Superrigidity of Hyperbolic Buildings

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Abstract

In this paper we study the behavior of harmonic maps into complexes with branching differentiable manifold structure. Main examples of such target spaces are Euclidean and hyperbolic buildings. We show that a harmonic map from an irreducible symmetric space of noncompact type other than real or complex hyperbolic into these complexes are non-branching. As an application, we prove rank one and higher rank superrigidity for the isometry groups of a class of complexes which includes hyperbolic buildings as a special case.

1 Introduction.

Let Y be a locally compact Riemannian cell complex. By this, we mean a complex where each cell is endowed with a Riemannian metric smooth up to the boundary and such that any of its faces is totally geodesic. Furthermore, we assume that Y has non-positive curvature (NPC) and a *branching* **D***ifferentiable* **M***anifold structure*. Roughly speaking, this means that for any two adjacent cells there exists an isometric and totally geodesic embedding

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of a complete Riemannian manifold containing both cells. We henceforth refer to such complexes as DM-complexes and the images of the differentiable manifolds as DM's. For precise definitions, we refer to Section 2. The main examples of such spaces are Euclidean and hyperbolic buildings. Our main theorem is to show the following non-branching behavior of harmonic maps into such spaces.

Theorem 1 *Let $\tilde{X} = G/K$ be an irreducible symmetric space of noncompact type other than $SO_0(p,1)/SO(p) \times SO(1)$, $SU_0(p,1)/S(U(p) \times U(1))$. Let Γ be a lattice in G and let $\rho : \Gamma \to Isom(Y)$ a group homomorphism where Y is an NPC DM-complex. If the rank of $\tilde{X}$ is ≥ 2, we assume additionally that Γ is cocompact. If the rank of $\tilde{X}$ is $= 1$, we assume additionally that the curvature operator of any DM in Y is non-positive. Then any ρ-equivariant harmonic map $u : \tilde{X} \to Y$ is non-branching and totally geodesic. In other words, the image of u is contained in a single DM of Y and u is totally geodesic as a map into that DM.*

Excluding the real and complex hyperbolic spaces from the domain is essential. For example, the projection map onto the leaf space of a quadratic differential on a Riemann surface (which is harmonic) branches near the zero of the quadratic differential. From the above theorem, we deduce the following superrigidity result for hyperbolic buildings.

Theorem 2 *Let $\tilde{X} = G/K$ be an irreducible symmetric space of noncompact type, other than $SO_0(p,1)/SO(p) \times SO(1)$, $SU_0(p,1)/S(U(p) \times U(1))$. Let Γ be a lattice in G and let $\rho : \Gamma \to Isom(Y)$ a reduced homomorphism where Y is a hyperbolic building. If the rank of $\tilde{X}$ is ≥ 2 we assume additionally that Γ is cocompact. Then $\rho(\Gamma)$ fixes a point of Y.*

For the definition of *reduced*, we refer to Section 2. Finally, recall that a harmonic map from a Kähler manifold to a Riemannian complex is called *pluriharmonic* if it is pluriharmonic in the usual sense on the regular set. Our method also yields the following theorem.

Theorem 3 *Let $\tilde{X}$ be the universal cover of a complete finite volume Kähler manifold (X, ω). Let $\Gamma = \pi_1(X)$, Y a hyperbolic building and $\rho : \Gamma \to Isom(Y)$ a group homomorphism. Then any finite energy ρ-equivariant harmonic map $u : \tilde{X} \to Y$ is pluriharmonic.*

We note that the above theorems for the case of Euclidean buildings are due to Gromov and Schoen (cf. [5]). The theory of harmonic maps into singular spaces and its application to rigidity theory has been used by a variety of authors that includes Gromov-Schoen [5], Korevaar-Schoen [10], [11] and Jost [6], [7], [8], [9].

Remarks and Acknowledgements. The question of superrigidity of hyperbolic buildings was first suggested by Mikhail Gromov. For alternative approaches in the case of product domains, we refer to Gelander-Karlsson-Margulis [4], Monod-Shalom [13] and Mineyev-Monod-Shalom [14]. We are thankful to Piere Pansu for these references and to Nicolas Monod for many useful discussions. We also thank the referee for many useful comments that significantly improved the exposition of the paper. Finally, in order to keep the conceptual transparency of this article, we defer the bulk of the technical work analyzing the singular set of harmonic maps to the companion article [3].

2 Harmonic maps

Recall that a metric space (Y, d) is called an *NPC space* if: (i) The space (Y, d) is a length space. That is, for any two points P and Q in Y, there exists a rectifiable curve c so that the length of c is equal to $d(P, Q)$. We call such distance realizing curve a geodesic. (ii) For any three points $P, R, Q \in Y$, let $c : [0, l] \to Y$ be the arclength parameterized geodesic from Q to R and let $Q_t = c(tl)$ for $t \in [0, 1]$. Then

$$d^2(P, Q_t) \leq (1-t)d^2(P, Q) + td^2(P, R) - t(1-t)d^2(Q, R).$$

We now define our class of target spaces.

Definition 4 *Let $\mathbf{E}^d$ be an affine space. A convex piecewise linear polyhedron S with interior in some $\mathbf{E}^i \subset \mathbf{E}^d$ is called a cell. We will use the notation S^i to indicate the dimension of S. A convex cell complex or simply a complex Y in $\mathbf{E}^d$ is a finite collection $\mathcal{F} = \{S\}$ of cells satisfying the following properties: (i) the boundary ∂S of $S^i \in \mathcal{F}$ is a union of $T^j \in \mathcal{F}$ with $j < i$ (called the faces of S) and (ii) if $T^j, S^i \in \mathcal{F}$ with $j < i$ and $S^i \cap T^j \neq \emptyset$, then*

$T^j \subset S^i$. For example a simplicial complex is a cell complex whose cells are all simplices. We will denote by $Y^{(i)}$ the i-dimensional skeleton of Y, i.e. the union of all cells S^j where $j \leq i$. Y is called k-dimensional or simply a k-complex if $Y^{(k+1)} = \emptyset$ but $Y^{(k)} \neq \emptyset$.

