z_0 by the lower semicontinuity of the dilatation function. We bump-up the metric $g_0 = \rho_0|dw|^2$ in this neighborhood to construct a one-parameter family of metrics $g_t = \rho_t|dw|^2$ with $\rho_t > \rho_0$ near z_0 for $t > 0$. In doing so, we are careful to control the upper curvature bound $\kappa(t)$ of g_t. Let $a(t)$ be the area of Σ_2 with respect to g_t. Next, we check that there is a sequence $t_i \to 0$ so that if f_{t_i} is the energy minimizing maps with respect to the metric g_{t_i}, then $\{K_{f_{t_i}}\}$ converges a.e. to K_{f_0}

This is the key to showing the first variation type inequality;

$$\liminf_{t_i\to 0} \frac{^{g_{t_i}}E^{f_{t_i}} - {}^{g_0}E^{f_0}}{t_i} > \frac{1}{2}\left(K^* + \frac{1}{K^*}\right) a'(0).$$

Let

$$\mathcal{E}(a,\kappa) = \sup_{\mathcal{M}_{a,\kappa}} \inf_{f\in\mathcal{F}_h} {}^{g}E^{f}.$$

By the definition of $\mathcal{E}(a,\kappa)$, $a \mapsto \mathcal{E}(a,\kappa)$ and $\kappa \mapsto \mathcal{E}(a,\kappa)$ are monotone functions so $\mathcal{E}(a,\kappa)$ is differentiable for a.e. a and a.e. κ. (In fact, it is not difficult to see that $a \mapsto \mathcal{E}(a,\kappa)$ is a linear map.) Therefore, for a.e. pair (a,κ), we have

$$\frac{d}{dt}\mathcal{E}(a(t),\kappa(t))|_{t=0} \geq \liminf_{t_i\to 0} \frac{^{g_{t_i}}E^{f_{t_i}} - {}^{g_0}E^{f_0}}{t_i}$$

where $a(0) = a$ and $\kappa(0) = \kappa$. Hence, by the chain rule,

$$\frac{\partial\mathcal{E}}{\partial a}(a(0),\kappa(0)) \cdot a'(0) + \frac{\partial\mathcal{E}}{\partial\kappa}(a(0),\kappa(0)) \cdot \kappa'(0) > \frac{1}{2}\left(K^* + \frac{1}{K^*}\right) a'(0). \qquad (1)$$

Recall that curvature bound can be adjusted by re-scaling the metric. Rescaling the metric just changes the area of the metric, and we see that

$$\frac{\partial\mathcal{E}}{\partial\kappa} \leq \frac{1}{\kappa}\frac{\partial\mathcal{E}}{\partial a} \qquad (2)$$

for a.e. pair (a,κ).

Because we can control the curvature bound, $\kappa'(0)$ can be made sufficiently small and inequalities (1) and (2) then imply

$$\frac{\partial\mathcal{E}}{\partial a} > \frac{1}{2}\left(K^* + \frac{1}{K^*}\right)$$

for a.e. a. Since $a \mapsto \mathcal{E}(a,\kappa)$ is a linear map with $\lim_{a\to 0}\mathcal{E}(a,\kappa) = 0$, we have

$$\mathcal{E}(a,\kappa) = \int_0^a \frac{\partial\mathcal{E}}{\partial a} da > \frac{1}{2}\left(K^* + \frac{1}{K^*}\right) a.$$

On the other hand, if f^* is an extremal map (i.e. $K[f^*] = K^*$), then

$$\mathcal{E}(a,\kappa) = \sup_{g\in\mathcal{M}_{a,\kappa}} \inf_{f\in\mathcal{F}_h} {}^{g}E^{f} \leq \sup_{g\in\mathcal{M}_{a,\kappa}} {}^{g}E^{f^*} \leq \left(K^* + \frac{1}{K^*}\right) a.$$

This contradiction shows that $K_{f_0} \leq K^*$ for all $z \in \Sigma_1$. A similar argument shows $\mathcal{E}(a,\kappa) = \left(K^* + \frac{1}{K^*}\right) a$, which in turn implies that $K_{f_0} \equiv K^*$ a.e. This is enough to show that f_0 is a Teichmüller map.

2. Energy of maps into metric spaces

In [**KS1**], Korevaar and Schoen define the energy of maps into complete metric spaces. Let (M,g) be a compact Riemannian manifold, d_M be the distance function on M induced by g and (X,d) be a complete metric space. For $p \geq 1$, a Borel measurable map $f : M \to X$ is said to be in $L^p(M,X)$ if

$$\int_M d^p(f(x),P)d\mu < \infty$$

for some $P \in X$. By the triangle inequality, this definition is independent of P chosen.

Let $M_\epsilon = \{x \in M : d_M(x,\partial M) > \epsilon\}$, $S_\epsilon(x) = \{y \in M : d_M(x,y) = \epsilon\}$ and $d\sigma_{x,\epsilon}$ be the induced volume form on $S_\epsilon(x)$. For $u \in L^p(\Omega,X)$, construct an $\epsilon-$approximate energy density function $e_\epsilon : M \to \mathbf{R}$ by setting

$$e_\epsilon(x) = \begin{cases} \frac{1}{\omega_n}\int_{S_\epsilon(x)} \frac{d^p(u(x),u(y))}{\epsilon^p}\frac{d\sigma_{x,\epsilon}}{\epsilon^{n-1}} & \text{for } x \in M_\epsilon \\ 0 & \text{for } x \in M - M_\epsilon \end{cases}$$

where ω_n is the volume of the unit n-sphere. Define a linear functional $E_\epsilon : C_c(M) \to \mathbf{R}$ on the set of continuous functions with compact support in M by setting

$$E_\epsilon(\varphi) = \int \varphi e_\epsilon d\mu.$$

The map $u \in L^p(M,X)$ has finite p-energy (or $u \in W^{1,p}(M,X)$) if

$$E^u \equiv \sup_{\varphi \in C_c(M), 0 \leq \varphi \leq 1} \limsup_{\epsilon \to 0} E_\epsilon(\varphi) < \infty.$$

The quantity E^u is defined to be the p-energy of the map u. If u has finite p-energy, the measures $e_\epsilon(x)d\mu(x)$ converge weakly to a measure which is absolutely continuous with respect to the Lebesgue measure ([**KS1**] Theorem 1.10). Therefore, there exists a function $|\nabla u|_p$, which we call the p-energy density function, so that $e_\epsilon(x)d\mu \rightharpoonup |\nabla u|_p d\mu$.

The p-energy for $p = 2$ will be simply referred to as energy. In analogy to the case of real-valued functions and maps into Riemannian manifolds, we write $|\nabla u|^2(x)$ in place of $|\nabla u|_2(x)$. In particular, for $p = 2$,

$$E^u = \int_M |\nabla u|^2 d\mu.$$

It is not true that $|\nabla u|^2$ is equal to $|\nabla u|_1^2$ (c.f. comments after Theorem 1.10 of [**KS1**]).

The map u is called energy minimizing if it is locally energy minimizing; i.e. for any Lipschitz domain $\Omega \subset M$ and map $v : \Omega \to X$ with $u = v$ on $\partial\Omega$ (here $u = v$ in the sense of [**KS1**] Theorem 1.12),

$$\int_\Omega |\nabla u|^2 d\mu \leq \int_\Omega |\nabla v|^2 d\mu.$$

Let $\Gamma(TM)$ be the set of Lipschitz vector field on M. Then for $V \in \Gamma(TM)$, the directional energy measures can be defined as the weak* limit of measures ${}^V e_\epsilon(x)dx$ where

$$ {}^V e_\epsilon(x) = \begin{cases} \frac{d^p(u(x),u(\bar{x}(x,\epsilon))}{\epsilon^p} & \text{for } x \in M_\epsilon \\ 0 & \text{for } x \in M - M_\epsilon. \end{cases} $$

and $\bar{x}(x,\epsilon)$ denotes the flow along V at time ϵ starting at point x. Again, it can be shown that if u has finite energy, the measures ${}^V e_\epsilon(x)d\mu(x)$ converge weakly to a measure which is absolutely continuous with respect to the Lebesgue measure ([**KS1**] Theorem 1.9.6). Therefore, there exists a function ${}^V e_p(x)$, which we call the p-energy density function, so that ${}^V e_\epsilon(x)d\mu \rightharpoonup {}^V e_p(x)d\mu$. If we write the directional energy function for $p=1$ as $|u_*(Z)|$, then ${}^V e_p = |u_*(Z)|^p$ ([**KS1**] Theorem 1.9.6, contrast with the fact that $|\nabla f|^2$ is not equal to $|\nabla f|_1^2$). Furthermore,

$$ \lim_{\epsilon \to 0} {}^V e_\epsilon(x) = |u_*(x)|^p(x) $$

for almost every $x \in M$. If $\{\epsilon_1, \epsilon_2, ..., \epsilon_n\}$ is a local orthonormal frame on M and if we identify $S^{n-1} \subset \mathbf{R}^n$ with $S^{n-1}_x \subset TM_x$ by

$$ \omega = (\omega^1, \omega^2, ..., \omega^n) \mapsto w^i \epsilon_i $$

then

$$ e_p(x) = \frac{1}{\omega_n} \int_{\omega \in S^{n-1}} |u_*(\omega)|^p d\sigma(\omega). $$

Finally, if X has curvature bounded from above by κ, then we can also make sense of the notion of the pull back metric

$$ \pi_u : \Gamma(TM) \times \Gamma(TM) \to L^1(M) $$

for $u \in W^{1,2}(M,X)$ (see [**Me1**] Lemma 3.7 and Proposition 3.8 which extends the result of [**KS1**] Lemma 2.3.1 and Theorem 2.3.2), defined by

$$ \pi_u(V,W) = \frac{1}{4}|u_*(V+W)|^2 - \frac{1}{4}|u_*(V-W)|^2 \quad \text{for} \quad V, W \in \Gamma(TM). $$

Letting $(x_1, x_2, ..., x_n)$ be the local coordinates and $\{\partial_1, \partial_2, ..., \partial_n\}$ be its corresponding tangent basis, if we write $(\pi_u)_{ij} = \pi_u(\partial_i, \partial_j)$ then $|\nabla u|^2 = g^{ij}(\pi_u)_{ij}$ ([**KS1**] Theorem 2.3.2). Furthermore, if $\psi : M \to M$ is a $C^{1,1}$ map, then writing $v = u \circ \psi$, we have the chain rule formula,

$$ (\pi_v)_{ij} = (\pi_u)_{lm} \psi^l_{,i} \psi^m_{,j}. \tag{3} $$

We have the following compactness theorem of energy minimizing maps.

THEOREM 1. *Let $\{d_i\}$ be distance functions on X with curvature bounded from above by κ. Assume X is compact with respect to the metric topology induced by d_i. Let $h : M \to X$ be a continuous map and let $f_i : M \to (X, d_i)$ be continuous energy minimizing maps in the homotopy class of h with $f_i = h$ on ∂M if $\partial M \neq \emptyset$. Let δ_i be the pull back distance function of d_i under f_i, i.e.*

$$ \delta_i(\cdot,\cdot) = d_i(f_i(\cdot), f_i(\cdot)). $$

If the energy of f_i is bounded from above by K for each i and if the distance functions d_i converge uniformly to a distance function d_0, then there exists a subsequence $\{i'\} \subset \{i\}$ and an energy minimizing map f_0 with respect to d_0 so that the maps $f_{i'}$ converge pointwise to f_0, the pull back distance functions $\delta_{i'}(\cdot,\cdot)$ converge uniformly to $d_0(f_0(\cdot), f_0(\cdot))$ and the energies of $f_{i'}$ converge to that f_0. In fact, the energy density functions of $f_{i'}$ converge a.e. to that of f_0.