Definition 5 *A complex Y along with a metric $G = \{G^S\}$ is called a Riemannian complex if each cell S of Y is equipped with a smooth Riemannian metric G^S such that for each cell S, the component functions of G^S extend smoothly all the way to the boundary of S. Furthermore, if S' is a face of S then the restriction G^S to S' is equal to $G^{S'}$ and S' is totally geodesic in S.*

Definition 6 *A k-dimensional Riemannian complex (Y, G) is said to have the* branching **D**ifferentiable **M**anifold structure *if given any two cells S_1 and S_2 of Y such that $S_1 \cap S_2 \neq \emptyset$, there exists a k-dimensional C^∞-differentiable complete Riemannian manifold M and an isometric and totally geodesic embedding $J : M \to Y$ such that $S_1 \cup S_2 \subset J(M)$. Such complexes will be referred to as DM-complexes. Furthermore, by an abuse of notation, we will often denote $J(M)$ by M and call it a DM (short for Differentiable Manifold).*

For the rest of the paper, we will also assume that the DM-complex Y is locally compact and NPC with respect to the distance function d induced from G^S. Note that if Y is a Euclidean complex and we require that all the DM's to be isometric to a k-dimensional Euclidean space, then Y is F-connected in the sense of [5], Section 6.1.

We now review the notion of harmonic map. Let Ω be a smooth bounded n-dimensional Riemannian domain and Y an NPC complex. A map $u : \Omega \to Y$ is said to be an L^2-map (or that $u \in L^2(\Omega, Y)$) if for some (and hence all) $P \in Y$, we have

$$\int_\Omega d^2(u(x), P) d\mu < \infty.$$

For $u \in L^2(\Omega, Y)$, define the energy density $|\nabla u|^2$ as in [5]. Set

$$E(u) = \int_\Omega |\nabla u|^2 d\mu$$

and call a map u of Sobolev class $W^{1,2}(\Omega, Y)$ if $E(u) < \infty$. If $u \in W^{1,2}(\Omega, Y)$, then there exists a well-defined notion of a trace of u, denoted $Tr(u)$, which is an element of $L^2(\partial\Omega, Y)$. Two maps $u, v \in W^{1,2}(\Omega, Y)$ have the same trace (i.e. $Tr(u) = Tr(v)$) if and only if $d(u,v) \in W_0^{1,2}(\Omega)$. For details we refer to [10]. A map $u : \Omega \to Y$ is said to be *harmonic* if it is energy minimizing among all $W^{1,2}$-maps with the same trace.

Similarly, there is the notion of equivariant harmonic map. Let $\tilde{X}$ be the universal cover of a complete, finite volume Riemannian manifold X, $\Gamma = \pi_1(X)$, Y an NPC Riemannian complex and $\rho : \pi_1(X) \to Isom(Y)$ a homomorphism. Let $u : \tilde{X} \to Y$ be a ρ-equivariant map that is locally of Sobolev class $W^{1,2}$. Since the energy density $|\nabla u|^2$ is Γ-invariant it descends to the quotient and we define

$$E(u) = \int_X |\nabla u|^2 d\mu.$$

An equivariant finite energy map u is called harmonic if it is energy minimizing among all finite energy ρ-equivariant maps $v : \tilde{X} \to Y$ which are locally of Sobolev class $W^{1,2}$.

The main regularity result of [5] and [10] is that harmonic maps are locally Lipschitz continuous. The key to Lipschitz regularity is the order function that we shall briefly review. Let $u : \Omega \to Y$ be a harmonic map. By Section 1.2 of [5], given $x \in \Omega$ there exists a constant $c > 0$ depending only on the C^2 norm of the metric on Ω such that

$$\sigma \mapsto Ord^u(x,\sigma) := e^{c\sigma^2} \frac{\sigma \ E_x(\sigma)}{I_x(\sigma)}$$

is non-decreasing for any $x \in \Omega$. In the above, we set

$$E_x(\sigma) := \int_{B_\sigma(x)} |\nabla u|^2 d\mu \ \text{ and } \ I_x(\sigma) := \int_{\partial B_\sigma(x)} d^2(u, u(x)) d\Sigma(x).$$

As a non-increasing limit of continuous functions,

$$Ord^u(x) := \lim_{\sigma \to 0} Ord^u(x,\sigma)$$

is an upper semicontinuous function. By following the proof of Theorem 2.3 in [5], we see that $Ord^u(x) \geq 1$ (this is equivalent to the Lipschitz property of u). The value $Ord^u(x)$ is called the order of u at x.

We now recall the existence result for harmonic maps. First, we make the following definition.

Definition 7 *Let Γ be a discrete group, Y an NPC space and $Isom(Y)$ the group of isometries of Y. A homomorphism $\rho : \Gamma \to Isom(Y)$ is called* reduced *if (i) ρ does not fix a point at infinity of Y and (ii) there is no unbounded closed convex $Z \subset Y$, $Z \neq Y$ preserved by $\rho(\Gamma)$.*

The above definition is the generalization in the NPC setting of the condition that ρ has Zariski dense image in Margulis' theorem. Our Definition appears to be slightly different than the one in [12]. However, it follows from [1] Corollary 3.8 that for the target spaces considered in this paper the two definitions are equivalent, but we will not use this fact. The following existence statement of equivariant harmonic maps is well-known; for example, see [5], [6] and [11].

Lemma 8 *Under the same assumptions as in Theorems 2, there exists a finite energy ρ-equivariant harmonic map $u : \tilde{X} \to Y$.*

PROOF. As in [5] Lemma 8.1 in the rank one case, there is a finite energy equivariant map $u : \tilde{X} \to Y$. In the higher rank case, this is automatically satisfied by the assumption of cocompactness. Now since the property of ρ being reduced in particular implies that $\rho(\Gamma)$ doesn't fix a point at infinity in Y, we obtain as in [5] Theorem 7.1 that u can be deformed to a finite energy equivariant harmonic map. Q.E.D.

3 The singular set

For a k-dimensional NPC DM-complex Y and a point $P \in Y$, let $T_P Y$ denote the (Alexandrov) tangent cone of Y at P. As explained in [3], $T_P Y$ is an unbounded F-connected Euclidean cell complex obtained by taking the tangent spaces to all the DM's passing through P with the appropriate identifications. Furthermore, the exponential map

$$\exp_P^Y : B_r(0) \subset T_P Y \to B_r(P) \subset Y$$

is defined by piecing together the exponential maps of all the DM's containing P.

Now let $u : \Omega \to Y$ be a harmonic map. Recall from [3] that a point $x_0 \in \Omega$ is called a *regular point* if $Ord^u(x_0) = 1$ and there exists $\sigma_0 > 0$ such that

$$u(B_{\sigma_0}(x_0)) \subset \exp^Y_{u(x_0)}(X_0), \tag{1}$$

where $X_0 \subset T_{u(x_0)}Y$ is isometric to $\mathbf{R}^k$. In particular, x_0 has a neighborhood mapping into a DM. A point $x_0 \in \Omega$ is called a *singular point* if it is not a regular point. Denote the set of regular points by $\mathcal{R}(u)$ and the set of singular points by $\mathcal{S}(u)$. One of the main results of [3] is the following theorem.

Theorem 9 *(cf. [3]) Let Ω be an n-dimensional Riemannian domain, Y a k-dimensional NPC DM-complex and $u : \Omega \to Y$ a harmonic map. Then the singular set $\mathcal{S}(u)$ of u has Hausdorff co-dimension 2 in Ω; i.e.*

$$\dim_{\mathcal{H}}(\mathcal{S}(u)) \leq n - 2.$$

Following [3] we will stratify the singular set further into the following subsets. Set

$$\mathcal{S}_0(u) = \{x_0 \in \Omega : Ord^u(x_0) > 1\},$$

$k_0 := min\{n, k\}$ and $\mathcal{S}_j(u) = \emptyset$ if $j \geq k_0 + 1$ or $j \leq -1$. For $j = 1, \ldots, k_0$, we define $\mathcal{S}_j(u)$ inductively as follows. Having defined $\mathcal{S}_m(u)$ for $m = j+1, j+2, \ldots$, define $\mathcal{S}_j(u)$ to be the set of points

$$x_0 \in \mathcal{S}(u) \backslash \left(\bigcup_{m=j+1}^{k_0} \mathcal{S}_m(u) \cup \mathcal{S}_0(u) \right)$$

with the property that there exists $\sigma_0 > 0$ such that

$$u(B_{\sigma_0}(x_0)) \subset \exp^Y_{u(x_0)}(X_0) \tag{2}$$

where $X_0 \subset T_{u(x_0)}Y$ is isometric to $\mathbf{R}^j \times Y_2^{k-j}$ and Y_2 is $(k-j)$-dimensional F-connected complex. Set

$$\mathcal{S}^-_m(u) = \bigcup_{j=0}^{m} \mathcal{S}_j(u) \quad \text{and} \quad \mathcal{S}^+_m(u) = \bigcup_{j=m}^{k} \mathcal{S}_j(u).$$

It was shown in [3] that

Lemma 10 *The sets $\mathcal{S}_0(u)$, $\mathcal{S}_1(u)$, ..., $\mathcal{S}_{k_0-1}(u)$, $\mathcal{S}_{k_0}(u)$ form a partition of $\mathcal{S}(u)$. Furthermore, the sets $\mathcal{R}(u)$, $\mathcal{R}(u)\cup\mathcal{S}_m^+(u)$ are open and the sets $\mathcal{S}_m^-(u)$ are closed.*

Consider the point $(0, P_0) \in \mathbf{R}^j \times Y_2^{k-j}$, where P_0 is the lowest dimensional stratum of Y_2^{k-j}. Define a metric G on $\mathbf{R}^j \times Y_2^{k-j}$ by pulling back the metric on Y via the exponential map in a neighborhood of $(0, P_0)$. When studying local properties of the harmonic map $u : \Omega \to Y$ at $x_0 \in \Omega$, we may assume that u maps $B_{\sigma_0}(x_0) \subset \Omega$ into $(\mathbf{R}^j \times Y_2^{k-j}, G)$, where $j \geq 0$. We need the following two lemmas.

Lemma 11 *Let $u = (u^1, u^2) : (B_{\sigma_0}(x_0), g) \to (\mathbf{R}^j \times Y_2^{k-j}, G)$ be the harmonic map with $u(x_0) = (0, P_0)$ given above and $j > 0$. If we write $u^1 = (u^1_I)_{I=1,\dots,j} : B_{\sigma_0}(x_0) \to \mathbf{R}^j$, then $u^1_I \in W^{2,p}_{loc}(B_{\frac{\sigma_0}{2}}(x_0))$ for any $p \in (0, \infty)$ and any $I = 1, \dots, j$. In particular, $|\nabla u^1|$ is continuous in $B_{\frac{\sigma_0}{2}}(x_0)$.*

Proof. Let $R > 0$ such that $u(B_{\sigma_0}(x_0))$ is contained in a ball B_R of radius R about $(0, P_0)$. For any DM M of $(\mathbf{R}^j \times Y_2^{k-j}, G)$, extend coordinates on $\mathbf{R}^j$ to define coordinates of $B_{2R} \cap M$. Since $\mathbf{R}^j \times Y_2^{k-j}$ is a locally finite complex there exist a finite number of distinct DM's $M_1, \dots, M_L$ contained in $\mathbf{R}^j \times Y_2^{k-j}$. Define

$$C_1 := \max_{l=1,\dots,L} \max_{i,r,s=1,\dots,k} \sup_{M_l \cap \overline{B_R}} \left| {}^{M_l}\Gamma^i_{rs} \right|$$

where ${}^{M_l}\Gamma^i_{rs}$ is the Christoffel symbols of M_l with respect to the coordinates in $B_{2R} \cap M_l$ as above. Let $u_i^{M_l}$ for $i = 1, \dots, k$ denote the ith coordinate function. Here, we emphasize that

$$u_I^{M_l} = u_I^1 \;\text{ for }\; I = 1, \dots, j$$

by the construction above. In particular, note that $u_I^{M_l} = u_I^{M_{l'}}$ for any $l, l' = 1, \dots, L$. Define

$$C_2 := \max_{l=1,\dots,L} \max_{i=1,\dots,k} \max_{\alpha=1,\dots,n} \left| \frac{\partial u_i^{M_l}}{\partial x^\alpha} \right|_{L^\infty(B_{\frac{3\sigma_0}{4}}(x_0) \cap u^{-1}(M_l))}.$$

For $x \in \mathcal{R}(u)$, let M_l be the DM containing $u(B_\delta(x))$ for some $\delta > 0$. We have the harmonic map equation

$$\triangle u_i^{M_l} = -g^{\alpha\beta M_l}\Gamma^i_{rs}(u)\frac{\partial u_r^{M_l}}{\partial x^\alpha}\frac{\partial u_s^{M_l}}{\partial x^\beta}$$

in $B_\delta(x)$. Thus, for δ sufficiently small,

$$|\triangle u_I^1|_{L^\infty(B_\delta(x))} = |\triangle u_I^{M_l}|_{L^\infty(B_\delta(x))} \leq CC_1C_2^2 \ \forall I = 1, \ldots, j, \tag{3}$$

where the constant C depends only on the dimension n and the metric g of the domain. Since $\dim_{\mathcal{H}}(B_{\frac{3\sigma_0}{4}}(x_0)\backslash\mathcal{R}(u)) \leq n-2$, we see that the inequality (3) implies $\triangle u_I^1 \in L^p(B_{\frac{3\sigma_0}{4}}(x_0))$ which in turn implies $u_I^1 \in W^{2,p}(B_{\frac{\sigma_0}{2}}(x_0))$. Q.E.D.