PROOF. In [**Me3**], we have shown that there exists a subsequence $\{d_{i'}\}$ and an energy minimizing map f_0 with respect to d_0 so that the pull back distance functions $d_{i'}(f_{i'}(\cdot), f_{i'}(\cdot))$ converge uniformly to $d_0(f_0(\cdot), f_0(\cdot))$ and the energies converge weakly as measures to that of f_0. Since there is no loss of energy, the directional energies of $f_{i'}$ converge weakly as measures to that of f_0. Thus, for any $Z \in TM$ and any $\varphi \in C_0^\infty(M)$,

$$\lim_{i' \to \infty} \int \varphi |(f_{i'})_*(Z)|^2 d\mu = \int \varphi |(f_0)_*(Z)|^2 d\mu$$

Furthermore, by the lower semicontinuity of (the $p = 1$ directional) energy ([**KS1**] Theorem 1.6.1),

$$\int \varphi |(f_0)_*(Z)|^2 d\mu \leq \liminf_{i' \to \infty} \int \varphi |(f_0)_*(Z)||(f_{i'})_*(Z)| d\mu \tag{4}$$

Therefore,

$$\begin{aligned} &\liminf_{i' \to \infty} \int \varphi \left(|(f_0)_*(Z)| - |(f_{i'})_*(Z)| \right)^2 d\mu \\ = \quad & \liminf_{i' \to \infty} \int \varphi \left(|(f_0)_*(Z)|^2 + |(f_{i'})_*(Z)|^2 - 2|(f_0)_*(Z)||(f_{i'})_*(Z)| \right) d\mu \\ \leq \quad & \int \varphi \left(|(f_0)_*(Z)|^2 + |(f_0)_*(Z)|^2 - 2|(f_0)_*(Z)|^2 \right) d\mu \\ = \quad & 0, \end{aligned}$$

and we obtain

$$\liminf_{i' \to \infty} \int \left(|(f_0)_*(Z)| - |(f_{i'})_*(Z)| \right)^2 d\mu = 0.$$

Hence there exists a subsequence $\{f_{i''}\}$ so that

$$\lim_{i'' \to \infty} \int \left(|(f_0)_*(Z)| - |(f_{i''})_*(Z)| \right)^2 d\mu = 0,$$

i.e. $|(f_{i''}) * (Z)| \to |(f_0)_*(Z)|^2$ in L^2, which implies almost everywhere convergence of $|(f_{i''}) * (Z)|$ to $|(f_0)_*(Z)|^2$. Finally, letting $(x^1, x^2, ..., x^n)$ be the local coordinates and and $\{\partial_1, \partial_2, .., \partial_n\}$ the corresponding tangent basis, we have

$$|\nabla f_i|^2 = g^{lm}(|(f_i)_*(\partial_l + \partial_m)|^2 - |(f_i)_*(\partial_l - \partial_m)|^2)$$

and

$$|\nabla f_0|^2 = g^{lm}(|(f_0)_*(\partial_l + \partial_m)|^2 - |(f_0)_*(\partial_l - \partial_m)|^2)$$

which implies that $|\nabla f_{i''}|^2$ converges to $|\nabla f|^2$ almost everywhere. □

3. Metrics of curvature bounded from above on a surface

We consider (possibly singular and degenerate) metrics g on Σ_2 with the property that $g = \rho |dw|^2$ locally with ρ a non-negative bounded function satisfying

$$\triangle \log \rho \geq -2\kappa\rho \quad \text{weakly.} \tag{5}$$

The identity map of Σ_2 with a metric $g = \rho|dw|^2$ on the target can be thought of as a weakly conformal harmonic map with conformal factor ρ. By Theorem 4.1 of [**Me1**], ρ satisfies $\triangle \rho \geq -2\kappa\rho^2$ weakly. Since ρ is locally bounded, there exist smooth subharmonic functions s_1 and s_2 so that $\triangle(\rho + s_1) \geq 0$ and $\triangle(\log \rho + s_2)$ weakly. We first prove a couple of lemmas regarding subharmonic functions.

LEMMA 2. *Let s be a non-negative weakly subharmonic function defined in an open set $\Omega \subset \mathbf{R}^n$ so that s is locally bounded from above and not identically equal to $-\infty$ in any neighborhood. Then $f \in H^1_{loc}(\Omega)$.*

PROOF. Let K be a compactly contained subset of Ω and M be such that $f(x) < M$ for all $x \in K$. For sufficiently small σ and $x \in K$, $\triangle f_\sigma \geq 0$ where f_σ is a mollification of f. It will be enough to show that f_σ has uniformly bounded H^1 norm in K. Let $K_1 \subset \Omega$ be a set compactly containing K. Fix $\zeta \in C^\infty_c(\Omega)$ so that $\zeta \equiv 1$ in K and $\zeta \equiv 0$ outside of K_1. We have,

$$0 \leq \int_\Omega \zeta^2 f_\sigma \triangle f_\sigma .$$

Integrating by parts and applying the Cauchy-Schwarz inequality, we obtain,

$$\int_K |\nabla f_\sigma|^2 \leq \int_\Omega \zeta^2 |\nabla f_\sigma|^2 \leq 4M^2 \int_\Omega |\nabla \zeta|^2 < C$$

where C is a constant depending on K. □

LEMMA 3. *Let s be a weakly subharmonic function defined in Ω not identically equal to $-\infty$ in any neighborhood. Then $s \in W^{1,1}_{loc}(\Omega)$.*

PROOF. By the Riesz Representation Theorem, there exists a positive measure μ, a harmonic function h and a compact subset K of Ω so that

$$s(x) = \int_\Omega \log|x-\xi| d\mu(\xi) + h(x)$$

for all $x = (x^1, x^2, ..., x^n) \in K$. Differentiate with respect to x^i to obtain

$$\frac{\partial s}{\partial x^i} = \int_K \frac{x^i - \xi^i}{|z-\xi|} d\mu(\xi) + \frac{\partial h}{\partial x^i}.$$

Then

$$\begin{aligned}
\int_K |\nabla s|(x) dx &\leq 2\int_K \int_\Omega \frac{1}{|x-\xi|} d\mu(\xi) dx + 2\int_K |\nabla h| dx \\
&\leq 2\int_\Omega \int_K \frac{1}{|x-\xi|} dx d\mu(\xi) + 2\int_K |\nabla h| dx \\
&\leq C\int_\Omega d\mu(\xi) + 2\int_D |\nabla h| dx \\
&< \infty
\end{aligned}$$

where C is a constant dependent on the diameter of K. □

From Lemmas 2 and 3, we see that ρ and $\log\rho$ are H^1_{loc} and $W^{1,1}_{loc}$ functions respectively. Although H^1_{loc} functions are only defined up to a set of measure zero, we will always consider ρ to be the representative function in its equivalence class satisfying

$$\rho(w_0) = \lim_{r\to 0} \frac{1}{\pi r^2} \int_{B_r(w_0)} \rho(w) du dv$$

for all $w_0 \in \Sigma_2$. Note that the above limit exists because of the subharmonicity of $\rho + s_1$.

If we define

$$d_g(w_1, w_2) = \inf\{\int_\gamma \sqrt{\rho} ds : \gamma \text{ is a smooth curve from } w_1 \text{ to } w_2\},$$

then (Σ_2, d_g) is a metric space of curvature bounded from above by κ in the sense of Alexandrov. (See [**Hu**] or [**Me1**] for more details.) Simply connected metric space of curvature bounded from above by κ are referred to as CAT(κ) spaces in literature. We let $A(g)$ be the area of Σ_2 with respect to g; in other words,

$$A(g) = \int_{\Sigma_2} \rho du dv.$$

Let $\mathcal{M}_{a,\kappa}$ be the set of all metrics $g = \rho|dw|^2$ satisfying inequality (5) with $A(g) = a$. We will say that a sequence g_i converges to g in the sense of distance functions if the corresponding distance functions d_{g_i} converges uniformly to d_g. In [**Me3**], we have shown that smooth, nondegenerate metrics are dense in $\bigcup_{a,\kappa} \mathcal{M}_{a,\kappa}$ in the following way:

LEMMA 4. *For any $g \in \mathcal{M}_{a,\kappa}$, there exists a one-parameter family of non-degenerate smooth metrics $\sigma \mapsto g^\sigma$, $\sigma \geq 0$, so that $g^\sigma \in \mathcal{M}_{a^\sigma,\kappa^\sigma}$ with $a^\sigma \to a$, $\kappa^\sigma \to \kappa$, and g^σ converging to g in the sense of distance functions.*

In [**Me2**], we prove the theorem for distance functions.

THEOREM 5. *Let $\{g_i\}$ be a sequence in $\mathcal{M}_{a,\kappa}$. There is a subsequence $\{g_{i'}\}$ converging to $g_0 \in \mathcal{M}_{a,\kappa}$ in the sense of distance functions.*

Let $g = \rho|dw|^2 \in \mathcal{M}_{a,\kappa}$. In general, the set $\{z : \rho(z) = 0\}$ may be non-empty. On the other hand, it is a set of Hausdorff dimension 0 by a property of subharmonic functions (see [**HK**]). In fact, since $\log \lambda$ is a $W^{1,1}_{loc}$ function, we have:

LEMMA 6. *Let $g = \rho|dw|^2$ in a coordinate neighborhood U. For any set K compactly supported in U, the perimeter and the measure of the set $E_t = \{z \in K : \log \rho(z) < t\}$ goes to zero as $t \to -\infty$ where the perimeter $P(E)$ of a set $E \subset K$ is*

$$P(E) = \int_K |\nabla \varphi_E| dx dy = \sup\{\int_K \varphi_E div\psi dx : \psi \in C^1_c(U, \mathbf{R}^2), |\psi| \leq 1\}.$$

PROOF. Since $\log \rho \in W^{1,1}(K)$,

$$\int_{-\infty}^{\infty} \int_K |D\varphi_{E_t}| dx dy \; dt = \int_K |\nabla \log \lambda| dx dy < \infty$$

by the co-area formula for functions of bounded variation. The claims of the lemma follows immediately. □

Let $g \in \mathcal{M}_{a,\kappa}$ with $g = \rho|dw|^2$ in a coordinate chart U and s_2 a subharmonic function so that $\triangle \log \rho + s_2 \geq 0$ weakly in U. By the Riesz representation theorem for subharmonic functions, there exists a positive measure $\mathcal{K}_1$ in U so that for any compactly supported subset $K \subset U$ and $w \in K$,

$$\log \rho(w) + s_2 = \int_U \log |w - z| d\mathcal{K}_1(z) + h(w)$$

where $h(w)$ is a harmonic function. Let $\mathcal{K}_2$ be the positive measure defined by

$$\mathcal{K}_2(E) = \int_E \triangle s_2 dx dy$$

and $\mathcal{K} = \mathcal{K}_1 - \mathcal{K}_2$. Although $\mathcal{K}_1$ and $\mathcal{K}_2$ are only defined in U and depends on the choice of s_2, the measure $\mathcal{K}$ is independent of this choice and is defined on all of

Σ_2. We call $\mathcal{K}$ the curvature measure of g. If $g = \rho|dw|^2$ is a smooth metric, then

$$\mathcal{K}(E) = \int_E \triangle \log \rho(z) dxdy = -2 \int_E K_G(z) d\mu_g$$

where K_G is the Gauss curvature of the surface (Σ_2, g).

By the Hahn decomposition theorem, there exists two disjoint measurable sets A and B with $A \cup B = \Sigma_2$ so that for any measurable set $E \subset \Sigma_2$,

$$\mathcal{K}(E) \geq 0 \ \text{ if } \ E \subset A$$

and

$$\mathcal{K}(E) \geq 0 \ \text{ if } \ E \subset B.$$

For any measurable set $E \subset \Sigma_2$, we define

$$\mathcal{K}_-(E) = \mathcal{K}(E \cap A) \ \text{ and } \ \mathcal{K}_+(E) = -\mathcal{K}(E \cap B).$$

Thus, for any coordinate neighborhood U and a measurable set $E \subset B \cap U$, we have

$$\mathcal{K}_+(E) = \mathcal{K}_+(E) - \mathcal{K}_-(E) = -\mathcal{K}(E) = \mathcal{K}_2(E) - \mathcal{K}_1(E) \leq \mathcal{K}_2(E).$$

Therefore, for any measuable set $E \subset U$,

$$\mathcal{K}_+(E) = \mathcal{K}_+(E \cap B) \leq \mathcal{K}_2(E \cap B) = \mathcal{K}_2(E). \tag{6}$$

Definition 7. *The negative curvature set $\mathcal{C}$ is defined by*

$$\mathcal{C} = \{w_0 \in \Sigma_2 | \text{ for every } \epsilon > 0, \text{there exists } \overline{B_\delta(w)} \subset B_\epsilon(w_0) \text{ so that } \mathcal{K}(\overline{B_\delta(w)}) > 0\}.$$

Lemma 8. *The set $\mathcal{C}$ is closed.*

Proof. Let $\{w_i\} \subset \mathcal{C}$ and $w_i \to w_0$. For $\epsilon > 0$, there exists $w_i \in B_\epsilon(w_0)$. Choose $\epsilon' > 0$ so that $B_{\epsilon'}(w_i) \subset B_\epsilon(w_0)$. By the definition of $\mathcal{C}$, there is $\overline{B_\delta(w)} \subset B_{\epsilon'}(w_i)$ with $\mathcal{K}(\overline{B_\delta(w)}) > 0$. Since $\overline{B_\delta(w)} \subset B_\epsilon(w_0)$, this shows $w_0 \in \mathcal{C}$. □

Lemma 9. *The support of the singular part of $\mathcal{K}_-$ is contained in $\mathcal{C}$.*

Proof. Suppose not. Then, for some coordinate chart $U \subset \Sigma_2$, there exists $A \subset U$ so that $m(A) = 0$ and $\mathcal{K}_-(A) = \alpha > 0$ where m is the Lebesgue measure. Let M be so that $\triangle s_2 < M$ where s_2 is the subharmonic function defined above. Since $\mathcal{C}$ is closed, we may assume that A is at a positive distance from $\mathcal{C}$ and hence there exists a countable collection of balls $\{B_i\}$ so that $\overline{B_i} \subset U - \mathcal{C}$, $A \subset \bigcup \overline{B_i}$ and $m(\bigcup \overline{B_i}) < \frac{\alpha}{M}$. Then

$$\alpha = \mathcal{K}_-(A) \leq \mathcal{K}_-(\bigcup \overline{B_i})$$

and, by inequality 6,

$$\mathcal{K}_+(\bigcup \overline{B_i}) \leq \mathcal{K}_2(\bigcup \overline{B_i}) = \int_{\bigcup \overline{B_i}} \triangle s_2 < Mm(\bigcup \overline{B_i}) < \alpha.$$

Thus, $\mathcal{K}_+(\bigcup \overline{B_i}) < \mathcal{K}_-(\bigcup \overline{B_i})$ which implies that $\mathcal{K}_+(\overline{B_i}) < \mathcal{K}_-(\overline{B_i})$ for at least one i. Therefore, $\mathcal{K}(\overline{B_i}) > 0$ and this contradicts $\overline{B_i} \subset U - \mathcal{C}$. □

Lemma 10. *Let $w_0 \in \Sigma_2 - \mathcal{C}$. There exists $R > 0$ so that $\rho_0(w) \geq \epsilon > 0$ for all $w \in B_R(w_0)$.*

PROOF. Since $w_0 \in \Sigma_2 - \mathcal{C}$, there exists $0 < r_0 < 1$ so that $\mathcal{K}(\overline{B_\delta(w)}) \leq 0$ for all $(\overline{B_\delta(w)}) \subset B_{r_0}(w_0)$. In particular, for $R = \frac{r_0}{2}$, $\mathcal{K}(\overline{B_t(w)}) \leq 0$ for all $w \in \overline{B_R(w_0)}$ and $t < R$. Fix $w \in \overline{B_R(w_0)}$ and let $\psi(t) = \mathcal{K}(\overline{B_t(w)})$, $\psi_+(t) = \mathcal{K}_+(\overline{B_t(w)})$ and $\psi_-(t) = \mathcal{K}_-(\overline{B_t(w)})$. Since $t \to \mathcal{K}_+(\overline{B_t(w)})$ and $t \to \mathcal{K}_-(\overline{B_t(w)})$ are nondecreasing functions, ψ_+ and ψ_- are differentiable for almost every t and hence ψ is differentiable for almost every t. Let

$$\phi(t) = \int_{\overline{B_t(w)}} \log|w-z| d\mathcal{K}(z),$$

$$\phi_+(t) = \int_{\overline{B_t(w)}} \log|w-z| d\mathcal{K}_+(z),$$

and

$$\phi_-(t) = \int_{\overline{B_t(w)}} \log|w-z| d\mathcal{K}_-(z).$$

Since $\log|w-z| < 0$, ϕ_+ and ϕ_- are nonincreasing functions and are differentiable for almost every t. Hence ϕ is differentiable for almost every t. Furthermore, since

$$\phi_+(t+\epsilon) - \phi_+(t) = \int_{\overline{B_{R+\epsilon}(w)} - \overline{B_R(w)}} \log|w-z| d\mathcal{K}(z),$$

we have

$$\log t(\psi_+(t+\epsilon) - \psi_+(t)) \leq \phi_+(t+\epsilon) - \phi_+(t) \leq \log(t+\epsilon)(\psi_+(t+\epsilon) - \psi_+(t)).$$

Dividing by ϵ and letting $\epsilon \to 0$, we get $\phi'_+(t) = \log t \psi'_+(t)$. Similarly, $\phi'_-(t) = \log t \psi'_-(t)$ and hence $\phi'(t) = \log t \psi'(t)$. Since the singular part of $\mathcal{K}$ is contained in $\mathcal{C}$, ϕ is a continuous function. Thus integrating $\phi'(t) = \log t \psi'(t)$ over $[0, R]$ gives

$$\begin{aligned} \int_{\overline{B_R(w)}} \log|w-z| d\mathcal{K}(z) &= \int_0^R \log t \frac{d\psi}{dt} dt \\ &= \log R \cdot \psi(R) \cdot - \lim_{t\to 0} \log t \cdot \psi(t) - \int_0^R \frac{\psi(t)}{t} dt. \end{aligned}$$

Since $\psi(t) = \mathcal{K}(\overline{B_t(w)}) \leq 0$,

$$-\int_0^R \frac{\psi(t)}{t} dt \geq 0.$$

Furthermore,

$$\begin{aligned} \psi(t) &= \mathcal{K}(\overline{B_t(w)}) \\ &= \mathcal{K}_1(\overline{B_t(w)}) - \mathcal{K}_2(\overline{B_t(w)}) \\ &\geq -\mathcal{K}_2(\overline{B_t(w)}) \\ &= -\int_{\overline{B_t(w)}} \triangle s_2 dxdy \\ &\geq -M\pi t^2. \end{aligned}$$

Thus,

$$-\lim_{t\to 0} \log t \cdot \psi(t) \geq M\pi \lim_{t\to 0} \log t \cdot t^2 = 0.$$

Additionally,

$$\begin{aligned}\int_{\overline{B_{r_0}(w_0)}-\overline{B_R(w)}} \log|w-z|d\mathcal{K}(z) &\geq \int_{\overline{B_{r_0}(w_0)}-\overline{B_R(w)}} \log|w-z|d\mathcal{K}_-(z)\\ &\geq \log R\cdot\mathcal{K}_-\left(\overline{B_{r_0}(w_0)}-\overline{B_R(w)}\right).\end{aligned}$$

Hence, for $w\in\overline{B_R(w_0)}$,

$$\begin{aligned}\rho(w) &= \exp\left(\int_{\overline{B_{r_0}(w_0)}}\log|w-z|d\mathcal{K}(z)+h(w)\right)\\ &= \exp\left(\int_{\overline{B_R(w)}}+\int_{\overline{B_{r_0}(w_0)}-\overline{B_R(w)}}\log|w-z|d\mathcal{K}(z)+h(w)\right)\\ &\geq \log R\cdot\psi(R)+\log R\cdot\mathcal{K}_-\left(\overline{B_{r_0}(w_0)}-\overline{B_R(w)}\right)\\ &\geq \log R\cdot\psi_-(r_0)\end{aligned}$$

and ρ_0 is bounded below in $B_R(w_0)$. □

4. Energy minimizing maps into singular surfaces

For the rest of the paper, we will concentrate on maps between two Riemann surfaces Σ_1 and Σ_2 of the same genus. Let $g\in\mathcal{M}_{a,\kappa}$ and d_g be the distance function defined by $g=\rho|dw|^2$. We will denote the energy of a map $f:\Sigma_1\to\Sigma_2$ with respect to the distance function d_g by

$${}^gE^f=\int_{\Sigma_1}|\nabla f|_g^2dxdy$$

where $|\nabla f|_g^2dxdy$ is the energy density function and $z=x+iy$ is the local coordinates for Σ_1. Note that because Σ_1 is of dimension 2, energy ${}^gE^f$ and the energy density measure $|\nabla f|_g^2dxdy$ are independent of the metric on Σ_1.

The following is a restatement of Theorem 1 in our situation.

THEOREM 11. *Let $\{g_i\}$ be a sequence in $\mathcal{M}_{a,\kappa}$ and $g\in\mathcal{M}_{a,\kappa}$ so that g_i converge to g_0 in the sense of the distance functions. Let f_i be an energy minimizing map with respect to g_i and assume the energies ${}^{g_i}E^{f_i}$ are uniformly bounded. Then there exists a subsequence $\{g_{i'}\}$ and an energy minimizing map f_0 with respect to g_0 so that the pull back distance functions $d_{g_i}(f_{i'}(\cdot),f_{i'}(\cdot))$ converge uniformly to $d_{g_0}(f_0(\cdot),f_0(\cdot))$ and the energies of $f_{i'}$ converge to that of f. More specifically, the energy density functions and directional energy density functions of $f_{i'}$ converge a.e. to those of f_0.*

With this, we give an alternative proof of the existence of energy minimizing maps between surfaces (cf. [**KS1**], [**Ser**]).

THEOREM 12. *Let $h:\Sigma_1\to\Sigma_2$ be a homeomorphism, $\mathcal{F}_h$ be the set of maps homotopic to h, and $g\in\mathcal{M}_{a,\kappa}$. There exists an energy minimizing map $f\in\mathcal{F}_h$ with respect to metric g which is a limit (in the sense of Theorem 11) of smooth energy minimizing maps.*

PROOF. This existence statement is an immediate corollary of Lemma 4 and Theorem 11. To see this, let g^σ as in Lemma 4 and f^σ be an energy minimizing map with respect to g^σ. By Theorem 11, there exists a sequence σ_i so that f^{σ_i} converges to an energy minimizing map f with respect to g. $\square$

We will assume that all energy minimizing maps in this paper were constructed as limits of smooth energy minimizing maps as in Theorem 12.

5. The stretch function $k_f(z)$

For any map $f : \Sigma_1 \to \Sigma_2$ with weak partial derivatives, the stretch function $k_f(z)$ is defined by

$$k_f(z) = \frac{|f_{\bar{z}}|}{|f_z|}.$$

The dilatation (geometrically described in the introduction) is then given by

$$K_f(z) = \frac{1 + k_f(z)}{1 - k_f(z)} = \frac{|f_z| + |f_{\bar{z}}|}{|f_z| - |f_{\bar{z}}|}.$$

A mapping f is called quasiconformal if $K_f(z)$ is bounded from above, or equivalently, there is a number $k < 1$ such that $|f_{\bar{z}}| \le k|f_z|$. Let $h : \Sigma_1 \to \Sigma_2$ be a orientation preserving homeomorphism and let $\mathcal{F}_h$ be the set of maps homotopic to h. As in the introduction, we let

$$K^* = \inf_{f \in \mathcal{F}_h} K[f]$$

where

$$K[f] = \sup_{z \in \Sigma_1} K_f(z).$$

We call f^* an extremal map if $K[f^*] = K^*$. The existence of an extremal map f^* can be shown by using the Arzela-Ascoli Theorem (see [**Be**]).