We now prove

Lemma 12 *Let Ω be an n-dimensional Riemannian domain, Y a k-dimensional NPC DM-complex and $u : \Omega \to Y$ a harmonic map. For any compact subdomain Ω_1 of Ω, there exists a sequence of smooth functions $\{\psi_i\}$ with $\psi_i \equiv 0$ in a neighborhood of $\mathcal{S}(u)\cap\overline{\Omega_1}$, $0 \leq \psi_i \leq 1$ and $\psi_i \to 1$ for all $x \in \Omega\backslash(\mathcal{S}(u)\cap\overline{\Omega_1})$ such that*

$$\lim_{i\to\infty}\int_\Omega |\nabla\nabla u||\nabla\psi_i| \ d\mu = 0.$$

Proof. The proof of the Lemma follows by induction from the following

Claim. *Assume that given any subdomain Ω_1' compactly contained in $\Omega\backslash\mathcal{S}_j^-(u)$, there exists a sequence of smooth functions $\{\hat{\psi}_i\}$ with $\hat{\psi}_i \equiv 0$ in a neighborhood of $\mathcal{S}_{j+1}^+(u) \cap \overline{\Omega_1'}$, $0 \leq \hat{\psi}_i \leq 1$, $\hat{\psi}_i \to 1$ for all $x \in \Omega\backslash(\mathcal{S}_{j+1}^+(u) \cap \overline{\Omega_1'})$ such that*

$$\lim_{i\to\infty}\int_\Omega |\nabla\hat{\psi}_i| \ d\mu = 0,$$

and

$$\lim_{i\to\infty}\int_\Omega \ |\nabla\nabla u||\nabla\hat{\psi}_i| \ d\mu = 0.$$

Then given any subdomain Ω_1 compactly contained in $\Omega\backslash\mathcal{S}_{j-1}^-(u)$, there exists a sequence of smooth functions $\{\psi_i\}$ with $\psi_i \equiv 0$ in a neighborhood of $\mathcal{S}_j^+(u)\cap\overline{\Omega_1}$, $0 \leq \psi_i \leq 1$, $\psi_i \to 1$ for all $x \in \Omega\backslash(\mathcal{S}_j^+(u) \cap \overline{\Omega_1})$ such that

$$\lim_{i\to\infty}\int_\Omega |\nabla\psi_i| \ d\mu = 0,$$

and

$$\lim_{i\to\infty}\int_\Omega |\nabla\nabla u||\nabla\psi_i|\ d\mu = 0.$$

We now prove the claim. For a subdomain Ω_1 compactly contained in $\Omega\backslash\mathcal{S}^-_{j-1}(u)$, let Ω_2 be such that $\Omega_1 \subset\subset \Omega_2 \subset\subset \Omega\backslash\mathcal{S}^-_{j-1}(u)$. Without the loss of generality, we can assume that $u=(u^1,u^2):\Omega_2\to(\mathbf{R}^j\times Y_2^{k-j},G)$ and that $\nabla u^1 \in W^{1,p}(\Omega_2)\cap C^0(\overline{\Omega_2})$ for any $p>0$ by Lemma 11. Furthermore, we claim that $|\nabla u^1|\neq 0$ in $S_j(u)\cap\overline{\Omega_1}$. Indeed, if $|\nabla u^1|(x)=0$ for some $x\in S_j(u)\cap\overline{\Omega_1}$, then the Gap theorem in [3] implies that $|\nabla u^2|(x)=0$ and therefore also $|\nabla u|(x)=0$, contradicting the fact that x is a point of order 1. In particular, this means that there exists a neighborhood $\mathcal{N}\subset\Omega_2$ of $\mathcal{S}_j(u)\cap\overline{\Omega_1}$ and a constant δ_0 such that

$$|\nabla u^1|\geq\delta_0>0 \ \text{ on } \mathcal{N}. \tag{4}$$

Below, we will use C to denote any generic constant which only depends on δ_0, the dimension of n of Ω and the Lipschitz constant of u. For $d\in(n-2,n)$ to be chosen later, fix a finite covering $\{B_{r_J}(x_J): J=1,\ldots,l\}$ of the compact set $\mathcal{S}_j(u)\cap\overline{\Omega_1}$ satisfying

$$\sum_{J=1}^{l} r_J^d \leq \epsilon. \tag{5}$$

We also assume

$$B_{3r_J}(x_J)\subset\mathcal{N} \tag{6}$$

which is true if $\epsilon>0$ is small and that $x_J\in\mathcal{S}_j(u)\cap\overline{\Omega_1}$. Let φ_J be a smooth function which is zero on $B_{r_J}(x_J)$ and identically one on $\Omega\backslash B_{2r_J}(x_J)$ such that $|\nabla\varphi_J|\leq Cr_J^{-1}$, $|\nabla\nabla\varphi_J|\leq Cr_J^{-2}$ and $|\nabla\nabla\nabla\varphi_J|\leq Cr_J^{-3}$. If φ is defined by

$$\varphi=\min\{\varphi_J : J=1,\ldots,l\},$$

then $\varphi\equiv 0$ in a neighborhood of $\mathcal{S}_j(u)\cap\overline{\Omega_1}$ and $\varphi\equiv 1$ on $\Omega_1\backslash\cup_{J=1}^l B_{2r_J}(x_J)$. Let

$$\Omega_1' = \Omega_1\backslash\cup_{J=1}^l B_{r_J}(x_J)$$

and let $\{\hat{\psi}_i\}$ be as in the inductive hypothesis. Now let $\psi_0=\varphi^2\hat{\psi}_i$. Then for i sufficiently large, we have

$$\int_\Omega|\nabla\psi_0|\ d\mu \ \leq\ 2\int_\Omega \varphi\hat{\psi}_i|\nabla\varphi|\ d\mu+\varphi^2|\nabla\hat{\psi}_i|\ d\mu$$

$$\leq \quad C\sum_{J=1}^{l} r_J^{n-1} + \int_\Omega |\nabla\hat{\psi}_i| \; d\mu \leq C\epsilon + \epsilon. \tag{7}$$

Furthermore, we have

$$\begin{aligned}
\int_\Omega |\nabla\nabla u||\nabla\psi_0| d\mu &= 2\int_\Omega \varphi\hat{\psi}_i|\nabla\nabla u||\nabla\varphi| d\mu + \int_\Omega \varphi^2|\nabla\nabla u||\nabla\hat{\psi}_i| d\mu \\
&\leq 2\int_\Omega \varphi\hat{\psi}_i|\nabla\nabla u||\nabla\varphi| d\mu + \int_\Omega |\nabla\nabla u||\nabla\hat{\psi}_i| d\mu.
\end{aligned}$$