For an energy minimizing map, the stretch function can be defined in the following way. Let $f : \Sigma_1 \to \Sigma_2$ be an energy minimizing map with respect to d_g. Its Hopf differential Φ_f is defined locally by $\varphi_f dz^2$ where

$$\varphi_f = \pi\left(\frac{\partial}{\partial x}, \frac{\partial}{\partial x}\right) - \pi\left(\frac{\partial}{\partial y}, \frac{\partial}{\partial y}\right) - 2i\pi\left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}\right).$$

Even when the target of f is nonsmooth, φ is holomorphic. To see this, we follow the argument of [**Sch**]. Let $F_t(x, y) = (x + t\eta(x, y), y)$ and $f_t = f \circ F_t$ where $\eta(x, y)$ is a C^∞ function with compact support in a coordinate neighborhood. By equation 3,

$$\begin{aligned}\int |\nabla f_t|^2 dxdy &= \int (\pi_{f_t})_{11} + (\pi_{f_t})_{22} dxdy \\ &= \int (\pi_f)_{11}(1 + t\eta_x)^2 + (\pi_f)_{22} + 2(\pi_f)_{12} t\eta_y \frac{d\xi d\tau}{1 + t\eta_x}\end{aligned}$$

Differntiating with respect to t and setting $t = 0$, we get

$$0 = \int ((\pi_f)_{11} - (\pi_f)_{22})\eta_x + 2(\pi_f)_{12}\eta_y \; dxdy.$$

Similarly,

$$0 = \int ((\pi_f)_{11} - (\pi_f)_{22})\eta_y - 2(\pi_f)_{12}\eta_x \; dxdy$$

for all η. In other words, φ_f satisfies the weak Cauchy-Riemann equation and by Weyl's lemma, φ_f is a smooth holomorphic function of z. Thus $\Phi_f = \varphi_f dz^2$ is a holomorphic quadratic differential on the Riemann surface Σ_1 and has finite zeroes counting multiplicities. Locally, Φ_f can be written as

$$\Phi_f = \left(\frac{m+2}{2}\right)^2 z^m dz^2$$

where the integer m is the vanishing order of Φ_f at 0. The coordinate z is then called the natural parameter of Φ_f. The integral curve of the distribution $\{v \in T\Sigma_1 : \Phi_f(v,v) \leq 0\}$ is called the vertical trajectory. The Hopf differential gives a geometric picture of harmonic maps as the vertical trajectories give the direction of minimal stretch. Using the natural parameter z, we set

$$K_f(z) = \sqrt{\frac{\pi_{11}}{\pi_{22}}}$$

and

$$k_f(z) = \frac{K_f - 1}{K_f + 1}.$$

Using the results of [**Me3**] and following the arguments of [**Ku**], we can define the stretch function k_f by considering the stretch factor for the tangent map of f. This stretch function can be written as

$$k_f(z) = \mu_f - \sqrt{\mu_f^2 - 1} \tag{7}$$

where

$$\mu_f = \lim_{\sigma\to 0} \frac{1}{\pi\sigma^{m+3}} I_f(\sigma),\ I_f(\sigma) = \int_{\{z:|z|=\sigma\}} d^2(f(z), f(0))ds$$

for some constant C. To see the equivalence of the two definitions, note that the natural parameter gives a normalization which requires that $\pi_{11} - \pi_{22} = 1$. Since $\mu_f = \pi_{11} + \pi_{22}$, this implies that

$$K_f = \frac{1+k_f}{1-k_f} = \sqrt{\frac{\mu_f + 1}{\mu_f - 1}} = \sqrt{\frac{\pi_{11}}{\pi_{22}}}.$$

LEMMA 13. *The function $k_f(z)$ is lower semicontinuous.*

PROOF. Let τ the order function for f defined by

$$\tau = Ord(z_0) = \lim_{\sigma\to 0} \frac{\sigma \int_{D_\sigma(z_0)} |\nabla f|^2 dxdy}{\int_{\partial D_\sigma(z_0)} d^2(f, f(z_0))ds}.$$

(See Sec 3 of [**Me3**] for a discussion regarding the order function in this setting.) By the proof of Theorem 2.3 of [**GS**] along with the relevant properties of harmonic maps into metric spaces of curvature bounded from above by κ shown in [**Me3**], we see that for some constant C,

$$\sigma \mapsto \frac{e^{C\sigma}}{\pi\sigma^{2\tau+1}} \int_{\partial D_\sigma} d^2(f(z), f(z_0))ds.$$

is a non-decreasing function. Thus

$$\bar{\mu}_f = \lim_{\sigma\to 0} \frac{e^{C\sigma}}{\pi\sigma^{2\tau+1}} \int_{\partial D_\sigma} d^2(f(z), f(z_0))ds$$

is an upper semicontinuous function since it is a non-increasing limit of a continuous functions

$$\frac{e^{C\sigma}}{\pi\sigma^{2\tau+1}}\int_{\partial D_\sigma} d^2(f(z), f(z_0))ds.$$

Hence,

$$\bar{k}_f(z) = \bar{\mu}_f - \sqrt{\bar{\mu}_f^2 - 1}$$

is lower semicontinous and satisfies

$$\bar{k}_f(z_0) \le \liminf_{z\to z_0} \bar{k}_f(z).$$

By definition, $k_f = \bar{k}_f$ when $\tau = \frac{m+2}{2}$. Furthermore, $\tau \le \frac{m+2}{2}$ and $\tau < \frac{m+2}{2}$ at z_0 implies $k_f(z_0) = 0$ by the argument of Lemma 5 of [**Ku**], this implies that k_f is lower semicontinous. $\square$

By definition, $\bar{\mu}_f = |\nabla f|^2$ for almost every z where $\tau = \frac{m+2}{2}$. Since $1 \le \tau \le \frac{m+2}{2}$ everywhere and $m > 0$ only at the finite zeroes of the Hopf differential Φ_f, we have the following lemma:

Lemma 14. *If f_i and f_0 are energy minimizing maps with respect to g_i and g_0 respectively and the energies of f_i converge to that of f_0 for almost every z then k_{f_i} converges to k_{f_0} for almost every z.*

6. The solution to the variational problem

Let f^* be the extremal map as defined in the previous section and $z = F^*(w)$ be the mapping inverse of $w = f^*(z)$. Then

$$K_{F^*}(w) = \frac{|F^*_w| + |F^*_{\bar{w}}|}{|F^*_w| - |F^*_{\bar{w}}|} = \frac{|f^*_z| + |f^*_{\bar{z}}|}{|f^*_z| - |f^*_{\bar{z}}|} = K_{f^*}(z) \le K^*.$$

For any $g = \rho|dw|^2 \in \mathcal{M}_{a,\kappa}$,

$$\begin{aligned}
\frac{1}{2}\left(K^* + \frac{1}{K^*}\right)\int_{\Sigma_2}\rho dudv \quad &\ge \quad \frac{1}{2}\int_{\Sigma_2}\left(K_{F^*}(w) + \frac{1}{K_{F^*}(w)}\right)\rho dudv \\
&= \quad \int_{\Sigma_2}\frac{|F^*_w|^2 + |F^*_{\bar{w}}|^2}{|F^*_w|^2 - |F^*_{\bar{w}}|^2}\rho dudv \\
&= \quad \int_{\Sigma_1}\frac{|f^*_z|^2 + |f^*_{\bar{z}}|^2}{|f^*_z|^2 - |f^*_{\bar{z}}|^2}\rho(f^*(z))(|f^*_z|^2 - |f^*_{\bar{z}}|^2)dxdy \\
&= \quad \int_{\Sigma_1}\rho(f^*(z))\left(|f^*_z|^2 + |f^*_{\bar{z}}|^2\right)dxdy \\
&= \quad {}^gE^{f^*}.
\end{aligned}$$

Thus the above inequality implies that for any $g \in \mathcal{M}_{a,\kappa}$,

$$\inf_{f\in\mathcal{F}_h} {}^\rho E^f \le {}^gE^{f^*} \le \frac{1}{2}\left(K^* + \frac{1}{K^*}\right)a.$$

and hence

$$\sup_{g\in\mathcal{M}_{a,\kappa}}\inf_{f\in\mathcal{F}_h} {}^gE^f \le \frac{1}{2}\left(K^* + \frac{1}{K^*}\right)a.$$

Fix $a_0 > 0$ and $\kappa_0 > 0$ and define

$$E : \mathcal{M}_{a_0,\kappa_0} \to \mathbf{R}$$

by setting

$$E(g) = \inf_{f \in \mathcal{F}_h} {}^g E^f.$$

Let $\{g_i\} \subset \mathcal{M}_{a_0,\kappa_0}$ be the maximizing sequence of $E(\cdot)$. In other words,

$$\lim_{i\to\infty} E(g_i) = \sup_{g \in \mathcal{M}_{a_0,\kappa_0}} E(g) = \sup_{g \in \mathcal{M}_{a_0,\kappa_0}} \inf_{f \in \mathcal{F}_h} {}^g E^f.$$

By Theorem 5, there exists a subsequence (which we will still call g_i by an abuse of notation) and $g_0 \in \mathcal{M}_{a_0,\kappa_0}$ so that g_i converges uniformly to g_0 in the sense of distance functions. Let f_i be an energy minimizing map with respect to g_i. By Theorem 11, there exists a subsequence (which again we will call g_i) so that ${}^{g_i}E^{f_i}$ converge to ${}^{g_0}E^{f_0}$ where f_0 is the energy minimizing map for g_0. Thus,

$${}^{g_0}E^{f_0} = \lim_{i\to\infty} {}^{g_i}E^{f_i} = \lim_{i\to\infty} \inf_{f \in \mathcal{F}_h} {}^{g_i}E^f$$

and we have found f_0 and g_0 satisfying

$${}^{g_0}E^{f_0} = \sup_{g \in \mathcal{M}_{a_0,\kappa_0}} \inf_{f \in \mathcal{F}_h} {}^g E^f. \tag{8}$$

We will show that f_0 is the Teichmüller map if $\kappa_0 > 0$.

7. The map f_0 is the Teichmüller map

Our strategy in showing that f_0 is a Teichmüller map is to first show that $k_{f_0}(z) \le k^*$ for every point $z \in \Sigma_1$, where $k^* = \sup k_{f^*}(z)$. This implies that f_0 is quasiconformal and hence its weak derivatives exist almost everywhere. Thus,

$$\begin{aligned}
{}^{g_0}E^{f_0} &= \int_{\Sigma_1} \rho_0(f_0(z))(|(f_0)_z|^2 + |(f_0)_{\bar z}|^2)dxdy \\
&= \int_{\Sigma_1} \rho_0(f_0(z)) \frac{|(f_0)_z|^2 + |(f_0)_{\bar z}|^2}{|(f_0)_z|^2 - |(f_0)_{\bar z}|^2}(|(f_0)_z|^2 - |(f_0)_{\bar z}|^2)dxdy \\
&= \int_{\Sigma_1} \rho_0(f_0(z)) \frac{1 + k_{f_0}^2(z)}{1 - k_{f_0}^2(z)}(|(f_0)_z|^2 - |(f_0)_{\bar z}|^2)dxdy \\
&\le \frac{1 + (k^*)^2}{1 - (k^*)^2} \int_{\Sigma_2} \rho_0 dudv.
\end{aligned} \tag{9}$$

Next, we will show

$${}^{g_0}E^{f_0} = \frac{1 + (k^*)^2}{1 - (k^*)^2} \int_{\Sigma_2} \rho_0 dudv,$$

This combined with inequality (9) shows that $k_{f_0} = k^*$ a.e.

We define

$$D_f(z) = \frac{1 + k_f^2(z)}{1 - k_f^2(z)}$$

and

$$D^* = \frac{1 + (k^*)^2}{1 - (k^*)^2}.$$

Proving that $k_{f_0} \le k^*$ is equivalent to showing $D_{f_0} \le D^*$. We also define

$$\mathcal{E}(a, \kappa) = \sup_{g \in \mathcal{M}_{a,\kappa}} \inf_{f \in \mathcal{F}_h} {}^g E^f.$$

Hence,

$$\mathcal{E}(a_0, \kappa_0) = {}^{g_0}E^{f_0} \le D^* a_0. \tag{10}$$

PART I: Proof that $k_{f_0} \le k^*$.

A key step in setting up a variational problem in the smooth setting is to establish the first variation formula for the functional. Because of the singular nature of our problem, we consider the following weaker version and show that it provides enough information for our purposes.

LEMMA 15. *Let $B_R(w_0) \subset \Sigma_2$ so that*

$$D_{f_0}(z) \ge C + \epsilon \quad \textit{for all } z \in f_0^{-1}(B_R(w_0)).$$

Assume that $t \mapsto \rho_t$, $0 \le t \le T$, is a one-parameter family of functions on $B_R(w_0)$ satisfying the following conditions: $\rho_t \ge \rho_0$, $\rho_t \to \rho_0$ uniformly, $\rho_t \equiv \rho_0$ near $\partial B_R(w_0)$, $\dot\rho_0 = \frac{d}{dt}\rho_t|_{t=0}$ is integrable and either $\frac{d^2}{dt^2}\rho_t \ge 0$ or $\frac{d^2}{dt^2}\rho_t \le 0$. Let g_t be defined by

$$g_t = \begin{cases} \rho_t |dw|^2 & \text{in } B_R(w_0) \\ \rho_0 |dw|^2 & \text{in } \Sigma_2 - B_R(w_0). \end{cases}$$

If $f_t \in \mathcal{F}_h$ be the energy minimizing map with respect to metric g_t as in Theorem 12, then there exists a sequence $t_i \to 0$ so that

$$\liminf_{t_i \to 0} \frac{{}^{g_{t_i}}E^{f_{t_i}} - {}^{g_0}E^{f_0}}{t_i} \ge (C + \epsilon_0)\, \dot a_0,$$

where $\epsilon_0 \in (0, \epsilon)$,

$$a_t = \int_{\Sigma_2} \rho_t du dv \quad \textit{and} \quad \dot a_0 = \frac{d}{dt} a_t|_{t=0}.$$

PROOF. If $\frac{d^2}{dt^2}\rho_t \ge 0$ then

$$\int_E \dot\rho du dv \le \int_E \frac{\rho_t - \rho_0}{t} du dv \le \int_E \frac{\rho_T - \rho_0}{T} du dv,$$

and if $\frac{d^2}{dt^2}\rho_t \le 0$ then

$$\int_E \frac{\rho_T - \rho_0}{T} du dv \le \int_E \frac{\rho_t - \rho_0}{t} du dv \le \int_E \dot\rho du dv$$

for any $E \subset \Sigma_2$. Furthermore,

$$\int_{\Sigma_2} \frac{\rho_T - \rho_0}{T} du dv < \infty \quad \text{and} \quad \int_{\Sigma_2} \dot\rho dv dy < \infty.$$

Hence, for $\kappa > 0$ to be chosen later, there exists $\delta_1 > 0$ so that

$$\int_E \frac{\rho_T - \rho_0}{T} du dv < \kappa \quad \text{and} \quad \int_E \dot\rho dv dy < \kappa$$

for any $E \subset \Sigma_2$ with $m(E) < \delta_1$. Therefore, for any $t \in (0, T)$,

$$\int_E \frac{\rho_t - \rho_0}{t} du dv < \kappa \quad \text{whenever} \quad m(E) < \delta_1. \tag{11}$$

Let

$$E_\delta = \{w \in B_R(w_0) : \rho_0(w) < \delta\}.$$

By Lemma 6, $m(E_\delta) \to 0$ as $\delta \to 0$. Since $\rho_t \geq \rho_0$, there exists $\delta_2 > 0$ and a set O of measure at most $\frac{\delta_1}{2}$ so that, $\rho_t(w) > \delta_2$ for $w \in B_R(w_0) - O$ and $t < T$.

By Theorem 11, there exists a sequence t_i so that $D_{f_{t_i}} \to D_{f_0}$ for almost every $z \in \Sigma_1$. For each t_i, let s be a smooth function so that $\triangle(\log \rho_{t_i} + s) \geq 0$ weakly. Therefore, for a symmetric smoothing function η_σ (i.e. $\eta_\sigma(z) = \frac{1}{\sigma^2}\eta(|z|)$ for a smooth function $\eta : \mathbf{R}^+ \cup \{0\} \to \mathbf{R}^+ \cup \{0\}$ with $\eta \equiv 1$ for $0 \leq t \leq \frac{1}{2}$, $\eta \equiv 0$ for $t \geq 1$ and $\int \eta = 1$), we have

$$\int (\log \rho_{t_i}(z+\zeta) + s(z+\zeta))\, \eta_\sigma(\zeta) d\xi d\tau \geq \log \rho_{t_i}(z) + s(z)$$

Let

$$\rho_{t_i}^\sigma = \exp\left(\int (\log \rho_{t_i}(z+\zeta) + s(z+\zeta))\, \eta_\sigma(\zeta) d\xi d\tau - s(z) \right).$$

Then $\rho_{t_i}^\sigma \geq \rho_{t_i} \geq \delta_2$. Let

$$g_{t_i}^\sigma = \begin{cases} \rho_{t_i}^\sigma |dw|^2 & \text{in } B_R(w_0) \\ \rho_0 |dw|^2 & \text{in } \Sigma_2 - B_R(w_0). \end{cases}$$

for σ sufficiently small and $f_{t_i}^\sigma$ be the map energy minimizing with respect to $g_{t_i}^\sigma$. By Theorem 11, there exists a sequence σ_j so that $D_{f_i}^{\sigma_j} \to D_{f_i}$. The maps $f_{t_i}^{\sigma_j}$ are Lipschitz with respect to metric $g_{t_i}^{\sigma_j}$ with local Lipschitz constant L which is independent of t_i or σ_j. (L is only dependent on the total energy.) This implies that for any $C \subset \Sigma_1$ and any t_i and sufficiently small σ_j,

$$\begin{aligned} \delta_2 m(f_{t_i}^{\sigma_j}(C) - O) &\leq \int_{f_{t_i}^{\sigma_j}(C) - O} \rho_{t_i}^{\sigma_j} dudv \\ &\leq \int_{C - (f_{t_i}^{\sigma_j})^{-1}(O)} |\nabla f_{t_i}^{\sigma_j}|^2_{g_{t_i}^{\sigma_j}} dxdy \\ &\leq L^2 m(C - (f_{t_i}^{\sigma_j})^{-1}(O)). \end{aligned}$$

In particular,

$$m(f_{t_i}^{\sigma_j}(C) - O) \leq \frac{L^2}{\delta_2} m(C - (f_{t_i}^{\sigma_j})^{-1}(O)). \tag{12}$$

Let $\delta_3 = \frac{\delta_2 \delta_1}{4L^2}$. By Egoroff's Theorem, there exists a set $A \subset \Sigma_1$ with $m(A) < \delta_3$ so that $D_{f_{t_i}} \to D_{f_0}$ uniformly on $\Sigma_1 - A$. Hence for sufficiently small t_i,

$$D_{f_{t_i}}(z) \geq C + \frac{\epsilon}{2}$$

for all $z \in \Sigma_1 - A$.

Again by Egoroff's Theorem, there exists a set $B \subset \Sigma_1$ with $m(B) < \delta_3$ so that $D_{f_{t_i}^{\sigma_j}} \to D_{f_{t_i}}$ uniformly on $\Sigma_1 - B$ and for a sufficiently small t_i and σ_j,

$$D_{f_{t_i}^{\sigma_j}}(z) \geq C + \frac{\epsilon}{4}$$

for all $z \in \Sigma_1 - (A \cup B)$. Noting that $f_{t_i}^{\sigma_j}$ is a smooth map, we have

$$\begin{aligned} {}^{g_{t_i}^{\sigma_j}}E^{f_{t_i}^{\sigma_j}} - {}^{g_0^{\sigma_j}}E^{f_0^{\sigma_j}} &\geq {}^{g_{t_i}^{\sigma_j}}E^{f_{t_i}^{\sigma_j}} - {}^{g_0^{\sigma_j}}E^{f_{t_i}^{\sigma_j}} \\ &= \int_{\Sigma_1} (\rho_{t_i}^{\sigma_j}(f_{t_i}^{\sigma_j}(z)) - \rho_0^{\sigma_j}(f_{t_i}^{\sigma}(z))) \left(|(f_{t_i}^{\sigma_j})_z|^2 + |(f_{t_i}^{\sigma_j})_{\bar z}|^2 \right) dxdy \end{aligned}$$

$$
\begin{aligned}
&\geq \int_{\Sigma_1-(A\cup B)} (\rho^{\sigma_j}_{t_i}(f^{\sigma}_{t_i}(z)) - \rho^{\sigma_j}_0(f^{\sigma}_{t_i}(z)))\left(|(f^{\sigma_j}_{t_i})_z|^2 + |(f^{\sigma_j}_{t_i})_{\bar z}|^2\right) dxdy \\
&= \int_{\Sigma_1-(A\cup B)} (\rho^{\sigma_j}_{t_i}(f^{\sigma}_{t_i}(z)) - \rho^{\sigma_j}_0(f^{\sigma}_{t_i}(z)))D_{f^{\sigma_j}_{t_i}}\left(|(f^{\sigma_j}_{t_i})_z|^2 - |(f^{\sigma_j}_{t_i})_{\bar z}|^2\right) dxdy \\
&\geq \left(C+\frac{\epsilon}{4}\right)\int_{\Sigma_1-(A\cup B)} (\rho^{\sigma_j}_{t_i}(f^{\sigma}_{t_i}(z)) - \rho^{\sigma_j}_0(f^{\sigma}_{t_i}(z)))\left(|(f^{\sigma_j}_{t_i})_z|^2 - |(f^{\sigma_j}_{t_i})_{\bar z}|^2\right) dxdy \\
&= \left(C+\frac{\epsilon}{4}\right)\int_{f^{\sigma_j}_{t_i}(\Sigma_1-(A\cup B))} (\rho^{\sigma_j}_{t_i} - \rho^{\sigma_j}_0) dudv \\
&= \left(C+\frac{\epsilon}{4}\right)\left(\int_{\Sigma_2} (\rho^{\sigma_j}_{t_i} - \rho^{\sigma_j}_0) dudv - \int_{f^{\sigma_j}_{t_i}(A\cup B)} (\rho^{\sigma_j}_{t_i} - \rho^{\sigma_j}_0) dudv\right)
\end{aligned}
$$

for sufficiently small t_i and σ_j.

By inequality (12) and because $\delta_3 = \frac{\delta_2\delta_1}{4L^2}$,

$$
\begin{aligned}
m((f^{\sigma_j}_{t_i}(A\cup A)\cup O) &\leq m(f^{\sigma_j}_{t_i}(A) - O)) + m(f^{\sigma_j}_{t_i}(B) - O) + m(O) \\
&\leq \frac{L^2}{\delta_2} m(A - (f^{\sigma_j}_{t_i})^{-1}(O)) + \frac{L^2}{\delta_2} m(B - (f^{\sigma_j}_{t_i})^{-1}(O)) + m(O) \\
&< \frac{L^2\delta_3}{\delta_2} + \frac{L^2\delta_3}{\delta_2} + \frac{\delta_1}{2} \\
&= \delta_1.
\end{aligned}
$$

Hence, by inequality 11,

$$
\int_{f^{\sigma_j}_{t_i}(A\cup B)} \frac{\rho_{t_i} - \rho_0}{t_i} dudv \leq \int_{f^{\sigma_j}_{t_i}(A\cup B)\cup O} \frac{\rho_{t_i} - \rho_0}{t_i} dudv < \kappa.
$$