For $\delta \in (0,1)$ to be chosen later, we write the first term as

$$\begin{aligned}
&2\int_\Omega \varphi\hat{\psi}_i|\nabla\nabla u||\nabla\varphi| \; d\mu \\
&\leq 2\int_{\cup_{J=1}^l B_{2r_J}(x_J)} \varphi\hat{\psi}_i|\nabla\nabla u||\nabla\varphi| \; d\mu \\
&\leq 2\left(\int_{\cup_{J=1}^l B_{2r_J}(x_J)} |\nabla\nabla u|^2|\nabla u|^{-1}\varphi^2\hat{\psi}_i|\nabla\varphi|^\delta \; d\mu\right)^{1/2} \\
&\quad \times \left(\int_{\cup_{J=1}^l B_{2r_J}(x_J)} |\nabla u||\nabla\varphi|^{2-\delta}\hat{\psi}_i \; d\mu\right)^{1/2} \\
&\leq 2\left(\int_{\cup_{J=1}^l B_{2r_J}(x_J)} |\nabla\nabla u|^2|\nabla u|^{-1}\varphi^2\hat{\psi}_i|\nabla\varphi|^\delta \; d\mu\right)^{1/2} \left(C\sum_{J=1}^{l} r_J^{n-2+\delta}\right)^{1/2}.
\end{aligned}$$

Let ρ_J be a Lipchitz function which is identically one on $B_{2r_J}(x_J)$ and identically zero on $\Omega\backslash B_{3r_J}(x_J)$ with $|\nabla\rho_J| \leq Cr_J^{-1}$ and $|\nabla\nabla\rho_J| \leq Cr_J^{-2}$. Define

$$\rho = \max\{\rho_J : J = 1, \ldots, l\}.$$

As in [5] Theorem 6.4 on $\mathcal{R}(u)$, we will use the pointwise inequalities

$$\frac{1}{2}\triangle|\nabla u|^2 \geq |\nabla\nabla u|^2 - c|\nabla u|^2 \text{ and } (1-\epsilon_n)|\nabla\nabla u|^2 \geq |\nabla|\nabla u||^2$$

with constant ϵ_n depending only on n which combine to imply

$$\epsilon_n|\nabla\nabla u|^2|\nabla u|^{-1} \leq \triangle|\nabla u| + c|\nabla u| \;\text{ on }\; \mathcal{R}(u).$$

Since $\varphi^2\rho^2\hat{\psi}_i \equiv 0$ in a neighborhood of $\mathcal{S}_j^+(u) \cap (\cup_{J=1}^l B_{2r_J}(x_J))$, we have that

$$\int_{\cup_{J=1}^l B_{2r_J}(x_J)} |\nabla\nabla u|^2|\nabla u|^{-1}\varphi^2\hat{\psi}_i|\nabla\varphi|^\delta \; d\mu$$

$$
\begin{aligned}
&\leq \int_\Omega |\nabla\nabla u|^2|\nabla u|^{-1}\varphi^2\rho^2\hat{\psi}_i|\nabla\varphi|^\delta \ d\mu \\
&\leq \frac{1}{\epsilon_n}\int_\Omega \triangle|\nabla u|\varphi^2\rho^2\hat{\psi}_i|\nabla\varphi|^\delta \ d\mu + \frac{c}{\epsilon_n}\int_\Omega |\nabla u|\varphi^2\rho^2\hat{\psi}_i|\nabla\varphi|^\delta \ d\mu.
\end{aligned}
$$

The second term of the right-hand side has the estimate

$$
\frac{c}{\epsilon_n}\int_\Omega |\nabla u|\varphi^2\rho^2\hat{\psi}_i|\nabla\varphi|^\delta \ d\mu \leq C\int_{\cup_{J=1}^l B_{2r_J}(x_J)} |\nabla\varphi|^\delta \leq C\sum_{J=1}^l r_J^{n-\delta}.
$$

The first term can be rewritten

$$
\begin{aligned}
&\frac{1}{\epsilon_n}\int_\Omega |\nabla u|\triangle(\varphi^2\rho^2\hat{\psi}_i|\nabla\varphi|^\delta) \ d\mu \\
&\quad\leq C\int_\Omega \hat{\psi}_i|\nabla u|\triangle(\varphi^2\rho^2|\nabla\varphi|^\delta) \ d\mu + C\int_\Omega \varphi^2\rho^2|\nabla\varphi|^\delta|\nabla u|\triangle\hat{\psi}_i \ d\mu \\
&\qquad +C\int_\Omega |\nabla u| < \nabla(\varphi^2\rho^2|\nabla\varphi|^\delta)\cdot\nabla\hat{\psi}_i > \ d\mu \\
&\quad= (a)+(b)+(c).
\end{aligned}
$$

By the mean value theorem,

$$
\frac{(|\nabla u^1|^2+s)^{\frac{1}{2}} - |\nabla u^1|}{s} = \frac{1}{2}(|\nabla u^1|^2+c)^{-\frac{1}{2}}
$$

for some $c \in (0,s)$. Letting $s = |\nabla u^2|^2 + 2 < \nabla u^1, \nabla u^2 >$, we have

$$
|\nabla u| = |\nabla u^1| + \frac{1}{2}(|\nabla u^1|^2+c)^{-\frac{1}{2}}(|\nabla u^2|^2 + 2 < \nabla u^1, \nabla u^2 >).
$$

Thus,

$$
\begin{aligned}
(a) &= C\int_\Omega \hat{\psi}_i|\nabla u|\triangle(\varphi^2\rho^2|\nabla\varphi|^\delta) \ d\mu \\
&= C\int_\Omega \hat{\psi}_i|\nabla u^1|\triangle(\varphi^2\rho^2|\nabla\varphi|^\delta) \ d\mu \\
&\quad +C\int_\Omega \frac{1}{2}\hat{\psi}_i(|\nabla u^1|^2+c)^{-\frac{1}{2}}(|\nabla u^2|^2 + 2 < \nabla u^1, \nabla u^2 >)\triangle(\varphi^2\rho^2|\nabla\varphi|^\delta) \ d\mu \\
&= (a)_1 + (a)_2.
\end{aligned}
$$

For p and q with $\frac{1}{p}+\frac{1}{q}=1$ to be chosen later, we have

$$\begin{aligned}
(a)_1 &= C\int_\Omega \hat{\psi}_i|\nabla u^1|\triangle(\varphi^2\rho^2|\nabla\varphi|^\delta)\ d\mu \\
&= -C\int_\Omega <\nabla(\hat{\psi}_i|\nabla u^1|),\nabla(\varphi^2\rho^2|\nabla\varphi|^\delta)>\ d\mu \\
&\leq C\int_\Omega |\nabla\hat{\psi}_i|\ |\nabla u^1|\ |\nabla(\varphi^2\rho^2|\nabla\varphi|^\delta)|\ d\mu + C\int_\Omega |\nabla|\nabla u^1||\ |\nabla(\varphi^2\rho^2|\nabla\varphi|^\delta)|\ d\mu \\
&\leq C\sum_{J=1}^{l} r_J^{-(1+\delta)}\int_\Omega |\nabla\hat{\psi}_i| + C\left(\int_\Omega |\nabla|\nabla u^1||^p\ d\mu\right)^{1/p}\left(\int_\Omega |\nabla(\varphi^2\rho^2|\nabla\varphi|^\delta)|^q\ d\mu\right)^{1/q} \\
&\leq C\sum_{J=1}^{l} r_J^{-(1+\delta)}\int_\Omega |\nabla\hat{\psi}_i| + C\left(\int_\Omega |\nabla|\nabla u^1||^p\ d\mu\right)^{1/p}\left(\sum_{J=1}^{l} r_J^{n-(1+\delta)q}\right)^{1/q}.
\end{aligned}$$

Furthermore, using (4) and (6), we have

$$\begin{aligned}
(a)_2 &= C\int_\Omega \frac{1}{2}\hat{\psi}_i(|\nabla u^1|^2+c)^{-\frac{1}{2}}(|\nabla u^2|^2+2<\nabla u^1,\nabla u^2>)\triangle(\varphi^2\rho^2|\nabla\varphi|^\delta)\ d\mu \\
&= C\int_\Omega (|\nabla u^2|^2+2<\nabla u^1,\nabla u^2>)\triangle(\varphi^2\rho^2|\nabla\varphi|^\delta)\ d\mu \\
&= C\sum_{J=1}^{l} r_J^{-(2+\delta)}\left(\int_\Omega |\nabla u^2|^2\ d\mu + \int_{B_{r_J}(x_J)} 2<\nabla u^1,\nabla u^2> d\mu\right) \\
&\leq C\sum_{J=1}^{l} r_J^{-(2+\delta)}\left(\int_\Omega |\nabla u^2|^2\ d\mu + 2\left(\int_\Omega |\nabla u^1|^2\ d\mu\right)^{1/2}\left(\int_\Omega |\nabla u^2|^2\ d\mu\right)^{1/2}\right) \\
&\leq C\sum_{J=1}^{l} r_J^{n-(2+\delta)+\frac{\epsilon_{gap}}{2}},
\end{aligned}$$

where ϵ_{gap} is the order gap for approximately harmonic maps into Y_2^{k-j} as described in [3]. Additionally,

$$\begin{aligned}
(b) &= C\int_\Omega \varphi^2\rho^2|\nabla\varphi|^\delta|\nabla u|\triangle\hat{\psi}_i\ d\mu \\
&\leq C\int_\Omega |\nabla(\varphi^2\rho^2|\nabla\varphi|^\delta)|\ |\nabla u|\ |\nabla\hat{\psi}_i|\ d\mu + C\int_\Omega \varphi^2\rho^2|\nabla\varphi|^\delta|\nabla|\nabla u||\ |\nabla\hat{\psi}_i|\ d\mu \\
&\leq C\sum_{J=1}^{l} r_J^{-(1+\delta)}\int_\Omega |\nabla\hat{\psi}_i|\ d\mu + C\sum_{J=1}^{l} r_J^{-\delta}\int_\Omega |\nabla\nabla u|\ |\nabla\hat{\psi}_i|\ d\mu
\end{aligned}$$

and

$$(c) = C\int_\Omega |\nabla u| < \nabla(\varphi^2\rho^2|\nabla\varphi|^\delta)\cdot\nabla\hat{\psi}_i > \ d\mu \le C\sum_{J=1}^{l} r_J^{-(1+\delta)}\int_\Omega |\nabla\hat{\psi}_i| \ d\mu.$$

Thus, we obtain

$$\begin{aligned}
&\frac{1}{\epsilon_n}\int_\Omega |\nabla u|\triangle(\varphi^2\rho^2\hat{\psi}_i|\nabla\varphi|^\delta) \ d\mu \\
\le\ & C\sum_{J=1}^{l} r_J^{-(1+\delta)}\int_\Omega |\nabla\hat{\psi}_i| + C\left(\int_\Omega |\nabla|\nabla u^1||^p \ d\mu\right)^{1/p}\left(\sum_{J=1}^{l} r_J^{n-(1+\delta)q}\right)^{1/q} \\
& +C\sum_{J=1}^{l} r_J^{n-(2+\delta)+\frac{\epsilon_{gap}}{2}} + C\sum_{J=1}^{l} r_J^{-\delta}\int_\Omega |\nabla\nabla u| \ |\nabla\hat{\psi}_i| \ d\mu.
\end{aligned}$$

Combining all the above estimates, we obtain

$$\begin{aligned}
&\int_\Omega |\nabla\nabla u| \ |\nabla\psi_0| \ d\mu \\
\le\ & 2\left(C\sum_{J=1}^{l} r_J^{-(1+\delta)}\int_\Omega |\nabla\hat{\psi}_i| + C\left(\int_\Omega |\nabla|\nabla u^1||^p \ d\mu\right)^{1/p}\left(\sum_{J=1}^{l} r_J^{n-(1+\delta)q}\right)^{\frac{1}{q}}\right. \\
& \left. +C\sum_{J=1}^{l} r^{n-(2+\delta)+\frac{\epsilon_{gap}}{2}} + C\sum_{J=1}^{l} r_J^{-\delta}\int_\Omega |\nabla\nabla u| \ |\nabla\hat{\psi}_i| \ d\mu + \sum_{J=1}^{l} r_J^{n-\delta}\right)^{1/2} \\
& \left(C\sum_{J=1}^{l} r_J^{n-2+\delta}\right)^{1/2} + \int_\Omega |\nabla\nabla u| \ |\nabla\hat{\psi}_i| \ d\mu.
\end{aligned}$$

First we choose $0 < \delta < \frac{\epsilon_{gap}}{2}$ and then choose d in (5) such that $d \in (n-2, n-(2+\delta)+\frac{\epsilon_{gap}}{2})$ and $d < n-2+\delta$. Then choose $q > 1$ such that $n-(1+\delta)q > d$ and p such that $\frac{1}{p}+\frac{1}{q} = 1$. Last fix i sufficiently large such that

$$\sum_{J=1}^{l} r_J^{-(1+\delta)}\int_\Omega |\nabla\hat{\psi}_i| \ d\mu, \ \sum_{J=1}^{l} r_J^{-\delta}\int_\Omega |\nabla\nabla u| \ |\nabla\hat{\psi}_i| \ d\mu < \epsilon.$$