Furthermore, since $\left|\frac{\rho^{\sigma_j}_{t_i} - \rho^{\sigma_j}_0}{t_i} - \frac{\rho_{t_i} - \rho_0}{t_i}\right| \to 0$ as $\sigma_j \to 0$,

$$
\int_{\Sigma_2} \left|\frac{\rho^{\sigma_j}_{t_i} - \rho^{\sigma_j}_0}{t_i} - \frac{\rho_{t_i} - \rho_0}{t_i}\right| dudv \leq \kappa
$$

for sufficiently small σ_j. Therefore,

$$
\begin{aligned}
&\int_{f^{\sigma_j}_{t_i}(A\cup B)} \frac{\rho^{\sigma_j}_{t_i} - \rho^{\sigma_j}_0}{t_i} dudv \\
&\leq \int_{f^{\sigma_j}_{t_i}(A\cup B)} \frac{\rho_{t_i} - \rho_0}{t_i} dudv + \int_{f^{\sigma_j}_{t_i}(A\cup B)} \frac{\rho^{\sigma_j}_{t_i} - \rho^{\sigma_j}_0}{t_i} - \frac{\rho_{t_i} - \rho_0}{t_i} dudv \\
&\leq \int_{f^{\sigma_j}_{t_i}(A\cup B)} \frac{\rho_{t_i} - \rho_0}{t_i} dudv + \int_{\Sigma_2} \left|\frac{\rho^{\sigma_j}_{t_i} - \rho^{\sigma_j}_0}{t_i} - \frac{\rho_{t_i} - \rho_0}{t_i}\right| dudv \\
&\leq 2\kappa
\end{aligned}
$$

for sufficiently small σ_j. This shows that

$$
{}^{g^{\sigma_j}_{t_i}}E^{f^{\sigma_j}_{t_i}} - {}^{g^{\sigma_j}_0}E^{f^{\sigma_j}_0} \geq \left(C+\frac{\epsilon}{4}\right)\left(\int_{\Sigma_2} (\rho^{\sigma_j}_{t_i} - \rho^{\sigma_j}_0) dudv - 2\kappa\right)
$$

and letting $\sigma_j \to 0$ and dividing by t_i, we get

$$
\frac{{}^{g_{t_i}}E^{f_{t_i}} - {}^{g_0}E^{f_0}}{t_i} \geq \left(C+\frac{\epsilon}{4}\right)\left(\int_{\Sigma_2} \frac{\rho_{t_i} - \rho_0}{t_i} dudv - 2\kappa\right)
$$

Therefore, letting $\kappa = \frac{\epsilon \dot{a}_0}{16(C+\frac{\epsilon}{4})}$, we have

$$\begin{aligned} \liminf_{i\to\infty} \frac{{}^{g_{t_i}}E^{f_{t_i}} - {}^{g_0}E^{f_0}}{t} &\geq \left(C+\frac{\epsilon}{4}\right)\left(\int_{\Sigma_2} \dot{\rho}_0 dudv - 2\kappa\right) \\ &> \left(C+\frac{\epsilon}{4}\right)\dot{a}_0 - \frac{\epsilon}{8}\dot{a}_0 \\ &= \left(C+\frac{\epsilon}{8}\right)\dot{a}_0. \end{aligned}$$

□

Let $z_0 \in \Sigma_1$. In order to show $D_{f_0}(z_0) \leq D^*$, we will assume that $D_{f_0}(z_0) \geq D^* + \epsilon$ for some $\epsilon > 0$, bump up the metric g_0 in a neighborhood of $f(z_0)$, and use Lemma 15 to seek a contradiction. In constructing a family of metrics which perturb g_0, it is important to control the distortion of the area and the curvature bound. For this construction, we will treat points in $\mathcal{C}$ and those not in $\mathcal{C}$ separately.

Case 1: $f(z_0) = w_0 \in \mathcal{C}$.

For $w_0 \in \mathcal{C}$, we will use the following lemma to bump up the metrics $g_0 = \rho_0|dw|^2$ in the neighborhood of w_0 in the sense of Lemma 15 without losing the curvature bound.

LEMMA 16. *Let $w_0 \in \Sigma_2$ and $R > 0$. There exists $0 < r_0 < R$ and $\phi \in \overline{C_c^\infty(B_R(w_0))}$ so that the following holds. If μ is any measure satisfying supp $\mu \subset \overline{B_{r_0}(w_0)}$, $\alpha = \mu(\overline{B_{r_0}(w_0)}) > 0$, $d\mu \geq -F(w)dudv$ for $F(w) \geq 0$,*

$$\rho = exp\left(\int log|w-z|d\mu\right)$$

and ν_t is a measure so that

$$(t\phi + \rho^{\frac{2}{\alpha}})^{\frac{\alpha}{2}} = \exp\left(\int \log|w-z|d\nu_t + h_t(z)\right) \tag{13}$$

where h_t is a harmonic function, then $d\nu_t \geq -F(w)dudv$ where $0 \leq t \leq 1$.

PROOF. It is sufficient to prove the statement assuming that $R = 1$ and w_0 is the origin of the unit disk. We let $\eta : [0,1] \to \mathbf{R}$ be a smooth function and $\phi(w) = \eta(r)$ where $r = |w|$. Also let $\psi = |w|^2$. Then

$$\begin{aligned} \nabla\psi(w) = 2r\frac{\partial}{\partial r} \qquad & \nabla\phi(w) = \eta'(r)\frac{\partial}{\partial r} \\ \triangle\psi = 4 \qquad & \triangle\phi = \eta''(r) + \frac{1}{r}\eta'(r). \end{aligned}$$

Using $\psi\triangle\psi - |\nabla\psi|^2 = \psi^2\triangle\log\psi = 0$, we have

$$\begin{aligned} (\phi+\psi)^2\triangle\log(\phi+\psi) &= (\phi+\psi)\triangle(\phi+\psi) - |\nabla(\phi+\psi)|^2 \\ &= \phi\triangle\phi - |\nabla\phi|^2 + \psi\triangle\psi - |\nabla\psi|^2 + \phi\triangle\psi - 2\nabla\psi\cdot\nabla\phi + \psi\triangle\phi \\ &= \eta\eta'' + \frac{1}{r}\eta\eta' - |\eta'|^2 + 4\eta - 3r\eta' + r^2\eta''. \end{aligned}$$

Let η be defined by

$$\eta(t) = \begin{cases} 2\epsilon e^{-2} & \text{for } t = 0 \\ 2\epsilon e^{-2} - \epsilon e^{-1/t} & \text{for } 0 < t \leq 0.5 \\ \epsilon e^{1/(t-1)} & \text{for } 0.5 < t < 1 \\ 0 & \text{for } t = 1 \end{cases}$$

It is a straightforward computation to check that

$$4\eta - 3t\eta' + t^2\eta'' \geq c_1\epsilon > 0$$

for some constant c_1 and since $\eta\eta'' + \frac{1}{t}\eta\eta' - |\eta'|^2$ is quadratic in ϵ,

$$\eta\eta'' + \frac{1}{t}\eta\eta' - |\eta'|^2 + 4\eta - 3t\eta' + t^2\eta'' \geq c_2\epsilon > 0$$

for some constant c_2 for sufficiently small ϵ. Now consider

$$\eta_{t_0}(t) = \begin{cases} 2\epsilon e^{-2} & \text{for } 0 \leq t \leq t_0 \\ \eta\left(\frac{t-t_0}{1-2t_0}\right) & \text{for } t_0 < t < 1-t_0, \\ 0 & \text{for } 1-t_0 \leq 1. \end{cases}$$

If we let $\tau = \frac{t-t_0}{1-2t_0}$, then $\frac{d\tau}{dt} = \frac{1}{1-2t_0}$ and $\frac{t}{1-2t_0} = \tau + \frac{t_0}{1-2t_0}$. Thus,

$$\begin{aligned} & 4\eta_{t_0}(t) - 3t\eta_{t_0}'(t) + t^2\eta_{t_0}''(t) \\ = \ & 4\eta(\tau) - \tfrac{3t}{1-2t_0}\eta'(\tau) + \left(\tfrac{t}{1-2t_0}\right)^2\eta''(\tau) \\ = \ & 4\eta(\tau) - 3\tau\eta'(\tau) + \tau^2\eta''(\tau) - \tfrac{3t_0}{1-2t_0}\eta'(\tau) + \tfrac{2t_0}{1-2t_0}\tau\eta''(\tau) + \left(\tfrac{t_0}{1-2t_0}\right)^2\eta''(\tau). \end{aligned}$$

Thus, for $t_0 > 0$ and $\epsilon > 0$ sufficiently small,

$$\eta_{t_0}\eta_{t_0}'' + \frac{1}{t}\eta_{t_0}\eta_{t_0}' - |\eta_{t_0}'|^2 + 4\eta_{t_0} - 3t\eta_{t_0}' + t^2\eta_{t_0}'' \geq c_3\epsilon \tag{14}$$

for some constant c_3. We now fix $t_0 > 0$ and $\epsilon > 0$ so that inequality (14) holds. Since we assume supp $\mu \subset \overline{B_{r_0}(0)}$, for $w \in B_1(0) - B_{r_0}(0)$, we have

$$\begin{aligned} \rho^{\frac{2}{\alpha}}(w) &= \exp\left(\frac{1}{\alpha}\int_{B_{r_0}(0)} \log|w-z|^2 d\mu(z)\right) \\ \nabla\rho^{\frac{2}{\alpha}}(w) &= \rho^{\frac{2}{\alpha}}\left(\frac{2}{\alpha}\int_{B_{r_0}(0)} \frac{u-x}{|w-z|^2}d\mu(z), \frac{2}{\alpha}\int_{B_{r_0}(0)} \frac{v-y}{|w-z|^2}d\mu(z)\right) \\ \triangle\rho^{\frac{2}{\alpha}}(w) &= 4\rho^{\frac{2}{\alpha}}\left[\left(\frac{1}{\alpha}\int_{B_{r_0}(0)} \frac{u-x}{|w-z|^2}d\mu(z)\right)^2 + \left(\frac{1}{\alpha}\int_{B_{r_0}(0)} \frac{v-y}{|w-z|^2}d\mu(z)\right)^2\right]. \end{aligned}$$

For any $\kappa > 0$, we can choose r_0 sufficienlty small so that for any $w \in B_1(0) - B_{\frac{t_0}{2}}(0)$ and $z \in B_{r_0}(0)$,

$$\left|\log|w-z|^2 - \log|w|^2\right| < \kappa.$$

Thus $w \in B_1(0) - B_{\frac{t_0}{2}}(0)$,

$$\frac{1}{\alpha}\int_{B_{r_0}(0)} (\log|w|^2 - \kappa)d\mu(z) < \frac{1}{\alpha}\int_{B_{r_0}(0)} \log|w-z|^2 d\mu(z) < \frac{1}{\alpha}\int_{B_{r_0}(0)} (\log|w|^2 + \kappa)d\mu(z).$$

Using the fact that $\mu(B_{r_0}(0)) = \alpha$, we have

$$\log|w|^2 - \kappa < \frac{1}{\alpha}\int_{B_{r_0}(0)} \log|w-z|^2 d\mu(z) < \log|w|^2 + \kappa$$

which implies

$$e^{-\kappa}|w|^2 < \rho^{\frac{2}{\alpha}} < e^{\kappa}|w|^2.$$

Thus, for any $\delta > 0$, we can choose r_0 sufficiently small so that

$$|\rho^{\frac{2}{\alpha}} - r^2| < \delta.$$

Similarly, by choosing r_0 sufficienlty small,

$$\begin{aligned} \left|\nabla\rho^{\frac{2}{\alpha}} - 2w\right| &< \delta \\ \left|\triangle\rho^{\frac{2}{\alpha}} - 4\right| &< \delta. \end{aligned}$$

for $w \in B_1(0) - B_{\frac{t_0}{2}}(0)$.