We then have

$$\begin{aligned}
&\int_\Omega |\nabla\nabla u| \ |\nabla\psi_0| \ d\mu \\
\le\ & 2\left(C\epsilon + C\left(\int_\Omega |\nabla|\nabla u^1||^p \ d\mu\right)^{1/p}\epsilon + 3C\epsilon\right)^{1/2}(C\epsilon)^{1/2} + \epsilon.
\end{aligned}$$

Finally, note that $\psi_i \equiv 0$ in a neighborhood $\mathcal{S}_{j+1}(u) \cap (\overline{\Omega_1} \backslash \cup_{J=1}^{l} B_{r_J}(x_J)$, $\varphi \equiv 0$ in $\cup B_{r_J}(x_J)$ and hence $\psi_0 \equiv 0$ in a neighborhood of $\mathcal{S}_j(u) \cap \overline{\Omega_1}$. Since $\epsilon > 0$ is arbitrary, this proves the claim and finishes the proof of the lemma. Q.E.D.

4 Proof of Main Theorems.

We first start with the following

Proposition 13 *Let $\tilde{X}$ be the universal cover of a complete finite volume Riemannian manifold X with parallel p-form ω and Y a k-dimensional NPC DM-complex where each DM has a non-positive curvature operator. If $\Gamma = \pi_1(X)$, $\rho : \Gamma \to Isom(Y)$ is a group homomorphism and $u : \tilde{X} \to Y$ a finite energy ρ-equivariant harmonic map, then $D^*(du \wedge \omega) = 0$ in a neighborhood of a regular point.*

PROOF. The proof is very similar to [5] Theorem 7.2 so we will only sketch the argument. Let $x_0 \in \tilde{X}$ be a regular point. As in [5], we will work on the quotient $X = \tilde{X}/\Gamma$. Fix $R > 0$ and a nonnegative smooth function ρ which is identically one in $B_R(x_0)$ and zero outside $B_{2R}(x_0)$ with $|\nabla \rho| \leq 2R^{-1}$. Let ψ be a nonnegative smooth function vanishing in a small neighborhood of $\mathcal{S}(u) \cap \overline{B_{2R}(x_0)}$. By Stokes theorem we obtain

$$\begin{aligned} 0 &= \int_X \langle D(\psi\rho^2 * (du \wedge \omega)), D * (du \wedge *\omega)\rangle \, d\mu \\ &+ (-1)^{m-p-1} \int_X \psi\rho^2 \langle *(du \wedge \omega), D^2 * (du \wedge *\omega)\rangle \, d\mu. \end{aligned}$$

Combining the above with the Corlette formula

$$*D * (du \wedge \omega) = (-1)^{m-1} D * (du \wedge *\omega)$$

in $\mathcal{R}(u)$, we obtain

$$\begin{aligned} 0 &= \int_X \langle d(\psi\rho^2) \wedge *(du \wedge \omega), *D * (du \wedge \omega)\rangle \, d\mu + \int_X \psi\rho^2 |D * (\omega \wedge du)|^2 \, d\mu \\ &+ (-1)^p \int_X \psi\rho^2 \langle *(du \wedge \omega), D^2 * (du \wedge *\omega)\rangle \, d\mu. \end{aligned}$$

By [2] the last two quantities are nonnegative. Hence for any $\epsilon > 0$, we have

$$
\begin{aligned}
0 &\leq \int_{B_R(x_0)} \psi |D * (\omega \wedge du)|^2 \, d\mu \\
&+ (-1)^p \int_{B_R(x_0)} \psi \langle *(du \wedge \omega), D^2 * (du \wedge *\omega) \rangle \, d\mu \\
&< \epsilon
\end{aligned}
$$

after taking R sufficiently large, ψ as in Lemma 12 and estimating the first term as in [5] Theorem 7.2. By letting $\psi \to 1$ we obtain $D^*(du \wedge \omega) = 0$ on $\mathcal{R}(u)$. Q.E.D.

Corollary 14 *Let $\tilde{X} = G/K$ be the quaternionic hyperbolic space or the Cayley plane, Γ a lattice in G, Y a k-dimensional hyperbolic building, $\rho : \Gamma \to Isom(Y)$ a group homomorphism and $u : \tilde{X} \to Y$ a finite energy ρ-equivariant harmonic map. Then in a neighborhood of a regular point, u is totally geodesic (i.e. $\nabla du = 0$).*

Proof. We apply Proposition 13 for ω either the Quaternionic Kähler 4-form or the Cayley 8-form to obtain in a neighborhood of a regular point that $D^*(du \wedge \omega) = 0$. The statement about totally geodesic follows from the above as in [2] Theorem 3.3. Q.E.D.

Proof of Theorem 3. We apply Proposition 13 for ω the Käler form to obtain in neighborhood of a regular point that $D^*(du \wedge \omega) = 0$. With the same notation as in [2] Theorem 3.3 this implies that $\nabla du(\omega) = 0$, which immediately implies that u is pluriharmonic.

We now treat the higher rank case.

Lemma 15 *Let $\tilde{X} = G/K$ be an irreducible symmetric space of noncompact type of rank ≥ 2, Γ a lattice in G, Y a k-dimensional NPC DM-complex, $\rho : \Gamma \to Isom(Y)$ a group homomorphism and $u : \tilde{X} \to Y$ a finite energy ρ-equivariant harmonic map. In addition, assume that Γ is cocompact. Then in a neighborhood of a regular point, u is totally geodesic (i.e. $\nabla du = 0$).*

Proof. The proof is similar to the one for the rank one case, once we establish the Bochner formula of Jost-Yau. Our goal is to verify [6] Lemma 5.2.1. As in Lemma 14 we will work on the quotient $X = \tilde{X}/\Gamma$ which by assumption is compact. We start with [6] equation (5.2.4) and note that all terms are tensorial except when we integrate by parts. Let ψ be a nonnegative smooth function vanishing in a small neighborhood of $\mathcal{S}(u)$. We multiply both sides of (5.2.4) by ψ, integrate over X and apply integration by parts on the first term on the right-hand side. The resulting equation is similar to [6] equation (5.2.5) except that all integrands get multiplied by the cutoff function ψ and has an extra term

$$\int_X \langle \frac{\partial \psi}{\partial \gamma} R^X_{\beta\alpha\gamma\delta} u_{x^\beta}, u_{x^\alpha x^\beta} \rangle \ d\mu$$

on the right-hand side. Since $R^X_{\beta\alpha\gamma\delta} u_{x^\beta}$ is bounded, this extra term is exactly of the form estimated in Lemma 12. Hence by taking $\psi \to 1$, we have verified [6] equation (5.2.5) and therefore also (5.2.8).

The other point is to justify the formula

$$0 = \int_X |\nabla\nabla u|^2 d\mu + \int_X R^X_{\alpha\beta} < u_{x^\alpha}, u_{x^\beta} > d\mu - \int_X < R^Y(u_{x^\alpha}, u_{x^\beta}) u_{x^\beta}, u_{x^\alpha} > d\mu. \tag{8}$$

To see this, multiply the Eells-Sampson Bochner formula [6] equation (5.2.1) by the cut-off function ψ as above and integrate over X. Applying integration by parts on the left-hand side, we obtain a term bounded by

$$\left| \int_X < \nabla\psi, \nabla |\nabla u|^2 > d\mu \right| \leq c \int_X |\nabla\nabla u||\nabla\psi|$$

where c depends on the Lipschitz constant of u. Note that the last integrand above is again of the form estimated in Lemma 12. Hence taking $\psi \to 1$, we have justfied (8). This verifies [6] Lemma 5.2.1 which in turn implies that u is totally geodesic (cf. proof of [6] Theorem 5.3.1). Q.E.D.

Proof of Theorem 1. The first step is to show that if $\gamma : [0,1] \to \tilde{X}$ is a constant speed parametrization of a geodesic, then $u \circ \gamma$ is a constant speed parametrization of a geodesic in Y. Let $x_0 = \gamma(0)$ and $x_1 = \gamma(1)$. Let S be a hypersurface perpendicular to $\gamma'(1)$ at x_1. For $r > 0$, let $\psi : B_r(0) \subset \mathbf{R}^{n-1} \to S$ be a parametrization of S near x_1. Define $\Psi : B_r(0) \times [0,1] \to \tilde{X}$ by setting $t \mapsto \Psi(\xi, t)$ to be the constant speed geodesic between x_0 and $\psi(\xi)$. This map is well-defined and smooth since $\tilde{X}$ has non-positive sectional curvature. In fact, its restriction to $B_r(0) \times (\epsilon, 1)$ is a diffeomorphism for any ϵ. We claim that there exist $x_{0i} \to x_0$ and $x_{1i} \to x_1$ such that if $\gamma_i : [0,1] \to \tilde{X}$ is a constant speed parametrization of a geodesic between x_{0i} and x_{1i}, then γ_i maps the open interval $(0,1)$ into $\mathcal{R}(u)$. To prove this claim, we observe that for $\epsilon_i \to 0$, there exists $\xi_i \in B_{\epsilon_i}(0)$ such that $\{\xi_i\} \times (\epsilon_i, 1) \cap \Psi^{-1}(\mathcal{S}(u)) = \emptyset$. Indeed, if $\{\xi\} \times (\epsilon_i, 1) \cap \Psi^{-1}(\mathcal{S}(u)) \neq \emptyset$ for all $\xi \in B_{\epsilon_i}(0) \subset \mathbf{R}^{n-1}$, then $\dim_{\mathcal{H}}(\Psi^{-1}(\mathcal{S}(u))) \cap B_{\epsilon_i}(0) \times (\epsilon_i, 1)) \geq n-1$; but on the other hand, Ψ restricted to $B_{\epsilon_i}(0) \times (\epsilon_i, 1)$ is a diffeomorphism, contradicting the fact that $\dim_{\mathcal{H}}(\mathcal{S}(u))) \leq n-2$. This proves the claim by letting $x_{0i} = \Psi(\xi_i, \epsilon_i)$, $x_{1i} = \Psi_1(\xi_i, 1)$ and $\gamma_i : [0,1] \to \tilde{X}$ be defined by $\gamma_i(t) = \Psi(\xi_i, \epsilon_i + (1-\epsilon_i)t)$. By Corollary 14 and Lemma 15, we have $\nabla du \equiv 0$ in $\mathcal{R}(u)$. Thus $u \circ \gamma_i$ is a constant speed parametrization of a geodesic. By the continuity of u, this then implies that $u \circ \gamma$ is also.

To complete the proof of the Lemma, it suffices to show that there is no point $x_0 \in \tilde{X}$ such that u bifurcates into different DM's at $u(x_0)$. Choose an arbitrary point $x_0 \in \tilde{X}$ and identify $x_0 = 0$ via normal coordinates. Assume without loss of generality that Y is locally isometrically embedded in $\mathbf{R}^K$, $u(0) = 0$ and that $T_{u(0)}Y$ is also isometrically embedded in $\mathbf{R}^K$. For $\lambda > 0$ sufficiently small, define $u_\lambda : B_1(0) \to \lambda^{-1}Y$ to be the map $u_\lambda(x) = \lambda^{-1}u(\lambda x)$. Since u maps geodesics to geodesics, u_λ maps geodesics to geodesics of $\lambda^{-1}Y$. From this we can see that u_λ, as a map into $\mathbf{R}^K$, is uniformly Lipschitz continuous and converges uniformly on every compact set to a degree 1 homogeneous minimizing map $u_* : B_1(0) \to T_{u(0)}Y \subset \mathbf{R}^K$. By Proposition 3.1 [5], u_* maps into a k-dimensional flat F of $T_{u(0)}Y$. Observe that $\exp_{u(0)} F$ and $u(B_\sigma(0))$ are both a union of geodesics emanating from $u(0)$. Thus if u bifurcates into different DM's at $x_0 = 0$ (i.e. $u(B_\sigma(0)) \not\subset \exp_{u(0)} F$ for any $\sigma > 0$), then there exists a geodesic γ emanating from 0 such the geodesic $u \circ \gamma$ only intersects $\exp_{u(0)} F$ at $u(0)$. On the other hand since u_* is of degree 1

$$(u \circ \gamma)'(0) = u_* \circ \gamma'(0) \in F$$

is a nonzero vector and since $u \circ \gamma$ is a geodesic this implies $u \circ \gamma \in \exp_{u(0)} F$, a contradiction. This shows that there exists no point at which u bifurcates into different DM's. Q.E.D.

Proof of Theorem 2. First, by Lemma 8 and under the same assumptions as in Theorem 2, there exists a finite energy ρ-equivariant harmonic map

$u : \tilde{X} \to Y$. Since by assumption there is no invariant unbounded convex closed subset of Y (other than Y itself) preserved by Γ, Theorem 1 implies that Y must be equal to its DM, hence $Y \simeq \mathbf{H}^k$. Now again by assumption the image of u must be bounded or equal to $Y \simeq \mathbf{H}^k$. In the first case u is constant, whereas the second case means that u is onto hence an isometry (cf. [2]), which is impossible by assumption. This implies that u is a constant map which in turn implies Theorem 2. Q.E.D.

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