If we set $\phi(w) = \eta_{t_0}(|w|)$, then for $w \in B_1(0) - B_{\frac{t_0}{2}}(0)$, we have

$$\begin{aligned} &(\phi+\rho^{\frac{2}{\alpha}})^2 \triangle \log(\phi+\rho^{\frac{2}{\alpha}}) \\ &= (\phi+\rho^{\frac{2}{\alpha}})\triangle(\phi+\rho^{\frac{2}{\alpha}}) - |\nabla(\phi+\rho^{\frac{2}{\alpha}})|^2 \\ &= \phi\triangle\phi - |\nabla\phi|^2 + \rho^{\frac{2}{\alpha}}\triangle\rho^{\frac{2}{\alpha}} - |\nabla\rho^{\frac{2}{\alpha}}|^2 + \phi\triangle\rho^{\frac{2}{\alpha}} - 2\nabla\phi\cdot\nabla\rho^{\frac{2}{\alpha}} + \rho^{\frac{2}{\alpha}}\triangle\phi \end{aligned} \tag{15}$$

and

$$\begin{aligned} \phi\triangle\rho^{\frac{2}{\alpha}} &\geq \eta_{t_0}\cdot(4-\delta) \\ -2\nabla\phi\cdot\nabla\rho^{\frac{2}{\alpha}} &\geq 4r\eta'_{t_0} + 2\delta\eta'_{t_0} \\ \rho^{\frac{2}{\alpha}}\triangle\phi &\geq r^2(\eta''_{t_0} + \frac{1}{r}\eta'_{t_0}) - \delta|\eta''_{t_0}|^2 + \frac{\delta}{r}\eta'_{t_0}. \end{aligned}$$

Thus

$$\begin{aligned} (\phi+\rho^{\frac{2}{\alpha}})^2\triangle\log(\phi+\rho^{\frac{2}{\alpha}}) \geq\ & \eta_{t_0}\eta''_{t_0} + \frac{1}{r}\eta_{t_0}\eta'_{t_0} - |\eta'_{t_0}|^2 + \rho^{4/\alpha}\triangle\log\rho^{\frac{2}{\alpha}} \\ & +4\eta_{t_0} - 3r\eta'_{t_0} + r^2\eta''_{t_0} \\ & +\delta(-\eta_{t_0} + 2\eta'_{t_0} - |\eta''_{t_0}| + \frac{1}{r}\eta'_{t_0}) \end{aligned}$$

for $w \in B_1(0) - B_{\frac{t_0}{2}}(0)$. By inequality (14), we see that

$$\eta_{t_0}\eta''_{t_0} + \frac{1}{r}\eta_{t_0}\eta'_{t_0} - |\eta'_{t_0}|^2 + 4\eta_{t_0} - 3r\eta'_{t_0} + r^2\eta''_{t_0} + \delta(-\eta_{t_0} + 2\eta'_{t_0} - |\eta''_{t_0}| + \frac{1}{r}\eta'_{t_0}) \geq 0$$

for $w \in B_1(0) - B_{\frac{t_0}{2}}(0)$ by choosing δ (and hence r_0) sufficiently small. Hence, it follows that

$$(\phi+\rho^{\frac{2}{\alpha}})^2\triangle\log(\phi+\rho^{\frac{2}{\alpha}}) \geq \rho^{4/\alpha}\triangle\log\rho^{\frac{2}{\alpha}} \tag{16}$$

for $w \in B_1(0) - B_{\frac{t_0}{2}}(0)$. Since supp $\mu \subset B_{r_0}(0)$, we have that $\triangle\log\rho \geq 0$ outside of $B_{r_0}(0)$ and this implies,

$$\triangle\log(\phi+\rho^{\frac{2}{\alpha}})^{\frac{\alpha}{2}} \geq 0 \tag{17}$$

for $w \in B_1(0) - B_{\frac{t_0}{2}}(0)$.

Recall that by construction, $\phi(w) \equiv 2\epsilon e^{-2}$ for $w \in B_{t_0}(0)$. Now for a smooth function $u \geq 0$ and constant $c \geq 0$,

$$\triangle \log(c+u) \geq \frac{u}{c+u} \triangle \log u.$$

Hence if $\triangle \log u \geq -F(w)$ with $F(w) \geq 0$, then $\triangle \log(u+c) \geq -F(w)$. Thus, by mollifying ρ and taking the limit, we see that

$$\int \triangle\psi \cdot \log(\phi + \rho^{\frac{2}{\alpha}})^{\frac{\alpha}{2}} dudv \geq -\int \psi F(w) dudv$$

for any $\psi \in C_c^\infty(B_{t_0}(0))$ and hence

$$d\nu_t \geq -F(w)dudv \tag{18}$$

in $B_{t_0}(0)$. Inequality 18 together with inequality 17 proves that $d\nu_t \geq -F(w)dudv$ on all of $B_1(0)$. The fact that we can replace ϕ with $t\phi$ for $0 \leq t \leq 1$ is evident from the proof. □

We are now ready to construct a one-parameter family of metrics $g_t = \rho_t|dw|^2$ and to apply the first variation argument of Lemma 15. With the aid of Lemma 16, we construct this one-parameter family so that we have control over the curvature bounds of g_t.

Suppose $D_{f_0}(z_0) \geq D^* + \epsilon$ for some $\epsilon > 0$. By the lower semicontinuity of D_{f_0}, there exists $B_R(f_0(z_0))$ so that $D_{f_0}(z) > D^* + \frac{\epsilon}{2}$ for any $z \in f_0^{-1}(B_R(f_0(z_0)))$. Let r_0 and ϕ be as in Lemma 16. By the definition of $\mathcal{C}$, there exists $\overline{B_\delta(\bar{w})} \subset B_{r_0}(w_0)$ so that $\alpha = \mathcal{K}(\overline{B_\delta(\bar{w})}) > 0$. Define $\mathcal{K}_1$ and $\mathcal{K}_2$ by setting $\mathcal{K}_1(E) = \mathcal{K}(E \cap \overline{B_\delta(\bar{w})})$ and $\mathcal{K}_2 = \mathcal{K} - \mathcal{K}_1$ and let

$$\rho_1(w) = \exp\left(\int \log|w-z| d\mathcal{K}_1(z)\right)$$

and

$$\rho_2(w) = \frac{\rho(w)}{\rho_1(w)} = \exp\left(\int \log|w-z| d\mathcal{K}_2(z) + h(w)\right)$$

for some harmonic function $h(w)$. Since supp $\mathcal{K}_1 \subset \overline{B_\delta(\bar{w})}$ and $\mathcal{K}_2(\overline{B_\delta(\bar{w})}) = 0$, $d\mathcal{K}_1 \geq -F(w)dudv$ where

$$F(w) = \begin{cases} 2\kappa\rho_0 & \text{in } \overline{B_\delta(\bar{w})} \\ 0 & \text{in } B_R(w_0) - \overline{B_\delta(\bar{w})}. \end{cases}$$

By Lemma 16, $d\nu_t \geq -F(w)dudv$ where ν_t is as in equality (13) with ρ_1 instead of ρ. Hence, if we let $\rho_t = (t\phi + \rho_1^{\frac{2}{\alpha}})^{\frac{\alpha}{2}}\rho_2$, then ρ_t satisfies

$$\begin{aligned}
\int (\triangle\psi) \log \rho_t dudv &= \int (\triangle\psi) \log(t\phi + \rho_1^{\frac{2}{\alpha}})^{\frac{\alpha}{2}} \rho_2 dudv \\
&= \int (\triangle\psi) \log(t\phi + \rho_1^{\frac{2}{\alpha}})^{\frac{\alpha}{2}} dudv + \int (\triangle\psi) \log \rho_2 dudv \\
&= \int \psi d\nu_t + \int \psi d\mathcal{K}_2 \\
&\geq -\int_{\overline{B_\delta(\bar{w})}} \psi F(w) dudv + \int_{B_R(w_0) - \overline{B_\delta(\bar{w})}} \psi d\mathcal{K}_2
\end{aligned}$$

$$
\begin{aligned}
&\geq -2\kappa_0 \int_{\overline{B_\delta(\bar{w})}} \psi\rho_0 dudv - 2\kappa_0 \int_{B_R(w_0)-\overline{B_\delta(\bar{w})}} \psi\rho_0 dudv \\
&= -2\kappa_0 \int \psi\rho_0 dudv \\
&\geq -2\kappa_0 \int \psi\rho_t dudv
\end{aligned}
$$

for any $\psi \in C_c^\infty(B_R(w_0))$. In other words,

$$\triangle \log \rho_t \geq -2\kappa_0 \rho_t$$

weakly. Let $g_t = \rho_t |dw|^2$ and $a(t) = \int_{\Sigma_2} \rho_t dudv$. Thus $g_t \in \mathcal{M}_{a(t),\kappa_0}$ We now show that:

LEMMA 17. *The function*

$$\dot{\rho}_0 = \frac{\alpha}{2} \rho_1^{1-\frac{2}{\alpha}} \rho_2 \phi$$

is an integrable function on Σ_2.

PROOF. Using the fact that $\frac{1}{\alpha}\mathcal{K}_1(B_{r_0}) = 1$,

$$
\begin{aligned}
\rho_1^{1-\frac{2}{\alpha}}(w) &= \exp\left(\left(1-\frac{2}{\alpha}\right)\int_{z\in B_{r_0}(w_0)} \log|w-z| d\mathcal{K}_1(z)\right) \\
&= \exp\left(\int_{z\in B_{r_0}(w_0)} \log|w-z|^{\alpha-2} \frac{d\mathcal{K}_1(z)}{\alpha}\right) \\
&\leq \int_{z\in B_{r_0}(w_0)} |w-z|^{\alpha-2} \frac{d\mathcal{K}_1(z)}{\alpha}
\end{aligned}
$$

by Jensen's Theorem. Hence,

$$
\begin{aligned}
\int_{w\in B_R(w_0)} \rho_1^{1-\frac{2}{\alpha}}(w) dudv &\leq \int_{w\in B_R(w_0)} \left(\int_{z\in B_{r_0}(w_0)} |w-z|^{\alpha-2} \frac{d\mathcal{K}_1(z)}{\alpha}\right) dudv \\
&= \int_{z\in B_{r_0}(w_0)} \left(\int_{w\in B_R(w_0)} |w-z|^{\alpha-2} dudv\right) \frac{d\mathcal{K}_1(z)}{\alpha}.
\end{aligned}
$$

If we let (r,θ) be the polar coordinates centered at z,

$$\int_{w\in B_{r_0}(w_0)} |w-z|^{\alpha-2} dudv \leq \int_0^{2\pi}\int_0^1 r^{\alpha-2} r dr d\theta = \frac{2\pi}{\alpha}.$$

Hence,

$$\int_{w\in B_R(w_0)} \rho_1^{1-\frac{2}{\alpha}}(w) dudv \leq \frac{2\pi}{\alpha}. \tag{19}$$

Since ρ_2 and ϕ are bounded functions, the assertion of the lemma follows from inequality (19). $\square$

Furthermore, we have that

$$\frac{d^2}{dt^2}\rho_t = \frac{\alpha}{2}\left(\frac{\alpha}{2}-1\right)(t\phi+\rho_1)^{\frac{\alpha}{2}-2}\phi^2\rho_2$$

so either $\frac{d^2}{dt^2}\rho_t \geq 0$ for all t or $\frac{d^2}{dt^2}\rho_t \leq 0$ for all t. Therefore, we can apply Lemma 15 and there exists a sequence $\{t_i\}$ so that

$$\liminf_{t_i \to 0} \frac{{}^{g_{t_i}}E^{f_{t_i}} - {}^{g_0}E^{f_0}}{t_i} > D^* a'(0).$$

We let

$$\mathcal{E}(a,\kappa) = \sup_{g \in \mathcal{M}_{a,\kappa}} \inf_{f \in \mathcal{F}_h} {}^{g}E^{f}.$$

Since $g_t \in \mathcal{M}_{a(t),\kappa_0}$, we have that $\mathcal{E}(a(t),\kappa_0) \geq {}^{g_t}E^{f_t}$, and thus

$$\begin{aligned}
\liminf_{t \to 0} \frac{\mathcal{E}(a(t),\kappa(0)) - \mathcal{E}(a(0),\kappa(0))}{\partial t} &= \liminf_{t \to 0} \frac{\mathcal{E}(a(t),\kappa(t)) - \mathcal{E}(a(0),\kappa(0))}{t} \\
&\geq \liminf_{t_i \to 0} \frac{{}^{g_{t_i}}E^{f_{t_i}} - {}^{g_0}E^{f_0}}{t_i} \\
&> D^* a'(0).
\end{aligned}$$

Recall that $\mathcal{E}(a,\kappa)$ is differentiable since $a \mapsto \mathcal{E}(a,\kappa)$ is a monotone function. Therefore, we have,

$$\liminf_{t \to 0} \frac{\mathcal{E}(a(t),\kappa(0)) - \mathcal{E}(a(0),\kappa(0))}{\partial t} = \frac{\partial}{\partial t}\mathcal{E}(a(t),\kappa_0)|_{t=0} = \frac{\partial \mathcal{E}}{\partial a}(a(0),\kappa_0) \cdot a'(0).$$

Hence

$$\frac{\partial \mathcal{E}}{\partial a}(a_0,\kappa_0) > D^*.$$

Since $\mathcal{E}(a_0,\kappa_0) = {}^{g_0}E^{f_0}$, we have that $\mathcal{E}(a,\kappa_0) = {}^{\left(\frac{a}{a_0}g_0\right)}E^{f_0} = \left(\frac{a}{a_0}\right) \cdot {}^{g_0}E^{f_0}$. Hence $a \mapsto \mathcal{E}(a,\kappa_0)$ is a linear function with $\lim_{a\to 0}\mathcal{E}(a,\kappa_0) = 0$. This implies that $\mathcal{E}(a_0,\kappa_0) > D^* a_0$, a contradiction to inequality 10. Therefore, we have shown that $D_{f_0}(z_0) \leq D^*$ if $f(z_0) \in \mathcal{C}$.

Case 2: $w_0 = f_0(z_0) \in \Sigma_2 - \mathcal{C}$.

Suppose there exists $z_0 \in \Sigma_1$ so that $D_{f_0}(z_0) \geq D^* + \epsilon$ and $w_0 = f_0(z_0) \in \Sigma_2 - \mathcal{C}$. By the lower semicontinuity of D_{f_0}, there exists $B_R(w_0)$ so that $D_{f_0}(z) > D^* + \frac{\epsilon}{2}$ for all $z \in f_0^{-1}(B_R(w_0))$. Let ψ be a smooth function not identically 1 so that $\psi \equiv 1$ in $\Sigma_2 - B_R(w_0)$ and $\psi \geq 1$ in $B_R(w_0)$.

Let $h = \psi\rho_0|dw|^2$, $\eta = \psi^{1/n}\rho_0$,

$$\rho_t = ((1-t)\rho_0^n + t\eta^n)^{1/n},$$

and $g_t = \rho_t|dw|^2$ with n to be chosen later. Now, since

$$\dot{\rho}_t = \frac{1}{n}((1-t)\rho_0^n + t\eta^n)^{\frac{1}{n}-1}(\eta^n - \rho_0^n)$$

we have

$$\dot{\rho}_0 = \frac{1}{n}\rho_0^{1-n}(\eta^n - \rho_0^n) = \frac{1}{n}(\psi - 1)\rho_0,$$

and thus $\dot{\rho}_0$ is integrable. Furthermore,

$$\frac{d^2}{dt^2}\rho_t = \frac{1}{n}(\frac{1}{n} - 1)((1-t)\rho_0^n + t\eta^n)^{\frac{1}{n}-2}(\eta^n - \rho_0^n) \leq 0.$$

Therefore, by Lemma 15,

$$\liminf_{t_i \to 0} \frac{{}^{g_{t_i}}E^{f_{t_i}} - {}^{g_0}E^{f_0}}{t_i} \geq (D^* + \epsilon_0)a'(0).$$

We now compute a curvature bound for g_t. If ρ_0 is smooth, then

$$\begin{aligned}
&-\frac{1}{2\rho_t}\triangle \log \rho_t \\
&= \frac{-\triangle \log \rho_0 + \frac{1}{n}\triangle \log(1-t+t\psi)}{2\rho_0(1-t+t\psi)^{\frac{1}{n}}} \\
&\leq -\frac{1}{2\rho_0(1-t+t\psi)^{\frac{1}{n}}}\left(-2\kappa_0\rho_0 + \frac{1}{n}\frac{t\triangle\psi}{1-t+t\psi} - \frac{1}{n}\frac{t^2|\nabla\psi|^2}{(1-t+t\psi)^2}\right) \\
&\leq \kappa_0 + \kappa_0\left(\frac{1}{(1-t+t\psi)^{\frac{1}{n}}} - 1\right) - \frac{1}{n}\frac{t\triangle\psi}{2\rho_0(1-t+t\psi)^{1+\frac{1}{n}}} + \frac{1}{n}\frac{t^2|\nabla\psi|^2}{2\rho_0(1-t+t\psi)^{2+\frac{1}{n}}}
\end{aligned}$$

Since

$$\frac{1}{(1-t+t\psi)^{\frac{1}{n}}} - 1 \leq 0$$

we have

$$\triangle \log \rho_t \geq -2\kappa(t)\rho_t$$

where

$$\kappa(t) = \kappa_0 + \frac{Ct}{n}$$

with constant C depending on the bounds for $|\triangle\psi|$ and $|\nabla\psi|^2$. By Lemma 10, $\rho_0 \geq \epsilon > 0$ and thus the curvature bound of $\kappa(t)$ remains valid even for singular metrics by using a smooth approximation and taking the limit. Therefore, $g_t \in \mathcal{M}_{a(t),\kappa(t)}$ and

$${}^{g_t}E^{f_t} \leq \mathcal{E}(\kappa(t), a(t)).$$

Since $\mathcal{E}(a,\kappa)$ is differentiable (since $a \mapsto \mathcal{E}(a,\kappa)$ is linear and $\kappa \mapsto \mathcal{E}(a,\kappa)$ is monotone), this implies

$$\frac{\partial\mathcal{E}}{\partial a}(a(0),\kappa(0)) \cdot a'(0) + \frac{\partial\mathcal{E}}{\partial\kappa}(a(0),\kappa(0)) \cdot \kappa'(0) = \frac{\partial\mathcal{E}}{\partial t}|_{t=0} \geq (D^* + \epsilon_0)a'(0). \tag{20}$$

Let $(\bar{g}_0, \bar{f}_0)$ be the critical pair for $(\kappa_0 + \epsilon, a_0)$. In particular, if $\bar{g}_0 = \bar{\rho}_0|dw|^2$, then

$$\triangle \log \bar{\rho}_0 \geq -2(\kappa_0+\epsilon)\bar{\rho}_0 = -2\kappa_0\left(\frac{\kappa_0+\epsilon}{\kappa_0}\right)\bar{\rho}_0$$

weakly. Using the assumption that $\kappa_0 > 0$, $A(\frac{\kappa_0+\epsilon}{\kappa_0}\bar{g}_0) = \frac{\kappa_0+\epsilon}{\kappa_0}A(\bar{g}_0)$ and thus $(\frac{\kappa_0+\epsilon}{\kappa_0})\bar{g}_0 \in \mathcal{M}_{\frac{\kappa_0+\epsilon}{\kappa_0}a_0,\kappa_0}$. Since $\bar{f}_0$ is an energy minimizing map with respect to metric $(\frac{\kappa_0+\epsilon}{\kappa_0})\bar{g}_0$, we have that

$$\mathcal{E}\left(\frac{\kappa_0+\epsilon}{\kappa_0}a_0,\kappa_0\right) \geq {}^{(\frac{\kappa_0+\epsilon}{\kappa_0})\bar{g}_0}E^{\bar{f}_0} = \left(\frac{\kappa_0+\epsilon}{\kappa_0}\right){}^{\bar{g}_0}E^{\bar{f}_0} = \frac{\kappa_0+\epsilon}{\kappa_0}\mathcal{E}(a_0,\kappa_0+\epsilon).$$

Thus,

$$\begin{aligned}
\frac{\partial\mathcal{E}}{\partial\kappa}(a_0,\kappa_0) &= \lim_{\epsilon\to 0}\frac{\mathcal{E}(a_0,\kappa_0+\epsilon) - \mathcal{E}(a_0,\kappa_0)}{\epsilon} \\
&\leq \lim_{\epsilon\to 0}\frac{\frac{\kappa_0}{\kappa_0+\epsilon}\mathcal{E}(\frac{\kappa_0+\epsilon}{\kappa_0}a_0,\kappa_0+\epsilon) - \mathcal{E}(a_0,\kappa_0)}{\epsilon}
\end{aligned}$$

$$\begin{aligned}
&= \frac{d}{dt}\left(\frac{\kappa_0}{t}\mathcal{E}\left(\frac{t}{\kappa_0}a_0,\kappa_0\right)\right)|_{t=\kappa_0} \\
&= \frac{d}{dt}\left(\frac{\kappa_0}{t}\right)|_{t=\kappa_0}\cdot\mathcal{E}\left(\frac{t}{\kappa_0}a_0,\kappa_0\right)+\frac{d}{dt}\mathcal{E}\left(\frac{t}{\kappa_0}a_0,\kappa_0\right)|_{t=\kappa_0} \\
&= -\frac{1}{\kappa_0}\mathcal{E}(a_0,\kappa_0)+\frac{\partial\mathcal{E}}{\partial a}\left(\frac{t}{\kappa_0}a_0,\kappa_0\right)\cdot\frac{d}{dt}\left(\frac{a_0 t}{\kappa_0}\right)|_{t=\kappa_0} \\
&= -\frac{1}{\kappa_0}\mathcal{E}(a_0,\kappa_0)+\frac{a_0}{\kappa_0}\frac{\partial\mathcal{E}}{\partial a}(a_0,\kappa_0) \\
&\leq \frac{a_0}{\kappa_0}\frac{\partial\mathcal{E}}{\partial a}(a_0,\kappa_0)
\end{aligned}$$

Furthermore,

$$\kappa'(0)=\frac{C}{n}.$$

and hence,

$$\begin{aligned}
&\frac{\partial\mathcal{E}}{\partial a}(a_0,\kappa_0)\cdot a'(0)+\frac{\partial\mathcal{E}}{\partial \kappa}(a_0,\kappa_0)\cdot\kappa'(0) \\
&\quad\leq \frac{\partial\mathcal{E}}{\partial a}(a_0,\kappa_0)\cdot a'(0)+\frac{Ca_0}{n\kappa_0}\frac{\partial\mathcal{E}}{\partial a}(a_0,\kappa_0) \\
&\quad\leq \frac{\partial\mathcal{E}}{\partial a}(a_0,\kappa_0)\cdot\left(a'(0)+\frac{Ca_0}{n\kappa_0}\right)
\end{aligned}$$

Combining this with inequality (20) gives

$$\frac{\partial\mathcal{E}}{\partial a}(a_0,\kappa_0)\cdot\left(a'(0)+\frac{Ca_0}{n\kappa_0}\right)\geq(D^*+\epsilon_0)\,a'(0)$$

Therefore, by choosing n sufficiently large,

$$\frac{\partial\mathcal{E}}{\partial a}(a_0,\kappa_0)>D^*.$$

As in Case 1, this is a contradiction.

PART 2: Proof that ${}^{g_0}E^{f_0}=\frac{1+(k^*)^2}{1-(k^*)^2}a_0$.

For any $\epsilon>0$, there exists z_0 so that $D_{f_0}(z_0)\geq D^*-\epsilon$ by the definition of f^* and D^*. Let $w_0=f(z_0)$. By the lower semicontinuity of D_{f_0}, there exists $B_R(w_0)$ so that $D_{f_0}(z)>D^*-2\epsilon$. Proceeding as in Case 1 or Case 2 above,

$$\liminf_{t\to 0}\frac{{}^{g_t}E^{f_t}-{}^{g_0}\,E^{f_0}}{t}\geq(D^*-3\epsilon)a'(0).$$

Since ϵ is arbitrary,

$$\frac{\partial\mathcal{E}}{\partial a}(a_0,\kappa_0)\geq D^*$$

By the linearity of $a\mapsto\mathcal{E}(a,\kappa_0)$ and the fact that $\mathcal{E}(a_0,\kappa_0)\geq D^*a_0$, this implies that

$$\mathcal{E}(a_0,\kappa_0)=D^*a_0.$$

This shows that $D_{f_0}=D^*$ a.e. and hence $k_{f_0}=k^*$ a.e.

The Hopf differential $\Phi_{f_0}=\varphi_{f_0}dz^2$ is the differential to f_0 associated in Teichmüller's theorem. The differential Ψ_{f_0} on the target is defined by the condition that $\Psi_{f_0} = \left(\frac{m+2}{2}\right)^2 w^m dw^2$ for the coordinate $w = f_0(z)$ whenever z is a natural parameter of Φ_{f_0}, i.e. $\Phi_{f_0} = \left(\frac{m+2}{2}\right)^2 z^m dz^2$. Kuwert [**Ku**] proved that f_0 is the unique minimizing map in its homotopy class with respect to the metric $|\Psi_{f_0}|$. Since Ψ_{f_0} is holomorphic, $|\Psi_{f_0}|$ defines a cone metric which is a smooth metric of curvature 0 except at finite number of degenerate points $\mathcal{D} = \{w : \Psi_{f_0}(w) = 0\}$. Therefore, f_0 is smooth on the set $\Sigma_1 - f_0^{-1}(\mathcal{D})$ which implies that $k_{f_0} \equiv k^*$ except possibly on $f_0^{-1}(\mathcal{D})$.

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